Arch. Math. 121 (2023), 731–744

0003-889X/23/5-60731-14
published online November 30, 2023
https://doi.org/10.1007/s00013-023-01944-0

Archiv der Mathematik

Nittka's invariance criterion and Hilbert space valued parabolic equations in L_p

W. Arendt, A.F.M. ter Elst, and M. Sauter

Abstract. Nittka gave an efficient criterion on a form defined on $L_2(\Omega)$ which implies that the associated semigroup is L_p-invariant for some given $p \in (1, \infty)$. We extend this criterion to the Hilbert space valued $L_2(\Omega, H)$. As an application, we consider elliptic systems of purely second order. Our main result shows that the induced semigroup is L_p-contractive for all $p \in [p_-, p_+]$ for some $1 < p_- < 2 < p_+ < \infty$.

Mathematics Subject Classification. 35B45, 35K15, 46B20.

Keywords. Semigroup, L_p-contraction, Sesquilinear forms, Strictly convex Banach spaces, Elliptic systems.

1. Introduction. Let $(S_t)_{t>0}$ be a C_0-semigroup on $L_2(\Omega)$, with $\Omega \subset \mathbb{R}^d$, which is associated with a closed sesquilinear form. Ouhabaz [16] gave a convenient criterion on the form characterising when the semigroup is L_∞-contractive. It is more complicated to describe L_p-contractivity if $1 < p < \infty$. The reason is the fact that there is no explicit formula which describes the orthogonal projection from $L_2(\Omega)$ to the closed convex set $\{u \in L_2(\Omega) : \|u\|_p \le 1\}$ if $1 < p < \infty$ and $p \ne 2$.

Nonetheless, Nittka [15] succeeded to overcome the difficulty by a structural analysis and developed an efficient criterion for the L_p-contractivity of a C_0-semigroup on $L_2(\Omega)$ which is associated with a form.

The purpose of this paper is twofold. Our first aim is to present Nittka's result. We do this within the more general setting of the vector-valued space $L_2(\Omega, H)$, where H is a Hilbert space. The result is the following.

Theorem 1.1. *Let $(\Omega, \mathcal{B}, \mu)$ be a σ-finite measure space. Let H be a Hilbert space. Fix $p \in (1, \infty)$. Define*

$$C = \{u \in L_2(\Omega, H) \cap L_p(\Omega, H) : \|u\|_{L_p(\Omega,H)} \le 1\}.$$

Birkhäuser

Let P be the orthogonal projection of $L_2(\Omega, H)$ onto C. Let $\mathcal{V}$ be a Hilbert space which is continuously and densely embedded in $L_2(\Omega, H)$. Let $\mathfrak{a}\colon \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ be a continuous elliptic sesquilinear form, let A be the operator in $L_2(\Omega, H)$ associated with $\mathfrak{a}$, and let S be the semigroup generated by $-A$. Then the following are equivalent.

(i) $\|S_t u\|_{L_p(\Omega,H)} \le \|u\|_{L_p(\Omega,H)}$ *for all* $u \in L_2(\Omega, H) \cap L_p(\Omega, H)$ *and* $t > 0$.
(ii) $P\mathcal{V} \subset \mathcal{V}$ *and* $\operatorname{Re} \mathfrak{a}(u, \|u\|_H^{p-1} \operatorname{sgn} u) \ge 0$ *for all* $u \in \mathcal{V}$ *with* $\|u\|_H^{p-1} \operatorname{sgn} u \in \mathcal{V}$.

Our proof in Sect. 2 is slightly different from Nittka's since we exploit the strict convexity of the space $L_p(\Omega, H)$ for all $1 < p < \infty$, which we prove in Appendix B.

Our second aim is to apply the criterion to purely second order Hilbert space valued elliptic operators with Neumann boundary conditions. They generate a contractive C_0-semigroup $(S_t)_{t>0}$ on $L_2(\Omega, H)$. Of particular interest are systems, that is, $H = \mathbb{C}^d$. In Sect. 3, we show that there is an interval $[p_-, p_+]$, with $1 < p_- < 2 < p_+ < \infty$, such that the semigroup S extends to a contractive C_0-semigroup on $L_p(\Omega, H)$ for all $p \in [p_-, p_+]$. To prove the needed estimates, we use a chain rule formula, which is quite delicate and will be proved in Appendix A. In [2], related results are obtained and further interesting references are given. In the scalar case, our results may be compared with Cialdea–Maz'ya [7], who introduced an algebraic version of L_p-dissipativity and presented an algebraic characterisation for scalar-valued elliptic operators. This algebraic characterisation was refined by Carbonaro–Dragičević [6] and they used the result of Nittka to describe contractive C_0-semigroups on $L_p(\Omega)$ via the notion that they called p-ellipticity.

2. Nittka's criterion for L_p-contractivity. Let $(\Omega, \mathcal{B}, \mu)$ be a σ-finite measure space. Let H be a Hilbert space. For all $p \in [1, \infty)$, we write $L_p = L_p(\Omega, H)$. If $u \in L_p$, then we write $\|u\|_p = \|u\|_{L_p(\Omega,H)}$ and $\|u\|_H\colon \Omega \to \mathbb{R}$ is the function from Ω into $\mathbb{R}$ such that

$$\|u\|_H(x) = \|u(x)\|_H$$

for all $x \in \Omega$. Further we write $\mathcal{H} = L_2 = L_2(\Omega, H)$. Throughout this paper, we fix $p \in (1, \infty)$. Define

$$C = \{u \in \mathcal{H} \cap L_p : \|u\|_p \le 1\}.$$

Clearly C is convex and it follows from Fatou's lemma that C is closed in $\mathcal{H}$. Let $P\colon \mathcal{H} \to C$ be the orthogonal projection. For all $u \in \mathcal{H}$, define

$$N(u) = \{h \in \mathcal{H} : \operatorname{Re}(h, v - u)_{\mathcal{H}} \le 0 \text{ for all } v \in C\}.$$

Then $N(u)$ is a closed cone in $\mathcal{H}$ with $0 \in N(u)$. We state an easy property regarding C and $N(u)$.

