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Generalized torsion elements in groups

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Abstract. A group element is called a generalized torsion element if a finite product of its conjugates is equal to the identity. We prove that in a nilpotent or FC-group, the generalized torsion elements are all torsion elements. Moreover, we compute the generalized order of an element in a finite group G using its character table.

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1. Introduction. For a pair of elements x and y in a group G, we write $x^y = y^{-1}xy$ for the conjugate of x by y. The order of x, denoted by $o(x)$, is the least positive integer k such that $x^k = 1$; the order is infinite if no such k exists. We say that x is a torsion element if $o(x)$ is finite. The set of all torsion elements of G will be denoted by $\mathsf{T}(G)$. An element $x \in G$ is said to be a *generalized torsion element* if there exist $g_1, \ldots, g_k \in G$ such that

$$x^{g_1} x^{g_2} \cdots x^{g_k} = 1.$$

We will denote by $\mathsf{T}_\bullet(G)$ the set of all generalized torsion elements in G. The *generalized order* of $x \in \mathsf{T}_\bullet(G)$, denoted by $o_\bullet(x)$, is defined to be the smallest positive integer n such $x^{g_1} \cdots x^{g_n} = 1$ for some $g_1, \ldots, g_n \in G$. Hence, the identity element, for example, has generalized order one. We say that G has *generalized exponent* k, writing $\exp_\bullet(G) = k$, if $\mathsf{T}_\bullet(G) = G$ and k is the smallest positive integer such that C^k contains 1 for every conjugacy class C of G. Here, $C^k = \{c_1 \cdots c_k \mid c_i \in C\}$.

The *maximal generalized order* $\max o_\bullet(G)$ of a group G is defined as $\max o_\bullet(G) = \max\{o_\bullet(x) \mid x \in G\}$. We observe that $\max o_\bullet(G) \leq \exp_\bullet(G) \leq \exp(G)$. Both inequalities can be strict, as shown by the example of $\mathrm{SL}(2,3)$ whose maximal generalized order is 3, its generalized exponent is 6, while its exponent is 12.

Note that if x is a torsion element of G, then $o_\bullet(x) \leq o(x)$. Thus $\mathsf{T}(G) \subseteq \mathsf{T}_\bullet(G)$. The reverse inclusion, however, does not hold. For example, in the infinite dihedral group D_∞, we have

$$\mathsf{T}(D_\infty) = \{g \mid g^2 = 1\} \quad \text{while} \quad \mathsf{T}_\bullet(D_\infty) = D_\infty.$$

Moreover, there are finitely generated torsion-free groups where all elements are generalized torsions (see [11, Problem 3.11], Gorchakov [5], or Goryushkin [6]). Osin [15, Corollary 1.2] constructed an example of a torsion-free 2-generator group G with exactly two conjugacy classes (in particular, $\exp_\bullet(G) = 2$). More recently, generalized torsion elements in knot groups were studied in a number of papers (see [9], [10], [13], [14], and the references therein for further results).

In Section 2, we focus on groups whose generalized torsions are torsions. We will prove that this holds in the class of FC-groups (that is, groups whose conjugacy classes are finite) as stated in the following theorem.

Theorem 1.1. *If G is an FC-group, then $\mathsf{T}_\bullet(G) = \mathsf{T}(G)$.*

In Section 3, we adapt some known results of Arad, Stavi, and Herzog [1] to obtain bounds for the generalized exponent of a finite group in terms of the conjugacy classes. Based on results of [1], we also present a practical method for calculating the generalized order for groups whose character table is known.

In Section 4, we will prove the following theorem showing that certain powers of generalized torsion elements lie deep in the lower central series.

Theorem 1.2. *Let x be an element of a group G.*

(1) *If $o_\bullet(x) = k$, then $x^{k^m} \in \gamma_m(G)$ for any positive integer m.*
(2) *If G is nilpotent, then $\mathsf{T}_\bullet(G) = \mathsf{T}(G)$.*

The first version of this text was published on arxiv.org on February 19, 2023. Later, in a private correspondence, T. Ito showed us alternative proofs for Theorems 1.1 and 1.2(b) which, among other related results, can be found in [8].

