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Cohen-Macaulay weighted oriented chordal and simplicial graphs

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Abstract. Herzog, Hibi, and Zheng classified the Cohen-Macaulay edge ideals of chordal graphs. In this paper, we classify Cohen-Macaulay edge ideals of (vertex) weighted oriented chordal and simplicial graphs, a more general class of monomial ideals. In particular, we show that the Cohen-Macaulay property of these ideals is equivalent to the unmixed one and hence, independent of the underlying field.

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1. Introduction. The study of combinatorial commutative algebra began with the pioneering work of R.P. Stanley [16] and G. Reisner [14] in 1975. The study of square-free monomial ideals grabbed the attention of researchers when it was seen in terms of simplicial complexes. Among the sub-classes of square-free monomial ideals, edge ideals of simple graphs, introduced by Villarreal in [17], stand out with their own identity.

The importance of the Cohen-Macaulay property in combinatorics comes through the proof of the "upper bound conjecture". Since Cohen-Macaulay simplicial complexes play an essential role in the study of algebraic geometry, algebraic topology, and combinatorics, finding good classes of Cohen-Macaulay monomial ideals is always in high demand. Because, via the polarization technique, corresponding to any Cohen-Macaulay monomial ideal, we get a Cohen-Macaulay simplicial complex. Corresponding to a simplicial complex Δ, we can associate a graph G such that $I(G)$ is Cohen-Macaulay if and only if Δ is Cohen-Macaulay, where $I(G)$ denotes the edge ideal of G. Therefore, the problem of classifying Cohen-Macaulay edge ideals of graphs is as difficult as classifying all Cohen-Macaulay simplicial complexes. There is evidence of edge ideals whose Cohen-Macaulay property depends on the base field. So, one can not expect a general classification, and people are interested in finding those classes of graphs whose Cohen-Macaulay property is independent of the underlying field. In this direction, some remarkable works are as follows: the classification of all Cohen-Macaulay trees by Villarreal [18]; all Cohen-Macaulay bipartite graphs by Herzog & Hibi [8]; all Cohen-Macaulay chordal graphs by Herzog, Hibi, and Zheng [9].

In the recent past, the notion of edge ideals of vertex-weighted directed graphs has been defined in [11], which is a generalization of the edge ideals of simple graphs.

A *weighted oriented graph* D_G, with an underlying simple graph G, is a directed graph on the vertex set $V(D_G) := V(G)$ with a weight function $w : V(D_G) \longrightarrow \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integers. We denote the edge set of D_G by $E(D_G)$. An edge of D_G is denoted by the ordered pair (x_i, x_j), which means the direction of the edge is from x_i to x_j.

Definition 1.1. Let D_G be a weighted oriented graph with $V(D_G) = \{x_1, \ldots, x_n\}$. Then the *edge ideal* of D_G, denoted by $I(D_G)$, is defined as

$$I(D_G) := \left\langle \{x_i x_j^{w(x_j)} \mid (x_i, x_j) \in E(D_G)\} \right\rangle$$

in the polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K. When $w(x_i) = 1$ for all i, $I(D_G)$ coincides with the usual edge ideal $I(G)$ of G. By saying D_G or $I(D_G)$ is Cohen-Macaulay, we mean the quotient ring $R/I(D_G)$ is Cohen-Macaulay.

The motivation behind studying weighted oriented edge ideals has its foundation in coding theory. Specifically, they appear as the initial ideals of certain vanishing ideals $I(\mathcal{X})$ of some sets of projective points $\mathcal{X}$ in the study of Reed-Muller-type codes (see [1], [10]). One can estimate some basic parameters and properties of Reed-Muller-type codes associated with $\mathcal{X}$ by studying weighted oriented edge ideals. For example, if $I(D_G)$ is Cohen-Macaulay, then $I(\mathcal{X})$ is Cohen-Macaulay.

Like the classification problem of Cohen-Macaulay simple graphs, people are interested in finding suitable classes of Cohen-Macaulay weighted oriented graphs (independent of the base field). For certain classes of weighted oriented edge ideals, Cohen-Macaulay ones are characterized combinatorially: (i) path and complete graphs [11]; (ii) forests [5]; (iii) bipartite graphs and graphs with perfect matching [6]; (iv) König graphs [12]; (v) cycles [15].

Chordal graphs play a significant role in the theory of edge ideals of simple graphs. This class is reflected in the celebrated Fröberg theorem [4], which states that $I(G)$ has a linear resolution if and only if G^c (complement of G) is chordal. Again, Francisco and Van Tuyl showed that if G is a chordal graph, then $R/I(G)$ is sequentially Cohen-Macaulay [3]. Herzog et al. gave the combinatorial characterization of Cohen-Macaulay chordal graphs in [9]. For chordal graphs, they proved that the Cohen-Macaulay property of $I(G)$ is equivalent to the unmixed one and, thus, independent of the field K.