Lemma 2.1. *For all $f \in \mathcal{H}$, there exist unique $u \in C$ and $h \in N(u)$ such that $f = u + h$. Actually, $u = Pf$.*

Proof. Let $u = Pf$ and $h = f - Pf$. Then $u \in C$ and $\operatorname{Re}(h, v-u)_{\mathcal{H}} = \operatorname{Re}(f - Pf, v - Pf)_{\mathcal{H}} \le 0$ for all $v \in C$. This proves existence. Let also $\tilde{u} \in C$ and $\tilde{h} \in N(\tilde{u})$ be such that $f = \tilde{u} + \tilde{h}$. Then $\operatorname{Re}(h, \tilde{u} - u)_{\mathcal{H}} \le 0$ and similarly $\operatorname{Re}(\tilde{h}, u - \tilde{u})_{\mathcal{H}} \le 0$. Hence $\operatorname{Re}(h - \tilde{h}, \tilde{u} - u)_{\mathcal{H}} \le 0$. Since $h - \tilde{h} = \tilde{u} - u$, this implies that $\operatorname{Re}\|h - \tilde{h}\|_{\mathcal{H}}^2 \le 0$ and the statement follows. □

If $u \colon \Omega \to H$ is a function, then define $\operatorname{sgn} u \colon \Omega \to H$ by

$$(\operatorname{sgn} u)(x) = \begin{cases} \frac{1}{\|u(x)\|_H} u(x) & \text{if } u(x) \neq 0, \\ 0 & \text{if } u(x) = 0. \end{cases}$$

Note that $\operatorname{sgn} u = \lim_{\varepsilon \downarrow 0} u_\varepsilon$ pointwise, where the function $u_\varepsilon \colon \Omega \to H$ is defined by $u_\varepsilon(x) = \frac{1}{\sqrt{\|u(x)\|_H^2 + \varepsilon}} u(x)$. Hence $\operatorname{sgn} u$ is (Bochner) measurable if u is (Bochner) measurable. Then also $\|u\|_H^{p-1} \operatorname{sgn} u$ is (Bochner) measurable whenever u is (Bochner) measurable.

For a description of $N(u)$, we use that $L_p(\Omega, H)$ is strictly convex. See Appendix B for a proof. Recall that a Banach space E is called *strictly convex* if for all $\xi, \eta \in E$ with $\|\xi\|_E = 1 = \|\eta\|_E$ and $\xi \neq \eta$, it follows that $\|\xi + \eta\|_E < 2$.

Lemma 2.2. *Let E be a Banach space. Assume that E^* is strictly convex and let $x \in E$. Then there exists a unique $f \in E^*$ such that $\operatorname{Re} f(x) = \|x\|_E^2 = \|f\|_{E^*}^2$. This unique f satisfies $f(x) = \|x\|_E^2$.*

Proof. The existence of an $f \in E^*$ such that $f(x) = \|x\|_E^2 = \|f\|_{E^*}^2$ is a well-known consequence of the Hahn–Banach theorem. For the uniqueness, we may assume without loss of generality that $\|x\|_E = 1$. Suppose also $g \in E^*$ with $\operatorname{Re} g(x) = \|x\|_E^2 = \|g\|_{E^*}^2$ and $g \neq f$. Define $h = \frac{1}{2}(f + g)$. Then $\|h\|_{E^*} < 1$ by the strict convexity of E^*. But then

$$1 = \tfrac{1}{2} \operatorname{Re}(f(x) + g(x)) = \operatorname{Re} h(x) \le \|h\|_{E^*} \|x\|_E < 1.$$

This is a contradiction. □

Now we are able to give a characterisation for $N(u)$.

Proposition 2.3. *Let $u \in C$. Then the following are equivalent.*

(i) $N(u) \neq \{0\}$.

(ii) $\|u\|_p = 1$ *and* $\|u\|_H^{p-1} \operatorname{sgn} u \in \mathcal{H}$.

If these conditions are valid, then $N(u) = \{t \, \|u\|_H^{p-1} \operatorname{sgn} u : t \in [0, \infty)\}$.

Proof. '(ii)⇒(i)'. Let $v \in C$. Then the Cauchy–Schwarz inequality and the Hölder inequality give

$$\begin{aligned} \operatorname{Re}(\|u\|_H^{p-1} \operatorname{sgn} u, v)_{\mathcal{H}} &\le \int_\Omega \|u\|_H^{p-1} \, \|v\|_H \\ &\le \Big(\int_\Omega \|u\|_H^{(p-1)p'} \Big)^{1/p'} \|v\|_p \\ &\le \|u\|_p^{p/p'} = 1 = \operatorname{Re}(\|u\|_H^{p-1} \operatorname{sgn} u, u)_{\mathcal{H}}. \end{aligned}$$

So $\|u\|_H^{p-1} \operatorname{sgn} u \in N(u)$ and $\{t\, \|u\|_H^{p-1} \operatorname{sgn} u : t \in [0,\infty)\} \subset N(u)$. In particular, $N(u) \neq \{0\}$.

'(i)⇒(ii)'. We first show that $\|u\|_p = 1$. Suppose that $\|u\|_p < 1$. Let $h \in N(u)$. Let $w \in L_2 \cap L_p$ with $\|w\|_p \le 1 - \|u\|_p$. Then $u + w \in C$. Since $h \in N(u)$, one deduces that $\operatorname{Re}\,(h, w)_{\mathcal{H}} \le 0$. Because $L_2 \cap L_p$ is dense in $\mathcal{H}$, it follows that $h = 0$. Hence (i) implies that $\|u\|_p = 1$.

Next let $h \in N(u)$ with $h \neq 0$. If $v \in L_2(\Omega, H) \cap L_p(\Omega, H)$, then $\operatorname{Re}\,(h, v)_{\mathcal{H}} \le \|v\|_p \operatorname{Re}\,(h, u)_{\mathcal{H}}$. So $h \in L_{p'}(\Omega, H)$ and $\|h\|_{p'} \le \operatorname{Re}\,(h, u)_{\mathcal{H}}$. If $\operatorname{Re}\,(h, u)_{\mathcal{H}} = 0$, then $\|h\|_{p'} = 0$ and $h = 0$, which is a contradiction. So $\operatorname{Re}\,(h, u)_{\mathcal{H}} \neq 0$. Multiplying h with a strictly positive constant, we may assume that $\operatorname{Re}\,(h, u)_{\mathcal{H}} = 1$. Then

$$\|h\|_{p'} \le 1 = \operatorname{Re}\,(h, u)_{\mathcal{H}} \le \|h\|_{p'}\, \|u\|_p \le \|h\|_{p'}.$$

So $\|h\|_{p'} = 1 = \operatorname{Re}\,(h, u)_{\mathcal{H}} = \operatorname{Re}\,\langle h, u\rangle_{L_{p'} \times L_p}$. We proved that

$$\operatorname{Re}\,\langle h, u\rangle_{L_{p'} \times L_p} = \|h\|_{p'}^2 = \|u\|_p^2.$$

On the other hand,

$$\int_\Omega \Big\| \|u\|_H^{p-1} \operatorname{sgn} u \Big\|_H^{p'} = \int_\Omega \|u\|_H^{(p-1)p'} = \int_\Omega \|u\|_H^p = 1,$$

so $\|u\|_H^{p-1} \operatorname{sgn} u \in L_{p'}$ and furthermore

$$\operatorname{Re}\,\langle \|u\|_H^{p-1} \operatorname{sgn} u, u\rangle_{L_{p'} \times L_p} = \int_\Omega \|u\|_H^p = 1 = \Big\| \|u\|_H^{p-1} \operatorname{sgn} u \Big\|_{p'}^2 = \|u\|_p^2.$$