2. Generalized torsion elements. Observe that if G is an abelian group, then $o_\bullet(g) = o(g)$ for all $g \in G$. However, if G is non-abelian, then $o_\bullet(g)$ need not be equal to $o(g)$. For example, in the symmetric group S_n, all elements are conjugate to their inverse and so S_n has generalized exponent 2 for all n. Thus, taking the n-cycle $\sigma = (12 \cdots n) \in S_n$, we have that $o_\bullet(\sigma) = 2$, while $o(\sigma) = n$, showing that an element of generalized order 2 can have arbitrarily large order.

According to Corollary 4.4, if G is a finite p-group with exponent p, then $o_\bullet(g) = o(g)$ for all $g \in G$. It is worth mentioning that if G is finite, then $o_\bullet(x)$ does not need to divide the order of G (see Example 3.5). Nevertheless, the following result holds.

Proposition 2.1. *Let G be a finite group.*

(1) *If G has an element with generalized order 2, then G has even order.*
(2) *G can be embedded into a finite group of generalized exponent 2.*

Proof. (1) Assume that a non-trivial element $x \in G$ has generalized order 2. By definition, there exists an element g in G such that $xx^g = 1$. If $g \in Z(G)$, then x has order 2 and so G has even order. Thus, in what follows, we may assume that $g \notin Z(G)$. The map

$$\rho_g : G \to G, \quad y \mapsto y^g,$$

is a permutation of the elements of G. Since $\rho_g(x) = x^{-1}$ and $\rho_g(x^{-1}) = x$, we may write $\rho_g \in \mathrm{Sym}(G)$ as a product of disjoint cycles and one of these cycles is (x, x^{-1}). Considering that disjoint cycles in $\mathrm{Sym}(G)$ always commute, ρ_g has even order. Now, the map

$$\rho : G \to \mathrm{Aut}(G), \quad y \mapsto \rho_y,$$

is a homomorphism whose kernel coincides with $Z(G)$. Since $g \notin Z(G)$ and ρ_g has even order, G has even order.

(2) Set $n = |G|$. It follows from Cayley's theorem [16, 1.6.8] that G can be embedded into the symmetric group S_n. Further, as was observed before this result, $\exp_\bullet(S_n) = 2$. □

Recall that a group element $x \in G$ is said to be *real* if $x^{-1} \in x^G$. In particular, a non-trivial element $x \in G$ has generalized order 2 if and only if there exists $g \in G$ such that $xx^g = 1$; that is, $x^{-1} = x^g \in x^G$. Thus, x is real if and only if $o_\bullet(x) = 2$.

In the next result, we collect some of the basic properties of generalized torsion elements.

Proposition 2.2. *Let G be a group.*

(1) *If H is a subgroup of G, then $\mathsf{T}_\bullet(H) \subseteq \mathsf{T}_\bullet(G)$.*
(2) *If K is a group and $\varphi\colon G \to K$ is a homomorphism, then $(\mathsf{T}_\bullet(G))^\varphi \subseteq \mathsf{T}_\bullet(K)$.*
(3) *$\mathsf{T}_\bullet(G)$ is a normal (and characteristic) subset of G.*
(4) *If $x \in \mathsf{T}_\bullet(G) \cap Z(G)$, then $x \in \mathsf{T}(G)$. Moreover, if G is abelian, then $\mathsf{T}_\bullet(G) = \mathsf{T}(G)$.*
(5) *If N is a normal subgroup of G and the quotient group G/N is abelian, then $\exp_\bullet(G/N)$ divides $\exp_\bullet(G)$.*

Proof. (1) If $x \in \mathsf{T}_\bullet(H)$, then there exist $h_1, \dots, h_k \in H$ such that

$$1 = x^{h_1} \cdots x^{h_k}$$

and so, $x \in \mathsf{T}_\bullet(G)$.