In this article, we characterize all Cohen-Macaulay chordal and simplicial weighted oriented graphs. In [2], the unmixed chordal and simplicial weighted oriented graphs were classified. We show in Theorem 3.1 that if the underlying graph G of a weighted oriented graph D_G is chordal or simplicial, then $I(D_G)$ is Cohen-Macaulay if and only if $I(D_G)$ is unmixed. This ensures that the Cohen-Macaulay property of edge ideals of weighted oriented chordal and simplicial graphs is independent of the base field. Our result generalizes the theorem of Herzog, Hibi, and Zheng [9] and includes a larger class of Cohen-Macaulay monomial ideals. Also, as a corollary, we identify all Gorenstein edge ideals of weighted oriented chordal and simplicial graphs.

2. Preliminaries. In this section, we recall some definitions and results related to our work. By a simple graph, we mean an undirected graph without multiple edges or loops.

Let G be a simple graph. A *vertex cover* C of G is a subset of $V(G)$ such that $e \cap C \neq \emptyset$ for all $e \in E(G)$. A *minimal vertex cover* of G is a vertex cover C of G such that if $C' \subsetneq C$, then C' can not be a vertex cover of G. Let D_G be a weighted oriented graph. For a vertex $v \in V(G)$, we call $\mathcal{N}_G(v) := \{u \in V(G) \mid \{u, v\} \in E(G)\}$ the *neighbour set* of v in G. We write $\mathcal{N}_G[v] := \mathcal{N}_G(v) \cup \{v\}$. A graph G is said to be *complete* if there is an edge between every pair of vertices. For $A \subseteq V(G)$, we denote the induced subgraph of G on the vertex set A by $G[A]$.

Definition 2.1. A *cycle* of length n, denoted by C_n, is a connected graph on n vertices such that $V(C_n) = \{x_1, \ldots, x_n\}$ and $E(C_n) = \{\{x_i, x_{i+1}\} \mid 1 \leq i \leq n \text{ and } x_{n+1} = x_1\}$. A graph G is called a *chordal graph* if G has no induced cycle of length > 3.

Definition 2.2. A vertex $v \in V(G)$ is said to be a *simplicial vertex* if the induced subgraph of G on $\mathcal{N}_G[v]$ is complete i.e., $G[\mathcal{N}_G[v]]$ is a complete graph and in this case, $G[\mathcal{N}_G[v]]$ is called a *simplex* of G. A graph G is said to be a *simplicial graph* if every vertex of G is a simplicial vertex of G or is adjacent to a simplicial vertex of G.

Definition 2.3 ([11]). Let D_G be a weighted oriented graph. Corresponding to a vertex cover C of G, consider the following sets

$$\mathcal{L}_1(C) = \{x \in C \mid \exists\, (x, y) \in E(D_G) \text{ such that } y \notin C\},$$
$$\mathcal{L}_2(C) = \{x \in C \mid x \notin \mathcal{L}_1(C) \text{ and } \exists\, (y, x) \in E(D_G) \text{ such that } y \notin C\},$$
$$\mathcal{L}_3(C) = C \setminus (\mathcal{L}_1(C) \cup \mathcal{L}_2(C)) = \{x \in C \mid \mathcal{N}_G(x) \subseteq C\}.$$

A vertex cover C of G is called a *strong vertex cover* of D_G if C is either a minimal vertex cover of G or for all $x \in \mathcal{L}_3(C)$, there is an edge $(y, x) \in E(D_G)$ with $y \in \mathcal{L}_2(C) \cup \mathcal{L}_3(C)$ and $w(y) \neq 1$.

Remark 2.4. Let D_G be a weighted oriented graph. Then the radical of $I(D_G)$ is $I(G)$. Thus, the minimal primes of $I(D_G)$ are the associated primes of $I(G)$, and they are precisely the ideals generated by the minimal vertex covers of G.

For a vertex cover C of D_G, consider the following irreducible ideal associated to C

$$Q_C := \langle \mathcal{L}_1(C) \cup \{x_j^{w(x_j)} \mid x_j \in \mathcal{L}_2(C) \cup \mathcal{L}_3(C)\}\rangle.$$

Theorem 2.5 ([11, Theorem 25]). *Let $\mathcal{C}_s$ denote the set of all strong vertex covers of D_G. Then the irredundant irreducible primary decomposition of $I(D_G)$ is given by*

$$I(D_G) = \bigcap_{C \in \mathcal{C}_s} Q_C.$$

Moreover, $\{P_C \mid P_C = \langle C\rangle,$ where $C \in \mathcal{C}_s\}$ is the set of associated primes of $I(D_G)$.

