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A reciprocity law in function fields

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Abstract. We generalize Gauss' lemma over function fields, and establish a reciprocity law for power residue symbols. As an application, a reciprocity law for power residue symbols is established in totally imaginary function fields.

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1. Introduction. Let p be an odd prime, and let S be a subset of $\mathbb{Z}$ such that $\{0, S, -S\}$ is a complete set of representatives modulo p. Let $s \in S$ and let $a \in \mathbb{Z}$ be coprime to p. Then, there exist $\epsilon(a, s) \in \{\pm 1\}$ and $s_a \in S$ such that $as = \epsilon(a, s)s_a$. Gauss' lemma states that the Legendre symbol $\left(\frac{a}{p}\right)$ can be written as follows:

$$\left(\frac{a}{p}\right) = \prod_{s \in S} \epsilon(a, s).$$

For distinct odd primes p and q, it holds that

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}},$$

which is called the *quadratic reciprocity law* in the field of rational numbers.

A quadratic reciprocity law for a quadratic number field was first established by Gauss, who provided the quadratic reciprocity law for $\mathbb{Q}(\sqrt{-1})$. The quadratic reciprocity law in imaginary quadratic number fields using elliptic functions was put forth by Herglotz, Niemeyer, Bayad [2], and Hajir and Villegas [8]. In particular, Bayad [2] constructed certain elliptic functions and proved the product formulas for them. As an application, he used the product formulas to establish the quadratic reciprocity law in imaginary quadratic number fields. Hayashi [11] corrected and refined Bayad's reciprocity law.

The analogies between number fields and function fields have several interesting aspects. Artin [1] established a rational function field analog of the quadratic reciprocity law, and Schmidt [16] proved a more general reciprocity law over rational function fields. Carlitz [3–5] provided another proof of the general reciprocity law using an analog of Gauss' lemma. In [10], we generalized the analog of Gauss' lemma over the rational function fields, and provided another proof of the general reciprocity law for power residue symbols. For details of the general reciprocity law over rational function fields, we refer to [15,17]. The purpose of this paper is to provide an analog of Gauss' lemma over general function fields, and establish a reciprocity law for power residue symbols. As an application, a reciprocity law for power residue symbols in totally imaginary function fields is established.

In Section 2, Gauss' lemma is generalized over function fields. In Section 3, a reciprocity law in function fields is established using Gauss' lemma. The last section is devoted to the proof of the theorems in the previous section.

2. Gauss' lemma. Let F be a function field in one variable over the field of constants $\mathbb{F}_q$, a finite field of q elements. Let ∞ be a place of F. We express R as the ring of elements of F that are regular outside ∞. Let F_∞ be the completion of F with respect to ∞, and let C_∞ be the completion of an algebraic closure of F_∞ with respect to ∞. In this section, we introduce certain symbols to establish Gauss' lemma. The ideas of Reichardt are used (see [12,13]).

2.1. Generalized Gauss' lemma. We assume that F contains a primitive n-th root of unity ζ_n. This implies that n divides $q-1$. The group $\mathbb{F}_q^*$ contains the n-th roots of unity $\mu_n := \{1, \zeta_n, \dots, \zeta_n^{n-1}\}$. Let $\mathfrak{p}$ be a prime ideal of R, and let $\varphi(\mathfrak{p})$ be the order of the unit group $(R/\mathfrak{p})^*$. When $\alpha \in R$ is coprime to $\mathfrak{p}$, there exists a unique element $\left(\frac{\alpha}{\mathfrak{p}}\right)_n \in \mu_n$ such that

$$\alpha^{\varphi(\mathfrak{p})/n} \equiv \left(\frac{\alpha}{\mathfrak{p}}\right)_n \pmod{\mathfrak{p}}.$$

When $\alpha \in R$ is contained in $\mathfrak{p}$, let $\left(\frac{\alpha}{\mathfrak{p}}\right)_n = 0$. For any ideal $\mathfrak{a}$ in R having the prime ideal decomposition $\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$, we extend the above symbol multiplicatively by setting $\left(\frac{\alpha}{\mathfrak{a}}\right)_n = \left(\frac{\alpha}{\mathfrak{p}_1}\right)_n^{e_1} \cdots \left(\frac{\alpha}{\mathfrak{p}_r}\right)_n^{e_r}$. This symbol is called the *n-th power residue symbol*.