(2) If $x \in \mathsf{T}_\bullet(G)$, then there exist $g_1, \dots, g_r \in G$ such that

$$1 = x^{g_1} \cdots x^{g_r}.$$

Since φ is a homomorphism, it follows that

$$1 = (x^\varphi)^{g_1^\varphi} \cdots (x^\varphi)^{g_r^\varphi}$$

and so, $x^\varphi \in \mathsf{T}_\bullet(K)$.

(3) Given an element $g \in G$, conjugation by g induces an automorphism on G. Now, the result follows from the previous item.

(4) If $x \in \mathsf{T}_\bullet(G) \cap Z(G)$, then there exist $g_1, \dots, g_r \in G$ such that

$$1 = x^{g_1} \cdots x^{g_r} = \underbrace{x \cdots x}_{r \text{ times}} = x^r.$$

In particular, $x \in \mathsf{T}(G)$.

(5) Let $\exp_\bullet(G) = n$ and let $x \in G$. By definition, there exist elements $g_1, \dots, g_n$ in G such that $x^{g_1} \cdots x^{g_n} = 1$. Since the quotient group G/N is abelian, we obtain $\overline{1} = (xN)^{g_1 N} \cdots (xN)^{g_n N} = x^n N$, and so $\exp(G/N) = \exp_\bullet(G/N)$ divides n. $\square$

We are now in the position to prove Theorem 1.1.

Proof of Theorem 1.1. It is clear that $\mathsf{T}(G) \subseteq \mathsf{T}_\bullet(G)$. Choose arbitrarily an element $x \in \mathsf{T}_\bullet(G)$. Then, there exist $g_1, \dots, g_k \in G$ such that $x^{g_1} x^{g_2} \cdots x^{g_k} = 1$. In particular, by construction, $x \in \mathsf{T}_\bullet(H)$, where $H = \langle x, g_1, \dots, g_k \rangle$. Since G is an FC-group, so is H. Thus, all centralizers of $x, g_1, \dots, g_k$ in H have finite index. Since the intersection of a finite set of subgroups each of which has finite index is itself of finite index [16, 1.3.12], the center $Z(H)$, being the intersection of the centralizers of the generators of H, has finite index in H. Therefore H is central-by-finite. Set $n = |H : Z(H)|$. Define the map $\theta^* : H \to H$ as follows:

$$\begin{aligned} \theta^* \ : H &\longrightarrow H, \\ h &\longmapsto h^n. \end{aligned}$$

By Schur's theorem [16, 10.1.3], θ^* is an endomorphism of H. By Proposition 2.2(2), $x^n = x^{\theta^*} \in \mathsf{T}_\bullet(H)$. Since $\mathrm{Im}(\theta^*) \leqslant Z(H)$, x^n is a torsion element (Proposition 2.2(4)) and so, x is also a torsion element. $\square$

We obtain the following result as a corollary; this result is somewhat similar to Dietzmann's lemma [16, 14.5.7] that if $X \subseteq G$ is a finite normal set consisting of torsion elements, then $\langle X \rangle$ is finite.

Corollary 2.3. *In a group G, a finite normal subset consisting of generalized torsion elements generates a finite normal subgroup.*

Proof. Let $X = \{x_1, \dots, x_k\} \subseteq \mathsf{T}_\bullet(G)$ be a finite normal subset of G and set $N = \langle X \rangle$. Since X is a normal set, all the conjugacy classes x_i^N have at most k elements and so the center $Z(N) = \bigcap_{i=1}^k C_N(x_i)$ has finite index. If $g \in N$, then $Z(N) \leq C_N(g)$, and hence $C_N(g)$ has finite index; that is, the conjugacy class g^N is finite. Thereby, N is an FC-group. By Theorem 1.1, $X \subseteq \mathsf{T}(G)$. Thus, Dietzmann's lemma implies that N is a finite normal subgroup. $\square$