By Lemma 2.2, we obtain that $\|u\|_H^{p-1} \operatorname{sgn} u = h \in \mathcal{H}$. Moreover, $N(u) \subset \{t\, \|u\|_H^{p-1} \operatorname{sgn} u : t \in [0,\infty)\}$. □

Let $\mathcal{V}$ be a Hilbert space that is continuously and densely embedded in $\mathcal{H} = L_2(\Omega, H)$. Let $\mathfrak{a}\colon \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ be a sesquilinear form such that $\mathfrak{a}$ is *continuous*, that is, there is an $M > 0$ such that

$$|\mathfrak{a}(u, v)| \le M\, \|u\|_{\mathcal{V}}\, \|v\|_{\mathcal{V}}$$

for all $u, v \in \mathcal{V}$, and $\mathfrak{a}$ is *elliptic*, that is, there are $\mu > 0$ and $\omega \in \mathbb{R}$ such that

$$\operatorname{Re} \mathfrak{a}(u, u) + \omega\, \|u\|_{\mathcal{H}}^2 \ge \mu\, \|u\|_{\mathcal{V}}^2 \tag{1}$$

for all $u \in \mathcal{V}$. Then there is a unique operator A in $\mathcal{H}$ whose graph is

$$\operatorname{graph}(A) = \{(u, f) : u \in \mathcal{V},\ f \in \mathcal{H},\ \text{and } \mathfrak{a}(u, v) = (f, v)_{\mathcal{H}} \text{ for all } v \in \mathcal{V}\}.$$

We call A the *operator associated with the form* $\mathfrak{a}$. Then $-A$ generates a holomorphic C_0-semigroup $(S_t)_{t>0}$ in $\mathcal{H}$ satisfying $\|S_t\|_{\mathcal{H} \to \mathcal{H}} \le e^{\omega t}$ for all $t > 0$, where ω is as in (1).

Proof of Theorem 1.1. '(i)⇒(ii)'. It follows from [16, Theorem 2.2 1)⇒2)], that $P\mathcal{V} \subset \mathcal{V}$ and $\operatorname{Re} \mathfrak{a}(Pf, f - Pf) \ge 0$ for all $f \in \mathcal{V}$. Since $\|u\|_H^{p-1} \operatorname{sgn} u \in L_2(\Omega, H)$ by assumption, one deduces that

$$\int_\Omega \|u\|_H^p = \int_\Omega (\|u\|_H^{p-1} \operatorname{sgn} u, \|u\|_H \operatorname{sgn} u)_H < \infty.$$

So $u \in L_p$. Without loss of generality, we may assume that $\|u\|_p = 1$. Then $u \in C$ and $\|u\|_H^{p-1} \operatorname{sgn} u \in \mathcal{H}$, so $\|u\|_H^{p-1} \operatorname{sgn} u \in N(u)$ by Proposition 2.3. Set $f = u + \|u\|_H^{p-1} \operatorname{sgn} u \in \mathcal{V} \subset \mathcal{H}$. Then the uniqueness of Lemma 2.1 gives $u = Pf$. Consequently $\operatorname{Re} \mathfrak{a}(u, \|u\|_H^{p-1} \operatorname{sgn} u) = \operatorname{Re} \mathfrak{a}(Pf, f - Pf) \geq 0$.

'(ii)⇒(i)'. Let $v \in \mathcal{V}$. We shall show that $\operatorname{Re} \mathfrak{a}(Pv, v - Pv) \geq 0$. If $v \in C$, then this is trivial, so we may assume that $v \notin C$. Set $u = Pv$ and $h = v - u$. Then $h \in N(u)$ by Lemma 2.1. Also $h \neq 0$, so Proposition 2.3 implies that $\|u\|_p = 1$ and $\|u\|_H^{p-1} \operatorname{sgn} u \in \mathcal{H}$. Moreover, there exists a $t \in [0, \infty)$ such that $h = t \|u\|_H^{p-1} \operatorname{sgn} u$. Then $t \neq 0$. Since $u = Pv \in P\mathcal{V} \subset \mathcal{V}$ by assumption, one deduces that $\|u\|_H^{p-1} \operatorname{sgn} u = \frac{1}{t}(v - u) \in \mathcal{V}$. Hence $\operatorname{Re} \mathfrak{a}(Pv, v - Pv) = t \operatorname{Re} \mathfrak{a}(u, \|u\|_H^{p-1} \operatorname{sgn} u) \geq 0$. Now it follows from [17, Theorem 2.2 2)⇒1)], that $S_t C \subset C$ for all $t > 0$. This obviously implies statement (i). □

3. Application to Hilbert space valued parabolic problems. Let $\Omega \subset \mathbb{R}^d$ be an open set. Let H be a separable Hilbert space. For all $k, l \in \{1, \ldots, d\}$, let $c_{kl} \colon \Omega \to \mathcal{L}(H)$ be a bounded function such that $x \mapsto (c_{kl}(x)\, \xi, \eta)_H$ is measurable from Ω into $\mathbb{C}$ for all $\xi, \eta \in H$. Let $\mu > 0$. We assume that

$$\operatorname{Re} \sum_{k,l=1}^{d} (c_{kl}(x)\, \xi_l, \xi_k)_H \geq \mu \sum_{k=1}^{d} \|\xi_k\|_H^2$$

for all $x \in \Omega$ and $\xi_1, \ldots, \xi_d \in H$. Further let $M > 0$ be such that

$$\sum_{k=1}^{d} \Big\| \sum_{l=1}^{d} c_{kl}(x)\, \xi_l \Big\|_H^2 \leq M^2 \sum_{k=1}^{d} \|\xi_k\|_H^2$$

for all $x \in \Omega$ and $\xi_1, \ldots, \xi_d \in H$. For simplicity, we consider Neumann boundary conditions. Define $\mathcal{V} = H^1(\Omega, H)$ and $\mathfrak{a} \colon \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ by

$$\mathfrak{a}(u, v) = \sum_{k,l=1}^{d} \int_{\Omega} (c_{kl}(x)\, (\partial_l u)(x), (\partial_k v)(x))_H \, dx.$$

Then $\mathfrak{a}$ is a continuous elliptic sesquilinear form. Let A be the operator associated with $\mathfrak{a}$ and let S be the semigroup generated by $-A$.

Theorem 3.1. *Let $p \in (1, \infty)$ and suppose that*

$$\frac{\mu}{M} \geq 2 \Big| \frac{p-2}{p} \Big| + \Big| \frac{p-2}{p} \Big|^2.$$

Then S extends consistently to a contraction semigroup in $L_p(\Omega, H)$.