The function field F contains the n-th roots of unity $\mu_n = \{1, \zeta_n, \dots, \zeta_n^{n-1}\}$. Let $\mathfrak{a}$ be a non-zero ideal of R. A subset $S = \{s_1, \dots, s_m\}$ of R such that $\{0, S, \zeta_n S, \dots, \zeta_n^{n-1} S\}$ is a complete set of representatives modulo $\mathfrak{a}$ is called a $1/n$-*system modulo* $\mathfrak{a}$. Let $\alpha \in R$ be coprime to $\mathfrak{a}$ and let S be a $1/n$-system modulo $\mathfrak{a}$. There exists a permutation π of $\{1, \dots, m\}$ such that for any $s_j \in S$, there exists an element $\zeta_n^{a(j)} \in \mu_n$ such that

$$\alpha s_j \equiv \zeta_n^{a(j)} s_{\pi(j)} \pmod{\mathfrak{a}}. \tag{2.1}$$

We write $\epsilon(\alpha, s_j)$ for $\zeta_n^{a(j)}$.

The following theorem is an analog of Gauss' lemma:

Theorem 2.1 (Generalized Gauss' lemma). *For any element $\alpha \in R$ coprime to $\mathfrak{a}$,*

$$\left(\frac{\alpha}{\mathfrak{a}}\right)_n = \prod_{s \in S} \epsilon(\alpha, s),$$

where S is a $1/n$-system modulo $\mathfrak{a}$.

2.2. Proof of Theorem 2.1. Set

$$\left\{\frac{\alpha}{\mathfrak{a}}\right\}_n = \prod_{s \in S} \epsilon(\alpha, s),$$

which is independent of the choice of S. Indeed, let $S' = \{s'_1, \ldots, s'_m\}$ be another $1/n$-system modulo $\mathfrak{a}$. There exist permutations π and π' of $\{1, \ldots, m\}$ such that for any j,

$$\alpha s_j \equiv \epsilon(\alpha, s_j) s_{\pi(j)} \pmod{\mathfrak{a}}, \quad \alpha s'_j \equiv \epsilon(\alpha, s'_j) s_{\pi'(j)} \pmod{\mathfrak{a}}.$$

There exists a permutation σ of $\{1, \ldots, m\}$ such that for any j,

$$s_j \equiv \zeta_n^{b(j)} s'_{\sigma(j)} \pmod{\mathfrak{a}}, \quad \alpha s_j \equiv \zeta_n^{b(j)} \alpha s'_{\sigma(j)} \equiv \zeta_n^{b(j)} \epsilon(\alpha, s'_{\sigma(j)}) s'_{\pi'\sigma(j)} \pmod{\mathfrak{a}}.$$

Applying these congruences to the exponential function $e_{\mathfrak{a}}$ for $\mathfrak{a}$, we obtain

$$e_{\mathfrak{a}}(\alpha s_j) = \epsilon(\alpha, s_j) e_{\mathfrak{a}}(s_{\pi(j)}), \quad e_{\mathfrak{a}}(\alpha s'_j) = \epsilon(\alpha, s'_j) e_{\mathfrak{a}}(s_{\pi'(j)}),$$
$$e_{\mathfrak{a}}(\alpha s_j) = \zeta_n^{b(j)} \epsilon(\alpha, s'_{\sigma(j)}) e_{\mathfrak{a}}(s_{\pi 7\sigma(j)}).$$