3. The generalized order in a finite group. In this section, we link the generalized order $o_\bullet(g)$ of an element $g \in G$ of a finite group to the characters of G. This also provides a practical method for computing the generalized order in groups for which the irreducible characters are known. Suppose in this section that G is a finite group and let $C_1, C_2, \dots, C_m$ be the conjugacy classes of G such that $C_1 = \{1\}$. A conjugacy class C is said to be *real* if $C^{-1} = C$, otherwise C is *non-real.* Following [1], we denote by λ the number of real conjugacy classes distinct from $\{1\}$ and by 2μ the number of non-real conjugacy classes (which is always an even number). Hence the number m of conjugacy classes of G can be written as $m = 1 + \lambda + 2\mu$.

The two assertions of the following proposition are proved in [1, Lemmas 7.3 and 7.4 of].

Proposition 3.1. *Suppose that G is a finite group and let $g \in G$.*

(1) *$o_\bullet(g)$ is less than or equal to the number of conjugacy classes in G that contain powers of g.*
(2) *$o_\bullet(g) \leq 2\mu + 2$.*

Corollary 3.2. *Let G be a finite group. If H is a core-free subgroup of G, then* $\max o_\bullet(G) \leq 2^{|G:H|-1}$.

Proof. Let R be the set of all right cosets of H. Every element g in G induces a permutation on R by right multiplication $(Hx)g = H(xg)$. Since H is core-free, G gets embedded into the symmetric group S_n where $n = |G : H|$. An important result due to Liebeck and Pyber [12, Theorem 2] states that the number of conjugacy classes of any subgroup of S_n is at most 2^{n-1}. Now, the result follows from Proposition 3.1(1). □

The generalized order of an element g of a finite group G can be calculated using the character table of G. Suppose that $\mathrm{Irr}(G)$ denotes the set of irreducible characters of G. For a conjugacy class $C \subseteq G$ and for $k \geq 1$, let $\alpha_{C,k}$ be the number of k-tuples $(g_1, \dots, g_k) \in C^k$ such that $g_1 \cdots g_k = 1$. That is, $\alpha_{C,k}$ counts how many ways the identity can be written as a product of k elements of C. For $g \in C$, we have that

$$o_\bullet(g) = \min\{k \geq 1 \mid \alpha_{C,k} > 0\}$$

and also that

$$\exp_\bullet(G) = \min\{k \geq 1 \mid \alpha_{C,k} > 0 \text{ for all conjugacy classes } C \subseteq G\}.$$

The following lemma appeared in [1, Lemma 10.10]; see also [17, Equation (1)].

Theorem 3.3. *Using the notation in the previous paragraph,*

$$\alpha_{C,k} = \frac{|C|^k}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g)^k}{\chi(1)^{k-2}}. \tag{1}$$

Theorem 3.3 gives an computationally efficient method for calculating the generalized order for elements in finite groups whose character tables are known.

Example 3.4. Suppose that G is the group number 3 among the groups of order 18 in GAP [4]. The group G has 9 conjugacy classes and 9 irreducible representations. Suppose that C is the conjugacy class number 8 according to the numbering given by GAP. Then one can compute, using (1), that $\alpha_{C,1} = \alpha_{C,2} = 0$, but $\alpha_{C,3} = 243$. Hence the identity element $1 \in G$ can be written as a product $g_1 g_2 g_3$ with $g_i \in C$ in 243 ways and in particular $o_\bullet(g) = 3$ for all $g \in C$.