Note that the condition in Theorem 3.1 is invariant by taking the dual exponent, that is, if $p \in (1, \infty)$ satisfies the condition, then so does q, where $\frac{1}{p} + \frac{1}{q} = 1$. If one is satisfied with some small interval, $1 < p_- < 2 < p_+ < \infty$ such that S is L_p-contractive for all $p \in [p_-, p_+]$, then a significantly easier and less technical proof than the following can be given.

Proof. Using duality, without loss of generality, we may assume that $p > 2$. We argue as in [7], [12], and [11]. Let $u \in H^1(\Omega, H)$. For all $n \in \mathbb{N}$, define

$$v_n = (\|u\|_H^{\frac{p-2}{2}} \wedge n)\, u, \quad w_n = (\|u\|_H^{p-2} \wedge n^2)\, u, \quad \text{and} \quad \chi_n = \mathbb{1}_{[\|u\|_H^{p-2} < n^2]}.$$

If follows from Proposition A.1 (cf. [4, Theorems 3.3 and 4.2]) that $v_n, w_n \in H^1(\Omega, H)$ with

$$\begin{aligned} \nabla v_n &= n\,(\mathbb{1} - \chi_n)\,\nabla u + \chi_n\,\|u\|_H^{\frac{p-2}{2}}\Big(\nabla u + \tfrac{p-2}{2}\,(\nabla\|u\|_H)\,\mathrm{sgn}\,u\Big), \text{ and} \\ \nabla w_n &= n^2\,(\mathbb{1} - \chi_n)\,\nabla u + \chi_n\,\|u\|_H^{p-2}\Big(\nabla u + (p-2)\,(\nabla\|u\|_H)\,\mathrm{sgn}\,u\Big). \end{aligned} \tag{2}$$

Note that $\chi_n\,\|v_n\|_H = \chi_n\,\|u\|_H^{p/2}$. Hence

$$\begin{aligned} (\mathbb{1} - \chi_n)\,\nabla u &= \frac{1}{n}\,(\mathbb{1} - \chi_n)\,\nabla v_n, \\ \chi_n\,\|v_n\|_H^{\frac{p-2}{p}}\,\nabla u &= \chi_n\Big(\nabla v_n - \tfrac{p-2}{p}\,(\nabla\|v_n\|_H)\,\mathrm{sgn}\,v_n\Big), \text{ and} \\ \nabla w_n &= n\,(\mathbb{1} - \chi_n)\,\nabla v_n + \chi_n\,\|v_n\|_H^{\frac{p-2}{p}}\Big(\nabla v_n + \tfrac{p-2}{p}\,(\nabla\|v_n\|_H)\,\mathrm{sgn}\,v_n\Big). \end{aligned}$$

Therefore, for almost all $x \in \Omega$, one obtains

$$\begin{aligned} &\sum_{k,l=1}^{d} \mathrm{Re}\,(c_{kl}\,\partial_l u, \partial_k w_n)_H \\ &= (\mathbb{1} - \chi_n)\sum_{k,l=1}^{d} \mathrm{Re}\,(c_{kl}\,\partial_l u, n\,\partial_k v_n)_H \\ &\quad + \chi_n \sum_{k,l=1}^{d} \mathrm{Re}\,\Big(c_{kl}\,\|v_n\|_H^{\frac{p-2}{p}}\,\partial_l u, \Big(\partial_k v_n + \tfrac{p-2}{p}\,(\partial_k\|v_n\|_H)\,\mathrm{sgn}\,v_n\Big)\Big)_H \\ &= (\mathbb{1} - \chi_n)\sum_{k,l=1}^{d} \mathrm{Re}\,(c_{kl}\,\partial_l v_n, \partial_k v_n)_H \\ &\quad + \chi_n \sum_{k,l=1}^{d} \mathrm{Re}\,\Big(c_{kl}\Big(\partial_l v_n - \tfrac{p-2}{p}\,(\partial_l\|v_n\|_H)\,\mathrm{sgn}\,v_n\Big), \\ &\quad \Big(\partial_k v_n + \tfrac{p-2}{p}\,(\partial_k\|v_n\|_H)\,\mathrm{sgn}\,v_n\Big)\Big)_H \\ &= \sum_{k,l=1}^{d} \mathrm{Re}\,(c_{kl}\,\partial_l v_n, \partial_k v_n)_H + \tfrac{p-2}{p}\,\chi_n\,\mathrm{Re}\,((c_{kl} - c_{lk}^*)\,\partial_l v_n, (\partial_k\|v_n\|_H)\,\mathrm{sgn}\,v_n)_H \\ &\quad - \Big(\tfrac{p-2}{p}\Big)^2\,\mathrm{Re}\,(c_{kl}\,(\partial_l\|v_n\|_H)\,\mathrm{sgn}\,v_n, (\partial_k\|v_n\|_H)\,\mathrm{sgn}\,v_n)_H \\ &\geq \Big(\mu - 2M\,\tfrac{p-2}{p} - M\,\Big(\tfrac{p-2}{p}\Big)^2\Big)\sum_{k=1}^{d} \|\partial_k v_n\|_H^2, \end{aligned}$$

where we used that $\sum_{k=1}^d |\partial_k \|v_n\|_H|^2 \le \sum_{k=1}^d \|\partial_k v_n\|_H^2$ and $\|\operatorname{sgn} v_n\|_H \le 1$ by Lemma A.5. By the assumption on p, we obtain that

$$\sum_{k,l=1}^d \operatorname{Re}\,(c_{kl}\,\partial_l u, \partial_k w_n)_H \ge 0$$

almost everywhere. This is for all $n \in \mathbb{N}$.

Next take the limit $n \to \infty$ and use (2). Then

$$\sum_{k,l=1}^d \operatorname{Re}\,(c_{kl}\,\partial_l u, \|u\|_H^{p-2}\,(\partial_k u + (p-2)\,(\partial_k \|u\|_H)\operatorname{sgn} u))_H \ge 0 \tag{3}$$

almost everywhere.