Hence, we have

$$\prod_{j=1}^{m} \epsilon(\alpha, s_j) = \prod_{j=1}^{m} \frac{e_{\mathfrak{a}}(\alpha s_j)}{e_{\mathfrak{a}}(s_j)} = \prod_{j=1}^{m} \frac{\zeta_n^{b(j)} \epsilon(\alpha, s'_{\sigma(j)}) e_{\mathfrak{a}}(s'_{\pi'\sigma(j)})}{\zeta_n^{b(j)} e_{\mathfrak{a}}(s'_{\sigma(j)})} = \prod_{j=1}^{m} \epsilon(\alpha, s'_j).$$

Let $S^* = \{s_j \in S \mid s_j \text{ coprime to } \mathfrak{a}\}$, and set

$$\left\{\frac{\alpha}{\mathfrak{a}}\right\}_n^* = \prod_{s \in S^*} \epsilon(\alpha, s),$$

which is independent of the choice of S. This can be proven in the similar way as the case $\left\{\frac{\alpha}{\mathfrak{a}}\right\}_n$. For any ideal $\mathfrak{b}$ of R containing $\mathfrak{a}$, set

$$S_{\mathfrak{b}} = \{s_j \in S \mid s_j R + \mathfrak{a} = \mathfrak{b}\}.$$

We observe that $S_{\mathfrak{a}} = \phi$ and $S_R = S^*$. The set S is a disjoint union of $S_{\mathfrak{b}}$ ($\mathfrak{a} \subset \mathfrak{b}$). A subset T of R such that $\{T, \zeta_n T, \ldots, \zeta_n^{n-1} T\}$ is a complete set of the representatives of $(R/\mathfrak{a})^*$ is referred to as a *prime $1/n$-system modulo $\mathfrak{a}$*. The following lemma is required to prove Gauss' lemma.

Lemma 2.2. *Let $\mathfrak{a}$ be a non-zero ideal of R. For any element $\alpha \in R$ coprime to $\mathfrak{a}$,*

$$\left\{\frac{\alpha}{\mathfrak{a}}\right\}_n = \prod_{\mathfrak{b} \mid \mathfrak{a}} \left\{\frac{\alpha}{\mathfrak{b}}\right\}_n^*.$$

Proof. Let $S_{\mathfrak{b}} = \{t_1, \ldots, t_k\}$. There exists an element $u_1 \in \mathfrak{b}^{-1}$ such that $t_1 u_1 \equiv 1 \pmod{\mathfrak{a}\mathfrak{b}^{-1}}$.

It holds that $\{t_1 u_1, \ldots, t_k u_1\}$ is a prime $1/n$-system modulo $\mathfrak{a}\mathfrak{b}^{-1}$. Indeed, we take any $\alpha \in R$ coprime to $\mathfrak{a}\mathfrak{b}^{-1}$. For each i, there exists $\zeta_n^{a(i)} \in \mu_n$ and $s_j \in S$ such that

$$\alpha t_i \equiv \zeta_n^{a(i)} s_j \pmod{\mathfrak{a}}. \tag{2.2}$$

We see easily that $s_j \in S_{\mathfrak{b}}$ and that

$$\alpha t_i u_1 \equiv \zeta_n^{a(i)} s_j u_1 \pmod{\mathfrak{a}\mathfrak{b}^{-1}}. \tag{2.3}$$

Next, we assume that there exist $\zeta, \zeta' \in \mu_n$ and $t_i, t_j \in S_{\mathfrak{b}}$ such that $\zeta t_i u_1 \equiv \zeta' t_j u_1 \pmod{\mathfrak{a}\mathfrak{b}^{-1}}$. Multiplying this congruence by t_1, we obtain $\zeta t_i \equiv \zeta' t_j \pmod{\mathfrak{a}}$, which implies $\zeta = \zeta'$ and $t_i = t_j$.