Example 3.5. Suppose that G is the Suzuki group Sz(8) and assume that C is the conjugacy class number three in the numbering by GAP. Using GAP, we computed that $\alpha_{C,1} = \alpha_{C,2} = 0$, but $\alpha_{C,3} = 196{,}560$. Thus the identity element of G can be written in $196{,}560$ ways as a product $g_1g_2g_3$ with $g_i \in C$. In particular, $o_\bullet(g) = 3$ for all $g \in C$. Interestingly, $196{,}560$ coincides with the kissing number of the 24-dimensional Leech lattice and is equal to the coefficient of the first non-constant term of the modular form the lattice; [3, Section 2].

Example 3.6. Let G be a group of order 2^k for $k = 1, \dots, 8$. Using GAP, we computed that the generalized exponent of G is a 2-power. However, there exists a finite 3-group whose generalized exponent is not a 3-power. For example, SmallGroup$(3^5, 4)$ has generalized exponent 6.

It is known that many finite non-abelian simple groups have generalized exponent less than or equal to 3 (see [18, Theorem 3] and [1, Chapters 1 and 2]). Using GAP, we computed that $\exp_\bullet(G) \leq 3$ whenever $G = A_n$ for $n \leq 15$ or $G = \mathrm{PSL}(2, q)$ for all $q \leq 49$. Further, in [17, Theorem 2.6], Shalev showed that if G is a finite non-abelian simple group and $x \in G$ is chosen at random, then the probability that $(x^G)^3 = G$ tends to 1 as $|G| \to \infty$. These facts support the following conjecture.

Conjecture 3.7. *If G is a finite non-abelian simple group, then* $\max o_\bullet(G) \leq 3$.

4. Relations between generalized torsion and the terms of the lower central series. We define recursively *commutators* of weight $1, 2, \dots$ in elements $x_1, x_2, \dots$ of a group G as follows. The elements $x_1, x_2, \dots$ are commutators of weight 1, $[x_i, x_j] = x_i^{-1}x_j^{-1}x_ix_j$, with $i \neq j$, are commutators of weight 2 and if c_1 and c_2 are commutators of weight w_1 and w_2, respectively, then $[c_1, c_2]$ is a commutator of weight $w_1 + w_2$. Here, c_1 and c_2 are called left and right *sub-commutators*, respectively. The *first entry* in a commutator $[c_1, c_2]$ is defined as the first entry of c_1, while the first entry of a commutator x of weight one is of course just x. In case brackets are omitted, the commutators are assumed left-normed, for example, $[x_1, x_2, x_3] = [[x_1, x_2], x_3]$. The terms $\gamma_i(G)$ of the lower central series of G are defined recursively as $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = [\gamma_i(G), G]$ for $i \geq 1$. In particular, $\gamma_2(G) = G'$ is the commutator (or derived) subgroup. It is well-known that $\gamma_i(G)$ is the subgroup of G generated by all commutators of weight i in the elements of G.

We quote the following well-known lemma (see [16, Lemma 5.1.5 and Exercise 5.1.4] and [7, Chapter III, Section 9.4]). It will be used in the rest of the paper, often without explicit reference.

Lemma 4.1. *For elements x, y, z of a group G and a positive integer k, the following identities are valid:*

(1) $xy = yx[x, y]$,
(2) $x^y = x[x, y]$,
(3) $[xy, z] = [x, z][x, z, y][y, z]$,
(4) $[x^k, y] = [x, y]^{x^{k-1}}[x, y]^{x^{k-2}} \cdots [x, y]^x[x, y]$,

(5) $x^k y^k = (xy)^k c_2^{\binom{k}{2}} \cdots c_i^{\binom{k}{i}} \cdots c_{k-1}^k c_k$, *where* $c_i \in \gamma_i(\langle x, y\rangle)$ *for each non-negative integer* i.

The item (5) above is known as the Hall-Petrescu formula.

Lemma 4.2. *Let* $k \geq 2$, *and let* $x, g_1, \ldots, g_k$ *be elements in a group* G.

(1) *We have that*

$$x^{g_1} x^{g_2} \cdots x^{g_k} = x^k \sigma_2$$

where σ_2 *is a product of commutators of weight at least* 2 *and the element* x *is the first entry of all the factors of* σ_2.