We assume from now on that in addition $\|u\|_H^{p-2}\,u \in \mathcal{V}$. Let $k \in \{1, \ldots, d\}$. We shall show that $\partial_k(\|u\|_H^{p-2}\,u) = f_k$ almost everywhere, where

$$f_k = \|u\|_H^{p-2}\,\Big(\partial_k u + (p-2)\,(\partial_k \|u\|_H)\operatorname{sgn} u\Big).$$

Write $r = 2\,\frac{p-1}{p-2} \in (2, \infty)$ and let $q \in (1,2)$ be such that $\frac{1}{2} + \frac{1}{r} = \frac{1}{q}$. Since $\|u\|_H^{p-1} \in L_2(\Omega)$, one deduces that $\int_\Omega (\|u\|_H^{p-2})^r = \int_\Omega (\|u\|_H^{p-1})^2 < \infty$. So $\|u\|_H^{p-2} \in L_r(\Omega)$. If $n \in \mathbb{N}$, then $n^2\,(\mathbb{1} - \chi_n)\,\|\partial_k u\|_H \le (\mathbb{1} - \chi_n)\,\|u\|_H^{p-2}\,\|\partial_k u\|_H$ and hence (2) gives

$$\|\partial_k w_n\|_H \le \|u\|_H^{p-2}\Big(\|\partial_k u\|_H + (p-2)\,\Big|\partial_k \|u\|_H\Big|\Big). \tag{4}$$

Note that the right hand side of (4) does not depend on n and is an element of $L_q(\Omega)$. Also $\lim \partial_k w_n = f_k$ almost everywhere. Hence $\lim \partial_k w_n = f_k$ in $L_q(\Omega, H)$. It is easy to see that $\lim w_n = \|u\|_H^{p-2}\,u$ in $L_2(\Omega, H)$ since $\|u\|_H^{p-2}\,u \in L_2(\Omega, H)$. Let $\varphi \in C_c^\infty(\Omega, H)$. Then

$$\begin{aligned}\int_\Omega (\partial_k(\|u\|_H^{p-2}\,u), \varphi)_H &= -\int_\Omega (\|u\|_H^{p-2}\,u, \partial_k\varphi)_H = -\lim_{n\to\infty}\int_\Omega (w_n, \partial_k\varphi)_H \\ &= \lim_{n\to\infty}\int_\Omega (\partial_k w_n, \varphi)_H = \int_\Omega (f_k, \varphi)_H.\end{aligned}$$

So $\partial_k(\|u\|_H^{p-2}\,u) = f_k$ almost everywhere.

Finally (3) implies that

$$\sum_{k,l=1}^d \operatorname{Re}\,(c_{kl}\,\partial_l u, \partial_k(\|u\|_H^{p-2}\,u))_H \ge 0$$

almost everywhere. Integrating over Ω gives $\operatorname{Re}\mathfrak{a}(u, \|u\|_H^{p-2}\,u) \ge 0$. Now apply Theorem 1.1.

By the way, with some more work, one can show that $\lim_{n\to\infty} \partial_k w_n = f_k$ in $L_2(\Omega, H)$. □

We comment on related results concerning the extension of S to a not necessarily contractive C_0-semigroup on L_p. Hofmann, Mayboroda, and McIntosh

[14] showed for $H = \mathbb{C}$, $\Omega = \mathbb{R}^d$, and $d \geq 3$ that the semigroup S can be extended to a C_0-semigroup on $L_p(\Omega)$ if $p \in [\frac{2d}{d+2}, \frac{2d}{d-2}]$. Conversely, for each $p \in (1, \frac{2d}{d+2}) \cup (\frac{2d}{d-2}, \infty)$, they construct an elliptic operator such that the associated semigroup $(S_t)_{t>0}$ cannot be extended consistently to a bounded semigroup $L_p(\mathbb{R}^d)$. The extension results are based on off-diagonal Davies–Gaffney estimates; cf. also [8, Theorem 25], and [5, Section 3.1].

Davies had already pointed out in the introduction of [9] that his proof of [8, Theorem 25] extends to the vector-valued case. Moreover, by [9, Theorem 10], for each $p \in (1, \frac{2d}{d+2}) \cup (\frac{2d}{d-2}, \infty)$, there exists an elliptic system with $H = \mathbb{C}^d$, $\Omega = \mathbb{R}^d$, $d \geq 3$, and with real symmetric coefficients such that the operator S_t does not continuously extend to L_p for any $t > 0$.

We shall give a corresponding extension result for our setting, which we obtain readily from standard estimates and tracing Auscher's proof of [5, Proposition 3.2].

Theorem 3.2. *Suppose $\Omega = \mathbb{R}^d$ or $\Omega \subset \mathbb{R}^d$ is open and Lipschitz. Then the semigroup S extends to a C_0-semigroup with growth bound 0 on $L_p(\Omega, H)$ for all $p \in (\frac{2d}{d+2}, \frac{2d}{d-2})$ if $d \geq 3$ and for all $p \in (1, \infty)$ if $d \in \{1, 2\}$.*

Proof. We outline the arguments for $d \geq 3$. Let $\omega > 0$. Since $\mathfrak{a}$ is elliptic for this choice of ω, by [18, Lemma 3.6.2 (3.60)] there exists a $c > 0$ such that $\|e^{-\omega t} S_t u\|_{\mathcal{V}} \leq c\, t^{-1/2} \|u\|_2$ for all $u \in L_2(\Omega, H)$ and $t > 0$. Combining this with the Sobolev embedding $\mathcal{V} \hookrightarrow L_{2d/(d-2)}$, we obtain that there exists a $C > 0$ such that

$$\|e^{-\omega t} S_t u\|_{2d/(d-2)} \leq C\, t^{-1/2} \|u\|_2$$

for all $u \in L_2(\Omega, H)$ and $t > 0$. Then, by duality,

$$\|e^{-\omega t} S_t^* u\|_2 \leq C\, t^{-1/2} \|u\|_{2d/(d+2)}$$

for all $u \in L_2 \cap L_{2d/(d+2)}$ and $t > 0$.

Next, it follows from inspection of the proof of [5, Proposition 3.2] that the parts (2) and (3) of [5, Proposition 3.2] extend to the vector-valued case and general open sets Ω, and are applicable to the C_0-semigroup $(T_t)_{t>0}$ given by $T_t = e^{-\omega t} S_t^*$. For the extension of part (2), one needs L_2–L_2 off-diagonal estimates that can be obtained, for example, as in [3, Theorem 4.2]. Moreover, the vector-valued version of the Riesz–Thorin theorem follows from [13, Lemma 2.6]. Let $q \in (\frac{2d}{d+2}, 2)$. By the extension of [5, Proposition 3.2 (2)], we obtain that T satisfies L_q–L_2 off-diagonal estimates, which implies by the extension of [5, Proposition 3.2 (3)] that T is uniformly bounded in L_q. Dualizing again, we obtain the statement for all $p \in (2, \frac{2d}{d-2})$. By considering the adjoint form, applying the result for $p > 2$, and taking the dual, we obtain the statement for $p \in (\frac{2d}{d+2}, 2)$. □

Remark 3.3. We comment on the admissible ranges for p in Theorems 3.1 and 3.2. Remarkably, it is possible that the range given in Theorem 3.1 for contractive extensions is larger than the one given in Theorem 3.2 for extensions with growth bound 0. For example, this occurs if $\frac{\mu}{M} \geq \frac{1}{2}$, say, and d is sufficiently large.