Observing that (2.2) and (2.3) are equivalent and that $S = \bigcup_{\mathfrak{b} \subset \mathfrak{a}} S_{\mathfrak{b}}$,

$$\left\{\frac{\alpha}{\mathfrak{a}}\right\}_n = \prod_{\mathfrak{b}|\mathfrak{a}} \left\{\frac{\alpha}{\mathfrak{a}\mathfrak{b}^{-1}}\right\}_n^* = \prod_{\mathfrak{b}|\mathfrak{a}} \left\{\frac{\alpha}{\mathfrak{b}}\right\}_n^*.$$

□

Lemma 2.3. *Let $\mathfrak{a}$ be a non-zero ideal of R. For any element $\alpha \in R$ coprime to $\mathfrak{a}$,*

$$\left\{\frac{\alpha}{\mathfrak{a}}\right\}_n^* \equiv \alpha^{\varphi(\mathfrak{a})/n} \pmod{\mathfrak{a}}. \tag{2.4}$$

Proof. Since $\{S^*, \zeta_n S^*, \ldots, \zeta_n^{n-1} S^*\}$ is a complete set of representatives for $(R/\mathfrak{a})^*$, $|S^*| = \varphi(\mathfrak{a})/n$. Using (2.1), we have

$$\begin{aligned}\left(\prod_{s_j \in S^*} s_j\right) \alpha^{\varphi(\mathfrak{a})/n} &\equiv \prod_{s_j \in S^*} \alpha s_j \equiv \prod_{s_j \in S^*} \epsilon(\alpha, s_j) s_{\pi(j)} \pmod{\mathfrak{a}} \\ &\equiv \left(\prod_{s_j \in S^*} s_{\pi(j)}\right)\left(\prod_{s_j \in S^*} \epsilon(\alpha, s_j)\right) \pmod{\mathfrak{a}} \\ &\equiv \left(\prod_{s_j \in S^*} s_j\right) \left\{\frac{\alpha}{\mathfrak{a}}\right\}_n^* \pmod{\mathfrak{a}},\end{aligned}$$

where π is a permutation of $\{1, \ldots, \varphi(\mathfrak{a})/n\}$. This yields (2.4). □

For any non-zero ideal $\mathfrak{b}$ of R, it holds that

$$\left\{\frac{\alpha}{\mathfrak{b}}\right\}_n^* = \begin{cases} \left(\frac{\alpha}{\mathfrak{p}}\right)_n & \text{if } \mathfrak{b} = \mathfrak{p}^a, \\ 1 & \text{otherwise.} \end{cases} \tag{2.5}$$

Indeed, when there exist coprime ideals $\mathfrak{c}, \mathfrak{d}$ of R such that $\mathfrak{b} = \mathfrak{c}\mathfrak{d}$, $\alpha^{\varphi(\mathfrak{c})} \equiv 1 \pmod{\mathfrak{c}}$ implies $\alpha^{\varphi(\mathfrak{b})/n} \equiv \left(\alpha^{\varphi(\mathfrak{c})}\right)^{\varphi(\mathfrak{d})/n} \equiv 1 \pmod{\mathfrak{c}}$. Similarly, we have $\alpha^{\varphi(\mathfrak{b})/n} \equiv 1 \pmod{\mathfrak{d}}$, which yields $\alpha^{\varphi(\mathfrak{b})/n} \equiv 1 \pmod{\mathfrak{b}}$. Next, we consider a case in which there exists a prime ideal $\mathfrak{p}$ and a positive number a such that $\mathfrak{b} = \mathfrak{p}^a$. If the cardinality of $R/\mathfrak{p}$ is p^f, then $\varphi(\mathfrak{b}) = p^{f(a-1)}\left(p^f - 1\right)$. As $p^f \equiv 1 \pmod{n}$, using Lemma 2.3,

$$\left\{\frac{\alpha}{\mathfrak{b}}\right\}_n^* \equiv \alpha^{\varphi(\mathfrak{b})/n} \equiv \left(\frac{\alpha}{\mathfrak{p}}\right)_n^{p^{f(a-1)}} \equiv \left(\frac{\alpha}{\mathfrak{p}}\right)_n \pmod{\mathfrak{p}},$$

which yields (2.5).