(2) *If* $o_\bullet(x) = k$, *then* $x^k = c_1 \cdots c_r$, *where each* $c_i = [c_{i,1}, c_{i,2}]$ *is a commutator of weight at least* 2 *such that the first entry of* $c_{i,2}$ *is* x.

(3) *If* $o_\bullet(x) = k$ *and* c_m *is a commutator of weight* m *with* x *in some entry, then* c_m^k *is a product of commutators of weight at least* $m+1$ *and* x *appears in some entry of all factors of* c_m^k.

(4) *If* $o_\bullet(x) = k$ *and* σ_m *is a product of commutators of weight at least* m *with* x *in some entry of all its factors, then* σ_m^k *is a product of commutators of weight at least* $m+1$ *and* x *appears in some entry of all its factors.*

(5) *If* $o_\bullet(x) = k$, *then* $(x^{g_1} x^{g_2} \cdots x^{g_k})^{k^m} = x^{k^{m+1}} \sigma_{m+1}$, *where* σ_{m+1} *is a product of commutators of weight at least* $m+1$ *and the element* x *appears in some entry of all factors of* σ_{m+1}.

Proof. (1) We proceed by induction on k. If $k = 2$, then

$$x^{g_1} x^{g_2} = x[x, g_1]x[x, g_2] = x^2[x, g_1][x, g_1, x][x, g_2].$$

Assuming the result holds for $k \geq 2$, we get

$$(x^{g_1} x^{g_2} \cdots x^{g_k})x^{g_{k+1}} = (x^k \sigma_2)x[x, g_{k+1}] = x^{k+1}\sigma_2[\sigma_2, x][x, g_{k+1}],$$

where σ_2 is a product of commutators of weight at least 2 and the element x is the first entry of the factors of σ_2. Now, the result follows by applying Lemma 4.1(3) several times to the commutator $[\sigma_2, x]$.

(2) If $o_\bullet(x) = k$, then there exist elements $g_1, \ldots, g_k \in G$ such that $1 = x^{g_1} x^{g_2} \cdots x^{g_k}$. By the previous item, we can write

$$1 = x^{g_1} x^{g_2} \cdots x^{g_k} = x^k \sigma_2,$$

where σ_2 is a product of commutators of weight at least 2 and the element x is the first entry of all the factors of σ_2. Thus, $x^k = \sigma_2^{-1}$.

(3) We proceed by induction on m. The basic step $m = 1$ follows by item (2). Assume the result holds for all positive integers up to m. Let c_{m+1} be a commutator of weight $m+1$ with x in some entry. Write $c_{m+1} = [c_i, c_j]$ where c_i, c_j are commutators of weight i and j, respectively, and $i + j = m + 1$. Without loss of generality, we can assume that x occurs in the left sub-commutator c_i. By Lemma 4.1(4), we obtain

$$\begin{aligned}[c_i^k, c_j] &= [c_i, c_j]^{c_i^{k-1}}[c_i, c_j]^{c_i^{k-2}} \cdots [c_i, c_j]^{c_i}[c_i, c_j] \\ &= ([c_i, c_j][c_i, c_j, c_i^{k-1}]) \cdots ([c_i, c_j][c_i, c_j, c_i])[c_i, c_j] \\ &= [c_i, c_j]^k \sigma_{m+2}\end{aligned}$$

where σ_{m+2} is a product of commutators of weight at least $m+2$ with x in some entry of all factors. By the induction hypothesis, c_i^k is a product of commutators of weight at least $i+1$ and x appears in some entry of all its factors, say $c_i^k = c_{i,1}\cdots c_{i,r}$. Since

$$c_{m+1}^k = [c_i, c_j]^k = [c_i^k, c_j]\sigma_{m+2}^{-1},$$

the result follows by applying several times Lemma 4.1(3) to the commutator $[c_i^k, c_j] = [c_{i,1}\cdots c_{i,r}, c_j]$.