A. The derivative of a truncation. Let $\Omega \subset \mathbb{R}^d$ be an open set. Let H be a Hilbert space. The principal aim in this section is to prove the following chain rule.

Proposition A.1. *Let $\alpha > 0$ and $M > 0$. Let $u \in H^1(\Omega, H)$. Define $v = (\|u\|_H^\alpha \wedge M)\, u$. Then $v \in H^1(\Omega, H)$ and*

$$\partial_k v = \alpha\, \mathbb{1}_{[\|u\|_H^\alpha < M]}\, \|u\|_H^\alpha \,\mathrm{Re}\,(\mathrm{sgn}\, u, \partial_k u)_H\, \mathrm{sgn}\, u + (\|u\|_H^\alpha \wedge M)\, \partial_k u$$

for all $k \in \{1, \ldots, d\}$.

The proof involves some work. We use the following approximation by smooth functions.

Lemma A.2. *The space $C^\infty(\Omega, H) \cap H^1(\Omega, H)$ is dense in $H^1(\Omega, H)$.*

Proof. This follows as in the scalar case in [1, Theorem 3.17]. ☐

For an approximation argument, the next lemma is useful.

Lemma A.3. *For all $n \in \mathbb{N}$, let $u_n \in H^1(\Omega, H)$. Let $u, g_1, \ldots, g_d \in L_2(\Omega, H)$. Suppose that $\lim u_n = u$ in $L_2(\Omega, H)$ and $\lim \partial_k u_n = g_k$ in $L_2(\Omega, H)$ for all $k \in \{1, \ldots, d\}$. Then $u \in H^1(\Omega, H)$ and $\partial_k u = g_k$ for all $k \in \{1, \ldots, d\}$.*

Proof. Let $\varphi \in C_c^\infty(\Omega)$. Let $k \in \{1, \ldots, d\}$. Then $-\int_\Omega u_n\, \partial_k \varphi = \int_\Omega (\partial_k u_n)\, \varphi$ for all $n \in \mathbb{N}$. Then the lemma follows by taking the limit $n \to \infty$. ☐

For the proof of Proposition A.1, we shall approximate the function $t \mapsto t^\alpha \wedge M$ with smooth functions. The next technical lemma gives sufficient conditions in order to apply a chain rule. Note that we do not require that f' is bounded.

Lemma A.4. *Let $f \in C^1(0, \infty)$. Suppose that f is bounded, $\lim_{t \downarrow 0} f(t) = 0$, $\lim_{t \downarrow 0} t\, f'(t) = 0$, and $\sup_{t \in (0,\infty)} t\, |f'(t)| < \infty$. Let $u \in H^1(\Omega, H)$. Define $v = f(\|u\|_H)\, u$. Then $v \in H^1(\Omega, H)$ and*

$$\partial_k v = \begin{cases} \|u\|_H\, f'(\|u\|_H)\, \mathrm{Re}\,(\mathrm{sgn}\, u, \partial_k u)_H\, \mathrm{sgn}\, u + f(\|u\|_H)\, \partial_k u & \text{on } [u \neq 0], \\ 0 & \text{on } [u = 0], \end{cases} \tag{5}$$

for all $k \in \{1, \ldots, d\}$.

Proof. Let $\varepsilon > 0$. Define $v_\varepsilon = f(\sqrt{\|u\|_H^2 + \varepsilon})\, u$. If $u \in C^1(\Omega, H)$, then $v_\varepsilon \in C^1(\Omega, H)$ and

$$\partial_k v_\varepsilon = \sqrt{\|u\|_H^2 + \varepsilon}\, f'(\sqrt{\|u\|_H^2 + \varepsilon})\, \frac{\mathrm{Re}\,(u, \partial_k u)_H}{\sqrt{\|u\|_H^2 + \varepsilon}}\, \frac{1}{\sqrt{\|u\|_H^2 + \varepsilon}}\, u + f(\sqrt{\|u\|_H^2 + \varepsilon})\, \partial_k u \tag{6}$$

for all $k \in \{1, \ldots, d\}$. Then, by Lemmas A.2 and A.3, this extends to all $u \in H^1(\Omega, H)$ and (6) is valid. Finally choose $\varepsilon = \frac{1}{n}$, take the limit $n \to \infty$, and use again Lemma A.3. ☐

Now we are able to prove Proposition A.1.

Proof of Proposition A.1. For all $n \in \mathbb{N}$, define $f_n, f\colon (0,\infty) \to \mathbb{R}$ by

$$f(t) = t^\alpha \wedge M,$$
$$f_n(t) = \tfrac{1}{2}\,(t^\alpha + \sqrt{M^2 + n^{-1}} - \sqrt{|t^\alpha - M|^2 + n^{-1}}).$$

Then $\lim f_n(t) = f(t)$ for all $t \in (0,\infty)$. Also $\lim_{t\downarrow 0} f_n(t) = 0$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. Then $f_n \in C^1(0,\infty)$ and

$$f_n'(t) = \tfrac{1}{2}\,\alpha\, t^{\alpha-1}\Big(1 - \frac{t^\alpha - M}{\sqrt{(t^\alpha - M)^2 + n^{-1}}}\Big)$$

for all $t \in (0,\infty)$. In particular, f_n is increasing. Moreover, $\lim_{t\downarrow 0} t\, f_n'(t) = 0$. In addition, $\lim_{n\to\infty} f_n'(t) = \alpha\, t^{\alpha-1}$ if $t^\alpha \le M$ and $\lim_{n\to\infty} f_n'(t) = 0$ if $t^\alpha > M$.

Let $n \in \mathbb{N}$ and $t \in (0,\infty)$. If $t^\alpha \le M$, then

$$0 \le t\, f_n'(t) = \tfrac{1}{2}\,\alpha\, t^\alpha\Big(1 + \frac{M - t^\alpha}{\sqrt{(t^\alpha - M)^2 + n^{-1}}}\Big) \le \alpha\, M.$$

Alternatively, if $t^\alpha > M$, then

$$\begin{aligned} 0 \le t\, f_n'(t) &= \tfrac{1}{2}\,\alpha\,\frac{t^\alpha - M + M}{\sqrt{(t^\alpha - M)^2 + n^{-1}}}\Big(\sqrt{(t^\alpha - M)^2 + n^{-1}} - \sqrt{(t^\alpha - M)^2}\Big) \\ &\le \tfrac{1}{2}\,\alpha\Big(1 + \frac{M}{\sqrt{n^{-1}}}\Big)\sqrt{n^{-1}} \le \tfrac{1}{2}\,\alpha\,(1 + M). \end{aligned}$$

So

$$\sup_{n\in\mathbb{N}}\ \sup_{t\in(0,\infty)}\ t\,|f_n'(t)| \le \alpha\,(M+1). \tag{7}$$

If $n \in \mathbb{N}$ and $t \in (0,\infty)$, then

$$\begin{aligned} 0 \le f_n(t) &\le \tfrac{1}{2}\,(t^\alpha + M + 1 - \sqrt{|t^\alpha - M|^2 + n^{-1}}) \\ &\le \tfrac{1}{2}\,(t^\alpha + M + 1 - |t^\alpha - M|) = \tfrac{1}{2} + f(t) \le \tfrac{1}{2} + M. \end{aligned}$$

So f_n is bounded and even

$$\sup_{n\in\mathbb{N}}\ \sup_{t\in(0,\infty)}\ |f_n(t)| \le \tfrac{1}{2} + M. \tag{8}$$

Hence all conditions of Lemma A.4 are satisfied for all the f_n.