When $\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$, using Lemma 2.2 and (2.5), we obtain

$$\left\{\frac{\alpha}{\mathfrak{a}}\right\}_n = \prod_{\mathfrak{b}|\mathfrak{a}} \left\{\frac{\alpha}{\mathfrak{b}}\right\}_n^* = \prod_{i=1}^{r}\prod_{j=1}^{e_i} \left\{\frac{\alpha}{\mathfrak{p}_i^j}\right\}_n^* = \prod_{i=1}^{r} \left(\frac{\alpha}{\mathfrak{p}_i}\right)_n^{e_i} = \left(\frac{\alpha}{\mathfrak{a}}\right)_n.$$

This completes the proof of Theorem 2.1. □

3. Reciprocity laws. Let ϕ be a rank one Drinfeld R-module corresponding to the R-lattice $L = \xi R$. For coprime $\alpha, \beta \in R$, let $\left(\frac{\alpha}{\beta}\right)_n = \left(\frac{\alpha}{\beta R}\right)_n$. The following is a reciprocity law for the n-th power residue symbol.

Theorem 3.1. *For coprime $\alpha, \beta \in R$,*

$$\left(\frac{\alpha}{\beta}\right)_n \left(\frac{\beta}{\alpha}\right)_n^{-1} = (-1)^{\frac{N(\alpha)-1}{n}\frac{N(\beta)-1}{n}} \nu(\alpha)^{\frac{N(\beta)-1}{n}} \nu(\beta)^{-\frac{N(\alpha)-1}{n}}, \tag{3.1}$$

where $N(\alpha) = q^{\deg(\alpha)}$ is the norm of α, and $\nu(\alpha)$ is the leading coefficient of ϕ_α.

Remark 3.2. The value $\nu(\alpha)^{\frac{N(\beta)-1}{n}} \nu(\beta)^{-\frac{N(\alpha)-1}{n}}$ in the above theorem belongs to μ_n. Indeed, for $\alpha, \beta \in R\backslash\{0\}$, the value $\phi_{\alpha\beta}$ of the Drinfeld module ϕ provides $\phi_{\alpha\beta} = \phi_\alpha\phi_\beta = \phi_\beta\phi_\alpha$, which yields $\nu(\alpha\beta) = \nu(\alpha)\nu(\beta)^{N(\alpha)} = \nu(\alpha)^{N(\beta)}\nu(\beta)$. Therefore, the claim follows from this.

From Theorem 3.1, we obtain another type of reciprocity law:

Theorem 3.3. *There exists a multiplicative function $\kappa : R\backslash\{0\} \to C_\infty^*$ such that for coprime $\alpha, \beta \in R$,*

$$\left(\frac{\alpha}{\beta}\right)_n \left(\frac{\beta}{\alpha}\right)_n^{-1} = (-1)^{\frac{N(\alpha)-1}{n}\frac{N(\beta)-1}{n}} \kappa(\alpha)^{\frac{N(\beta)-1}{n}} \kappa(\beta)^{-\frac{N(\alpha)-1}{n}}. \tag{3.2}$$

Remark 3.4. Let $A = \mathbb{F}_q[T]$ be the polynomial ring over $\mathbb{F}_q$, and let $K = \mathbb{F}_q(T)$ be its quotient field. Let $K_\infty = \mathbb{F}_q((1/T))$ be the completion of K with respect to $\infty = (1/T)$, and let C_∞ be the completion of an algebraic closure of K_∞ with respect to ∞. A separable extension F/K is called *totally imaginary* if ∞ has only one prime over F. For details of such extensions, we refer the reader to the papers Gekeler [6], Rosen [14], and Hamahata [9]. Let F be a totally imaginary extension of K, and let O_F be the integral closure of A in F. Using Theorem 3.1 for $R = O_F$, we have the analog of Bayad [2, Théorème 2.9].