Theorem 2.6 ([11, Theorem 31]). *$I(D_G)$ is unmixed if and only if $I(G)$ is unmixed and $\mathcal{L}_3(C) = \emptyset$ for any strong vertex cover C of D_G.*

3. The Cohen-Macaulay classification. In this section, we will show that the Cohen-Macaulay property of weighted oriented chordal and simplicial graphs is equivalent to the unmixed property. For these classes of graphs, we also characterize the Gorenstein ones.

Theorem 3.1. *Let D_G be a weighted oriented graph such that G is chordal or simplicial. Then $I(D_G)$ is Cohen-Macaulay if and only if $I(D_G)$ is unmixed.*

Proof. The "only if" part of the theorem is well-known. Let $I(D_G)$ be unmixed. Then $I(G)$ is unmixed. Therefore, by [13, Theorem 1 and 2], every vertex of G belongs to exactly one simplex of G, and the vertex sets of simplices form a partition of $V(G)$. Let $H_1, \ldots, H_m$ be all the simplices of D_G. Without loss of generality, let $V(H_1) = \{x_1, \ldots, x_{k_1}\}$, $V(H_2) = \{x_{k_1+1}, \ldots, x_{k_2}\}, \ldots, V(H_m) = \{x_{k_{m-1}+1}, \ldots, x_{k_m}\}$ and x_{k_i} is a simplicial vertex of G belonging to the simplex H_i for each $1 \leq i \leq m$. Let $h_i = x_{k_{i-1}+1} + \cdots + x_{k_i}$ for each $1 \leq i \leq m$, where $k_0 = 0$.
Claim: The polynomials $h_1, \ldots, h_m$ form a regular sequence on $R/I(D_G)$.
Proof of the claim. $I(D_G)$ is a monomial ideal, and so, every associated prime of $I(D_G)$ is generated by a set of variables. Since $I(D_G)$ is unmixed, all the associated primes of $I(D_G)$ are minimal, and they are the associated primes of the ideal $I(G)$, the radical of $I(D_G)$. Thus, by Remark 2.4, the associated primes of $I(D_G)$ are generated by the minimal vertex covers of G. Note that no minimal vertex cover of G contains the set $V(H_1)$ as $\mathcal{N}_G[x_{k_1}] = V(H_1)$. Therefore, h_1 does not belong to any associated prime of $I(D_G)$. Hence, h_1 is a regular element on $R/I(D_G)$. Now, we will show that h_i is regular on $R/\langle I(D_G), h_1, \ldots, h_{i-1}\rangle$. Let $I(D_G) = \bigcap_{C \in \mathcal{C}_s} Q_C$ be the primary decomposition of $I(D_G)$ as described in Theorem 2.5. Let P be an associated prime of the ideal $\langle I(D_G), h_1, \ldots, h_{i-1}\rangle$. Then $I(D_G) \subseteq P$ implies $Q_C \subseteq P$ for some $C \in \mathcal{C}_s$ and hence, $P_C \subseteq P$. Since H_j is a simplex for each $1 \leq j \leq m$, at least $|V(H_j)| - 1$ elements of $V(H_j)$ belong to P. Therefore, for each $1 \leq j \leq i-1$, we have $V(H_j) \subseteq P$ as $h_1, \ldots, h_{i-1} \in P$. Now, it is clear that $P_C + \langle \bigcup_{j=1}^{i-1} V(H_j)\rangle \subseteq P$. Suppose $P_C + \langle \bigcup_{j=1}^{i-1} V(H_j)\rangle \subsetneq P$. Then P has

an element g such that $g \notin P_C + \langle \bigcup_{j=1}^{i-1} V(H_j) \rangle$. Then we can choose g from the polynomial ring $K[V(G) \setminus (C \cup V(H_1) \cup \cdots \cup V(H_{i-1}))]$. Since P is an associated prime ideal of $\langle I(D_G), h_1, \ldots, h_{i-1} \rangle$ and $h_1, \ldots, h_{i-1}$ are linear forms, we can choose a homogeneous polynomial f such that $\langle I(D_G), h_1, \ldots, h_{i-1} \rangle : f = P$ and $g \in I(D_G) : f$. Then $I(D_G)$ will have an associated prime containing $\langle P_C, g \rangle$, which is a contradiction as $I(D_G)$, being unmixed, has no embedded prime. Hence, any associated prime of $\langle I(D_G), h_1, \ldots, h_{i-1} \rangle$ is of the form $P_C + \langle \bigcup_{j=1}^{i-1} V(H_j) \rangle$ for some $C \in \mathcal{C}_s$. Now, from this observation, it is clear that the element h_i does not belong to P, otherwise $V(H_i) \subseteq P$, which is a contradiction as $V(H_i) \cap V(H_j) = \emptyset$ for all $1 \leq j \leq i-1$ and C, being the minimal cover of D_G, can not contain the set $V(H_i)$. We have chosen $C \in \mathcal{C}_s$ in an arbitrary fashion and thus, h_i can not belong to any associated prime of $\langle I(D_G), h_1, \ldots, h_{i-1} \rangle$. Hence, h_i is a regular element on $R/\langle I(D_G), h_1, \ldots, h_{i-1} \rangle$. By induction hypothesis, we get that $h_1, \ldots, h_m$ form a regular sequence on $R/I(D_G)$. Therefore, $\operatorname{depth}(R/I(D_G)) \geq m$. Now, $I(D_G)$ is unmixed, which implies any associated prime of $I(D_G)$ is generated by some minimal vertex cover of G. Since $V(H_1), \ldots, V(H_m)$ form a partition of $V(D_G)$, each minimal vertex cover of G contains exactly $\sum_{i=1}^{m}(|V(H_i)| - 1) = |V(D_G)| - m$ vertices. Thus, $\operatorname{ht}(I(D_G)) = |V(D_G)| - m$ and $\dim(R/I(D_G)) = |V(D_G)| - \operatorname{ht}(I(D_G)) = m$. It is well-known that $\operatorname{depth}(R/I(D_G)) \leq \dim(R/I(D_G))$, which gives $\operatorname{depth}(R/I(D_G)) = \dim(R/I(D_G)) = m$. Hence, $R/I(D_G)$ is Cohen-Macaulay.