Let $u \in H^1(\Omega, H)$. For all $n \in \mathbb{N}$, define $v_n = f_n(u)\,u$. Then $v_n \in H^1(\Omega, H)$ with derivatives given by (5) and f replaced by f_n. The Lebesgue dominated convergence theorem and the uniform bounds (8) and (7) imply that $\lim v_n = v$ and $\lim \partial_k v_n = \partial_k v$ in $L_2(\Omega, H)$ for all $k \in \{1, \ldots, d\}$. Then the proposition follows from Lemma A.3. □

Almost the same arguments show that the norm of an $H^1(\Omega, H)$-function is in the Sobolev space.

Lemma A.5. *Let* $u \in H^1(\Omega, H)$*. Then* $\|u\|_H \in H^1(\Omega)$ *and furthermore* $\partial_k \|u\|_H = \mathrm{Re}\,(\mathrm{sgn}\, u, \partial_k u)_H$ *for all* $k \in \{1, \ldots, d\}$*.*

Proof. Let $\varepsilon > 0$. For all $u \in H^1(\Omega, H)$, define $u_\varepsilon \colon \Omega \to H$ by $u_\varepsilon = \sqrt{\|u\|_H^2 + \varepsilon}$. Let $\varphi \in C_c^\infty(\Omega)$ and $k \in \{1, \ldots, d\}$. If $u \in C^\infty(\Omega, H) \cap H^1(\Omega, H)$, then $u_\varepsilon \in C^\infty(\Omega, H)$ with classical partial derivative $\partial_k u_\varepsilon = \frac{\mathrm{Re}\,(u, \partial_k u)_H}{u_\varepsilon}$. Hence

$$- \int_\Omega u_\varepsilon \, \partial_k \varphi = \int_\Omega \frac{\mathrm{Re}\,(u, \partial_k u)_H}{u_\varepsilon} \, \varphi. \tag{9}$$

Using approximation and Lemma A.2, it follows that (9) is valid for all $u \in H^1(\Omega, H)$. Finally choose $\varepsilon = \frac{1}{n}$ and take the limit $n \to \infty$. □

B. Strict convexity of $L_p(\Omega, H)$. As before, let $(\Omega, \mathcal{B}, \mu)$ be a σ-finite measure space and H a Hilbert space. In order to make this paper more self-contained, we give a direct proof of the following theorem. At the end of this section, we give information on more general results.

Theorem B.1. *Let $p \in (1, \infty)$ and $u, v \in L_p(\Omega, H)$ with $\|u\|_p = \|v\|_p = 1$. If $\|u + v\|_p = 2$, then $u = v$.*

For the proof of Theorem B.1, we use three lemmas.

Lemma B.2. *Let $\xi, \eta \in H$ and suppose that $\|\xi + \eta\|_H = \|\xi\|_H + \|\eta\|_H$. If $\eta \neq 0$, then there is a $\lambda \in [0, \infty)$ such that $\xi = \lambda \, \eta$.*

Proof. The equality implies that $\mathrm{Re}\,(\xi, \eta)_H = \|\xi\|_H \, \|\eta\|_H$. This gives equality in the Cauchy–Schwarz inequality. Hence there is a $\lambda \in \mathbb{C}$ such that $\xi = \lambda \, \eta$. Using again the equality, one deduces that $|1 + \lambda| = |\lambda| + 1$ and therefore $\lambda \in [0, \infty)$. □

Let $p, q \in (1, \infty)$ and suppose that $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma B.3. *Let $a, b \in [0, \infty)$. Then $a \, b \leq \frac{1}{p} \, a^p + \frac{1}{q} \, b^q$ and the equality holds if and only if $a^p = b^q$.*

Proof. This follows from the concavity of the logarithm. □

Lemma B.4. *Let $f \in L_p(\Omega)$ and $g \in L_q(\Omega)$ with $f \neq 0$ and $g \neq 0$. Suppose that $\int_\Omega |f| \, |g| = \|f\|_{L_p(\Omega)} \, \|g\|_{L_q(\Omega)}$. Then there exists a $\lambda > 0$ such that $|f|^p = \lambda \, |g|^q$ almost everywhere.*

Proof. We may assume that $\|f\|_{L_p(\Omega)} = 1 = \|g\|_{L_q(\Omega)}$. Then

$$1 = \int_\Omega |f| \, |g| \leq \int_\Omega \tfrac{1}{p} \, |f|^p + \tfrac{1}{q} \, |g|^q = \tfrac{1}{p} \, \|f\|_{L_p(\Omega)} + \tfrac{1}{q} \, \|g\|_{L_q(\Omega)} = 1.$$

Hence $|f| \, |g| = \frac{1}{p} \, |f|^p + \frac{1}{q} \, |g|^q$ almost everywhere and the lemma follows from Lemma B.3. □

Proof of Theorem B.1. Using the triangle inequality on H, twice the Hölder inequality on $L_p(\Omega)$, and the assumption $\|u + v\|_p = \|u\|_p + \|v\|_p$, one obtains

$$
\begin{aligned}
\|u+v\|_p^p &= \int_\Omega \|u+v\|_H \,\|u+v\|_H^{p-1} \\
&\le \int_\Omega \|u\|_H \,\|u+v\|_H^{p-1} + \int_\Omega \|v\|_H \,\|u+v\|_H^{p-1} \\
&\le \Big(\int_\Omega \|u\|_H^p\Big)^{1/p}\Big(\int_\Omega \|u+v\|_H^{(p-1)q}\Big)^{1/q} \\
&\quad +\Big(\int_\Omega \|v\|_H^p\Big)^{1/p}\Big(\int_\Omega \|u+v\|_H^{(p-1)q}\Big)^{1/q} \\
&= (\|u\|_p + \|v\|_p)\,\|u+v\|_p^{p/q} = \|u+v\|_p^{\frac{p}{q}+1} = \|u+v\|_p^p.
\end{aligned}
$$