4. Proof of Theorems 3.1 and 3.3.

4.1. Proof of Theorem 3.1. Let $T_\beta = \{y_1, \ldots, y_m\}$ be a $1/n$-system modulo βR, and set $S_\beta = \xi\beta^{-1}T_\beta$. For each i, it holds that $\alpha y_i \equiv \epsilon(\alpha, y_i)y_{\pi(i)} \pmod{\beta R}$ if and only if $\alpha\xi\beta^{-1}y_i \equiv \epsilon(\alpha, y_i)\xi\beta^{-1}y_{\pi(i)} \pmod{L}$. Using $e_L(\alpha\xi\beta^{-1}y_i) = \epsilon(\alpha, y_i)e_L(\xi\beta^{-1}y_{\pi(i)})$ and Theorem 2.1,

$$\left(\frac{\alpha}{\beta}\right)_n = \prod_{z\in S_\beta} \frac{e_L(\alpha z)}{e_L(z)} = \prod_{z\in S_\beta} \frac{\phi_\alpha(e_L(z))}{e_L(z)}.$$

We can identify $C_\infty\{\tau\}$ with the non-commutative ring of additive polynomials of X with coefficients in C_∞, where the product is the composition of maps. For $\phi_\alpha \in C_\infty\{\tau\}$, we write $\phi_\alpha(X) = \alpha X + \cdots + \nu(\alpha)X^{N(\alpha)}$. As

$$\begin{aligned}\phi_\alpha(X) &= \nu(\alpha) \prod_{y\in L/\alpha L} (X - e_L(y/\alpha)) \\ &= \nu(\alpha)X \prod_{\zeta\in\mu_n}\prod_{y\in T_\alpha} (X - \zeta e_L(y/\alpha)) \\ &= \nu(\alpha)X \prod_{z\in S_\alpha} (X^n - e_L(z)^n)\,, \\ \left(\frac{\alpha}{\beta}\right)_n &= \prod_{x\in S_\beta} \nu(\alpha) \prod_{z\in S_\alpha} (e_L(x)^n - e_L(z)^n) \\ &= \nu(\alpha)^{\frac{N(\beta)-1}{n}}(-1)^{\frac{N(\alpha)-1}{n}\frac{N(\beta)-1}{n}} \prod_{x\in S_\beta}\prod_{z\in S_\alpha} (e_L(z)^n - e_L(x)^n)\,.\end{aligned}$$

Similarly, we have

$$\left(\frac{\beta}{\alpha}\right)_n = \nu(\beta)^{\frac{N(\alpha)-1}{n}} \prod_{x\in S_\beta}\prod_{z\in S_\alpha} (e_L(z)^n - e_L(x)^n)\,,$$

which yields (3.1), as desired. □

4.2. Proof of Theorem 3.3. We retain the notations used in Theorem 3.1. Fix a sign function sgn : $F_\infty^* \to \mathbb{F}_\infty^*$, where $\mathbb{F}_\infty$ is the field of constants of F_∞. Let $\operatorname{sgn}(0) = 0$. There exists an element $c \in C_\infty$ such that $\psi := c\phi c^{-1}$ is a sgn-normalized Drinfeld R-module. When $\kappa(\alpha)$ is the leading coefficient of $\psi_\alpha(X)$, $\kappa : R \to \mathbb{F}_\infty$ is a twisting of sgn. From $\kappa(\alpha) = c^{1-N(\alpha)}\nu(\alpha)$ ($\alpha \in R$), we have

$$\nu(\alpha)^{\frac{N(\beta)-1}{n}}\nu(\beta)^{-\frac{N(\alpha)-1}{n}} = \kappa(\alpha)^{\frac{N(\beta)-1}{n}}\kappa(\beta)^{-\frac{N(\alpha)-1}{n}},$$

which yields (3.2), as desired. □

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