□

Corollary 3.2. *Let D_G be a weighted oriented graph such that G is chordal or simplicial. Then $R/I(D_G)$ is Gorenstein if and only if D_G is a disjoint union of edges.*

Proof. Let G be a chordal or simplicial graph such that $I(G)$ is unmixed. Then by [13, Theorem 1 and 2], each vertex of G belongs to exactly one simplex of G, and the vertex sets of simplices form a partition of $V(G)$. Therefore, the same argument given for chordal graphs in [9, Corollary 2.1] is also applicable to simplicial graphs. Hence, it follows from [9, Corollary 2.1] that $R/I(G)$ is Gorenstein if and only, if G is a disjoint union of edges. Now, $I(D_G)$ being a monomial ideal, by [7, Theorem 2.6], $R/I(D_G)$ is Gorenstein implies $R/I(G)$ is Gorenstein. Thus, D_G is a disjoint union of edges.

Conversely, if D_G is a disjoint union of edges, then $I(D_G)$ is complete intersection, and hence, $R/I(D_G)$ is Gorenstein. □

Example 3.3. Consider the weighted oriented chordal graph D_G in Fig. 1. Note that the induced subgraphs $G[\{x_1, x_2, x_3, x_4\}]$, $G[\{x_5, x_6, x_7\}]$, and $G[\{x_8, x_9, x_{10}\}]$ are simplices of G and each vertex of G belongs to exactly one of these simplices. Then by [13, Theorem 2], $I(G)$ is unmixed. Let C be a strong vertex cover of D_G. If $\{x_1, x_2, x_3, x_4\} \subseteq C$, then $x_1 \in \mathcal{L}_3(C)$, but, there is no vertex x_i with $w(x_i) > 1$ and $(x_i, x_1) \in E(D_G)$. This gives a contradiction to the definition of strong vertex cover. Therefore, $\{x_1, x_2, x_3, x_4\} \nsubseteq C$ and similarly, $\{x_5, x_6, x_7\}, \{x_8, x_9, x_{10}\} \nsubseteq C$. Hence, $\mathcal{L}_3(C) = \emptyset$, and this is true for any

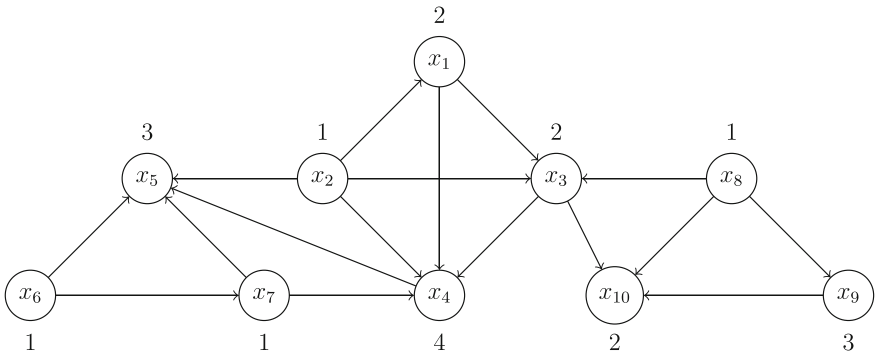

Figure 1. A Cohen-Macaulay weighted oriented chordal graph D_G

strong vertex cover of D_G. Thus, by Theorem 2.6, $I(D_G)$ is unmixed and from Theorem 3.1, we get $R/I(D_G)$ is Cohen-Macaulay.

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