Hence all three inequalities are equalities. The first gives that there is a null-set $N_1 \subset \Omega$ such that $\|u+v\|_H(x) = \|u\|_H(x) + \|v\|_H(x)$ for all $x \in \Omega\backslash N_1$ such that $\|u+v\|_H(x) \neq 0$. Recall that $\|u\|_p = 1$, so $\|u\|_H \neq 0 \in L_p(\Omega)$. Similarly $\|u+v\|_H \neq 0 \in L_p(\Omega)$ and therefore $\|u+v\|_H^{p-1} \neq 0 \in L_q(\Omega)$. Hence the equality in the first Hölder inequality together with Lemma B.4 gives that there are $\alpha > 0$ and a null-set $N_2 \subset \Omega$ such that $\|u\|_H^p(x) = \alpha\,\|u+v\|_H^{(p-1)q}(x)$ for all $x \in \Omega\backslash N_2$. Similarly there are $\beta > 0$ and a null-set $N_3 \subset \Omega$ such that $\|v\|_H^p(x) = \beta\,\|u+v\|_H^{(p-1)q}(x)$ for all $x \in \Omega\backslash N_3$. Hence $\|u\|_H^p = \gamma\,\|v\|_H^p$ on $\Omega \setminus (N_2 \cup N_3)$, where $\gamma = \frac{\alpha}{\beta}$. Since $\|u\|_p = \|v\|_p = 1$, one deduces that $\gamma = 1$.

Now let $x \in \Omega \setminus (N_1 \cup N_2 \cup N_3)$. If $\|u+v\|_H(x) = 0$, then subsequently $\|u\|_H^p(x) = \alpha\,\|u+v\|_H^{(p-1)q}(x) = 0$ and $u(x) = 0$. Similarly $v(x) = 0$ and therefore $u(x) = v(x)$. Alternatively, if $\|u+v\|_H(x) \neq 0$, then $v(x) \neq 0$ since $x \notin N_3$. Moreover, $\|u(x)+v(x)\|_H = \|u(x)\|_H + \|v(x)\|_H$ and Lemma B.2 implies that there is a $\lambda \in [0,\infty)$ such that $u(x) = \lambda\,v(x)$. But $\|u(x)\|_H^p = \|v(x)\|_H^p$ and hence $u(x) = v(x)$. Therefore $u = v$ almost everywhere. □

With a small modification, one can prove that $L_p(\Omega, E)$ is strictly convex if E is strictly convex and $p \in (1,\infty)$. In fact, a stronger result than Theorem B.1 is known. The space $L_p(\Omega, E)$ is uniformly convex if E is uniformly convex and $p \in (1,\infty)$. See [10] and the references therein.

Acknowledgements. The second-named author is most grateful for the hospitality extended to him during a fruitful stay at Ulm University. He wishes to thank Ulm University for financial support.

References

[1] Adams, R.A., Fournier, J.J.F.: Sobolev Spaces. Second Edition. Pure and Applied Mathematics (Amsterdam), 140. Elsevier/Academic Press, Amsterdam (2003)

[2] Angiuli, L., Lorenzi, L., Mangino, E.M.: Generation of semigroups associated to strongly coupled elliptic operator in $L^p(\mathbb{R}^d;\mathbb{R}^m)$. arXiv:2212.12784 (2022)

[3] Arendt, W., ter Elst, A.F.M.: Sectorial forms and degenerate differential operators. J. Oper. Theory **67**, 33–72 (2012)

[4] Arendt, W., Kreuter, M.: Mapping theorems for Sobolev spaces of vector-valued functions. Studia Math. **240**, 275–299 (2018)

[5] Auscher, P.: On necessary and sufficient conditions for L^p-estimates of Riesz transforms associated to elliptic operators on $\mathbb{R}^n$ and related estimates. Memoirs Amer. Math. Soc. **186**(871), xviii+75 pp. (2007)

[6] Carbonaro, A., Dragičević, O.: Convexity of power functions and bilinear embedding for divergence-form operators with complex coefficients. J. Eur. Math. Soc. (JEMS) **22**, 3175–3221 (2020)

[7] Cialdea, A., Maz'ya, V.: Criterion for the L^p-dissipativity of second order differential operators with complex coefficients. J. Math. Pures Appl. **9**(84), 1067–1100 (2005)

[8] Davies, E.B.: Uniformly elliptic operators with measurable coefficients. J. Funct. Anal. **132**, 141–169 (1995)

[9] Davies, E.B.: Limits on L^p regularity of self-adjoint elliptic operators. J. Differential Equations **135**, 83–102 (1997)

[10] Dowling, P.N., Hu, Z., Mupasiri, D.: Complex convexity in Lebesgue-Bochner function spaces. Trans. Amer. Math. Soc. **348**, 127–139 (1996)

[11] Egert, M.: On p-elliptic divergence form operators and holomorphic semigroups. J. Evol. Equ. **20**, 705–724 (2020)

[12] ter Elst, A.F.M., Liskevich, V., Sobol, Z., Vogt, H.: On the L^p-theory of C_0-semigroups associated with second-order elliptic operators with complex singular coefficients. Proc. Lond. Math. Soc. **115**, 693–724 (2017)

[13] Grafakos, L., Liu, L., Yang, D.: Vector-valued singular integrals and maximal functions on spaces of homogeneous type. Math. Scand. **104**, 296–310 (2009)

[14] Hofmann, S., Mayboroda, S., McIntosh, A.: Second order elliptic operators with complex bounded measurable coefficients in L^p, Sobolev and Hardy spaces. Ann. Sci. Éc. Norm. Supér. **44**, 723–800 (2011)

[15] Nittka, R.: Projections onto convex sets and L^p-quasi-contractivity of semigroups. Arch. Math. (Basel) **98**, 341–353 (2012)

[16] Ouhabaz, E.M.: L^∞-contractivity of semigroups generated by sectorial forms. J. Lond. Math. Soc. **46**, 529–542 (1992)

[17] Ouhabaz, E.M.: Analysis of Heat Equations on Domains. London Mathematical Society Monographs Series, 31. Princeton University Press, Princeton, NJ (2005)

[18] Tanabe, H.: Equations of Evolution. Translated from the Japanese by N. Mugibayashi and H. Haneda. Monographs and Studies in Mathematics, 6. Pitman (Advanced Publishing Program), Boston, Mass.-London (1979)

W. Arendt and M. Sauter
Institute of Applied Analysis
Ulm University
Helmholtzstr. 18
89081 Ulm
Germany
e-mail: wolfgang.arendt@uni-ulm.de

M. Sauter
e-mail: manfred.sauter@uni-ulm.de

A.F.M. ter Elst
Department of Mathematics
University of Auckland
Private Bag 92019, Auckland 1142
New Zealand
e-mail: t.terelst@auckland.ac.nz

Received: 20 October 2023

Revised: 25 October 2023

Accepted: 26 October 2023