(4) We proceed by induction on the number r of factors of σ_m. The basic step $r=1$ follows by the previous item. Assume that the result holds for all positive integers up to $r \geq 1$ and set $\sigma_m = \tau_1\cdots\tau_r\tau_{r+1}$ where each τ_i is a commutator of weight at least m with x in some entry. By Lemma 4.1(5), we can deduce that

$$\sigma_m^k = (\tau_1\cdots\tau_r\tau_{r+1})^k = (\tau_1\cdots\tau_r)^k\tau_{r+1}^k\sigma_{m+1}$$

where σ_{m+1} is a product of commutators of weight at least $m+1$ with x appearing in some entry in each factor. Thus, the result follows by applying the induction hypothesis to $(\tau_1\cdots\tau_r)^k$ and to τ_{r+1}^k.

(5) We show item (5) by induction on m. Firstly, we will show the basic step $m=1$. By item (1), we can write $x^{g_1}x^{g_2}\cdots x^{g_k} = x^k\sigma_2$. By parts (1) and (5) of Lemma 4.1, we have

$$(x^{g_1}x^{g_2}\cdots x^{g_k})^k = (x^k\sigma_2)^k = x^{k^2}\tilde{\sigma_2}$$

where $\tilde{\sigma_2}$ is a product of commutators of weight at least 2 and the element x appears in some entry of all factors.

Now, assume that the result holds for all positive integers up to $m \geq 1$. By the induction hypothesis, we get

$$(x^{g_1}x^{g_2}\cdots x^{g_k})^{k^{m+1}} = ((x^{g_1}x^{g_2}\cdots x^{g_k})^{k^m})^k = (x^{k^{m+1}}\sigma_{m+1})^k$$

where σ_{m+1} is a product of commutators of weight at least $m+1$ and the element x appears in some entry of all factors. By Lemma 4.1(5), we can deduce that $(x^{k^{m+1}}\sigma_{m+1})^k = x^{k^{m+2}}\sigma_{m+1}^k\sigma_{m+2}$, where σ_{m+2} is a product of commutators of weight at least $m+2$ and the element x appears in some entry of all the factors. Thus, the result follows applying item (4) to σ_{m+1}^k. $\square$

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. (1) Let x be an element in a group G with generalized order $o_\bullet(x) = k$. Thus, there exist elements $g_1,\ldots,g_k$ in G such that $x^{g_1}x^{g_2}\cdots x^{g_k} = 1$. By Lemma 4.2 (5), we have, for every positive integer m,

$$1 = (x^{g_1}x^{g_2}\cdots x^{g_k})^{k^{m-1}} = x^{k^m}\sigma_m$$

where σ_m is a product of commutators of weight at least m. Thus, $x^{k^m} \in \gamma_m(G)$.

(2) Let c be the nilpotency class of G. We need to show that $\mathsf{T}_\bullet(G) \subseteq \mathsf{T}(G)$. Choose arbitrarily $x \in \mathsf{T}_\bullet(G)$. By the previous item,

$$x^{k^{c+1}} \in \gamma_{c+1}(G) = \{1\}.$$

As $x \in \mathsf{T}_\bullet(G)$ has been chosen arbitrarily, we conclude that $\mathsf{T}_\bullet(G) \subseteq \mathsf{T}(G)$. $\square$

A group G is said to be orderable if there is a total order on G such that $a \leq b$ implies that $xay \leq xby$ for all $a, b, x, y \in G$. It is known that torsion-free nilpotent groups are orderable; see [2].

Remark 4.3. If G is an orderable group, then $\mathsf{T}_\bullet(G) = \{1\}$. It is known that the converse does not hold in general (see [2]). We can deduce from the previous result that if G is nilpotent with $\mathsf{T}_\bullet(G) = \{1\}$, then G is orderable.

Corollary 4.4. *If x is an element in a nilpotent p-group G, then p divides $o_\bullet(x)$.*

Proof. Let c be the nilpotency class of G. Since G is a p-group, we get that x is a generalized torsion element. Setting $o_\bullet(x) = k$, it follows from Theorem 1.2(1) that $x^{k^{c+1}} = 1$. Consequently, p divides k^{c+1} and so, p divides k. $\square$

Remark 4.5. The previous result cannot be improved. In general, if G is a p-group and $g \in G$, then $o_\bullet(g)$ need not be a p-power. Let G be the 8-th group of order 81 from the GAP Small Groups Library. Then G contains elements with generalized torsion order 6.

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References

[1] Arad, Z., Stavi, J., Herzog, M.: Powers and products of conjugacy classes in groups. In: Products of Conjugacy Classes in Groups, pp. 6–51. Lecture Notes in Math., 1112. Springer, Berlin (1985)

[2] Bludov, V.V.: An example of an unorderable group with strictly isolated identity. Algebra i Logika **11**, 619–632, 736 (1972)

[3] Conway, J.H., Sloane, N.J.A.: Sphere Packings, Lattices and Groups. Third Edition. With additional contributions by E. Bannai, R.E. Borcherds, J. Leech, S.P.

Norton, A.M. Odlyzko, R.A. Parker, L. Queen and B.B. Venkov. Grundlehren der mathematischen Wissenschaften, 290. Springer, New York (1999)

[4] The GAP Group. GAP – Groups, Algorithms, and Programming, Version 4.12.2, gap-system.org (2022)

[5] Gorčakov, J.M.: An example of a G-periodic torsion-free group. Algebra i Logika Sem. **6**(3), 5–7 (1967)

[6] Gorjuškin, A.P.: An example of a finitely generated G-periodic torsion-free group. Sibirsk. Mat. Ž. **14**, 204–207, 239 (1973)

[7] Huppert, B.: Endliche Gruppen I. Grundlehren der mathematischen Wissenschaften, vol. 134. Springer, Berlin-New York (1967)

[8] Ito, T.: On a group whose generalized torsion elements are torsion elements. arXiv:2303.05726v2 (2023)

[9] Ito, T., Motegi, K., Teragaito, M.: Generalized torsion and Dehn filling. Topology Appl. **301**, Paper No. 107515, 14 pp. (2021)

[10] Ito, T., Motegi, K., Teragaito, M.: Generalized torsion for hyperbolic 3-manifold groups with arbitrary large rank. Bull. London Math. Soc. **55**(3), 1203–1209 (2023)

[11] Khukhro, E.I., Mazurov, V.D.: Unsolved problems in group theory. The Kourovka notebook. arXiv:1401.0300v26 (2022)

[12] Liebeck, M.W., Pyber, L.: Upper bounds for the number of conjugacy classes of a finite group. J. Algebra **198**(2), 538–562 (1997)

[13] Motegi, K., Teragaito, M.: Generalized torsion elements and bi-orderability of 3-manifold groups. Canad. Math. Bull. **60**(4), 830–844 (2017)

[14] Naylor, G., Rolfsen, D.: Generalized torsion in knot groups. Canad. Math. Bull. **59**(1), 182–189 (2016)

[15] Osin, D.: Small cancellations over relatively hyperbolic groups and embedding theorems. Ann. of Math. (2) **172**(1), 1–39 (2010)

[16] Robinson, D.J.S.: A Course in the Theory of Groups. Second Edition. Graduate Texts in Mathematics, 80. Springer, New York (1996)

[17] Shalev, A.: Word maps, conjugacy classes, and a noncommutative Waring-type theorem. Ann. of Math. (2) **170**(3), 1383–1416 (2009)

[18] Vdovin, E.P., Gal't, A.A.: Strong reality of finite simple groups. Sib. Mat. J. **51**(4), 610–615 (2010)

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