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A short note on decay rates of odd partitions: an application of spectral asymptotics of the Neumann–Poincaré operators

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Abstract. Here, we introduce a theorem currently proved uniquely by the asymptotic behaviors of eigenvalues of a compact operator. Specifically, a problem of partitions is considered, and the Neumann–Poincaré operator is employed as the compact linear operator. Then a theorem is proved by the spectrum of the Neumann–Poincaré operator. Although the following proposed problem looks artificial, our result in the partitions seems to be proven uniquely by the spectral theory of the Neumann–Poincaré operators: Odd partitions of the unit interval $[0, 1]$ are considered, that is, we divide the unit interval $[0, 1]$ into $2N + 1$ disjoint *non-zero* intervals $L_{N,k}$ $(k = 1, \ldots, 2N + 1)$, and the sum of corresponding lengths $\sum_{k=1}^{2N+1} |L_{N,k}| = 1$ for each $N \in \mathbb{N}_{\geq 0}$. Thus we obtain a countable set of real numbers $P = \{|L_{N,k}| \;\; k = 1, 2, \ldots, 2N + 1, \;\; N \in \mathbb{N}_{\geq 0}\}$ by odd partitions of the unit interval. One can enumerate the set P in decreasing order to obtain the non-increasing sequence

$$a_1 = |L_{0,1}| = 1 > a_2 \geq a_3 \geq \cdots > 0.$$

We show that for any $C \geq 1/2$, there exist odd partitions of the unit interval such that

$$a_j \sim C j^{-1/2} \quad \text{as } j \to \infty.$$

Here, the coefficient $C = 1/2$ corresponds to the optimal decay. We prove this fact by a fundamental property of the Riemann zeta function and by eigenvalue asymptotics for some compact linear operators known as the Neumann–Poincaré operators.

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1. Introduction and Results. Although many sophisticated results have been presented on the spectral theory of compact operators, their applications in number theory often seem weaker than those in specific fields.

The well-known Gauss circle problem, which is the problem of determining how many integer lattice points $N(r)$ there are in a circle centered at the origin with radius $r > 0$, is a typical example, where $N(r) = \pi r^2 + O(r^{1/2+\epsilon})$. The estimate $0 < \epsilon \leq 1/2$ can be proven by eigenvalue asymptotics of the Laplace operator, whereas the improved estimate $0 < \epsilon \leq 27/208$ can be proven by analytic number theory [5,6,8]. Thus, spectral theory of linear operators has brought about superior results. However, when spectrtal theory is applied to mathematical problems of different fields, the obtained results often seem weaker than those in specific fields.

Our purpose here is to present an application that seems to be currently proven uniquely by spectral theory. This is shown by the behavior of partitions. More precisely, when we divide the unit interval $[0, 1]$ into $2N + 1$ *non–zero length* subintervals, finely divided partitions seem to appear. We call such partitions "Odd Partitions". Denoting odd partitions as $L_{N,k}$ $(k = 1. \dots, 2N+1)$ for each $N \in \mathbb{N}_{\geq 0}$, one can enumerate the countably infinitely many real numbers $\{|L_{N,k}| \ \ k = 1, 2, \dots, 2N + 1, \ N \in \mathbb{N}_{\geq 0}\}$ in decreasing order. Here, $|\cdot|$ denotes the Lebesgue measure (length). Thus, such procedure allows us to give the non-increasing sequence

$$1 = |L_{0,1}| = a_1 > a_2 \geq a_3 \geq \cdots > 0. \tag{1.1}$$

For instance, equi-partitions of the unit interval yield the following diagram, where the first partitioning yields 3 intervals, whose length is 1/3. Similarly, a non-increasing sequence is produced.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1/3 & 1/3 & 1/3 & & \\ & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & \\ 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 \\ & & \vdots & \vdots & \vdots & & \end{array}$$

In the above diagram, each row shows a partition of the unit interval, that is, the sum of each row equals one. Thus, we obtain the following enumerated sequence in decreasing order:

$$1, 1/3, 1/3, 1/3, 1/5, 1/5, 1/5, 1/5, 1/5, 1/7, 1/7, 1/7, 1/7, 1/7, 1/7, 1/7, \dots \tag{1.2}$$

After the N-th procedure, the diagram of partitions always consists of N^2 numbers. One can easily find from (1.2) that the $j = N^2$-th number a_j is $1/(2N - 1) \sim \frac{1}{2}j^{-1/2}$ for large j. It is strongly expected that the optimal decay rate is attained by such equi-partitions.

In fact, we have the desirable decay rates:

Theorem 1.1 (Main Theorem). *For all $C \geq \frac{1}{2}$, there exist odd partitions of the unit interval such that*

$$a_j \sim Cj^{-1/2} \quad \text{as } j \to \infty.$$

Here, we emphasize that $C = \frac{1}{2}$ is the minimum coefficient, namely, $a_j \not\sim \frac{1}{2}j^{-1/2}$ as $j \to \infty$

To prove this, we recall the concept of unconditional sums, which are convenient here:

Proposition 1.2. *For odd partitions, we define the infinite sum $\tau(p)$ by*

$$\tau(p) := \sum_{\substack{N\in\mathbb{N}_{\geq 0}\\ k=1,2,\ldots,2N+1}} |L_{N,k}|^p \quad (p > 2). \tag{1.3}$$

Then, $\tau(p) \geq (1-2^{1-p})\zeta(p-1)$ $(p > 2)$, where $\zeta(p)$ denotes the Riemann zeta function.
The equality holds only for the case of equi-partitions.

We remark that Proposition 1.2 holds true even in the case that the sum (1.3) diverges to ∞.

Proof of Proposition 1.2. Since the sum (1.3) consists only of positive values, the sum is unconditional and independent of rearrangements.

It follows by Hölder's inequality (e.g., see [13]) that

$$\begin{aligned} 1 &= |L_{N,1}| + |L_{N,2}| + \cdots + |L_{N,2N+1}| \\ &\leq (1+1+\cdots+1)^{1/q} \cdot \Big(|L_{N,1}|^p + |L_{N,2}|^p + \cdots + |L_{N,2N+1}|^p\Big)^{1/p} \\ &\leq (2N+1)^{\frac{p-1}{p}} \cdot \Big(|L_{N,1}|^p + |L_{N,2}|^p + \cdots + |L_{N,2N+1}|^p\Big)^{1/p} \end{aligned}$$

where $p, q \in [1,\infty]$ with $1/p + 1/q = 1$. Thus

$$|L_{N,1}|^p + |L_{N,2}|^p + \cdots + |L_{N,2N+1}|^p \geq (2N+1)^{(1-p)} \quad \text{for } p \geq 1, \tag{1.4}$$

and so

$$\tau(p) \geq \sum_{N\in\mathbb{N}_{\geq 0}} (2N+1)^{(1-p)} = (1-2^{1-p})\zeta(p-1) \quad \text{for } p > 2. \tag{1.5}$$

The equality holds only for the case of equi-partitions
(i.e., $|L_{N,1}| = |L_{N,2}| = \cdots = |L_{N,2N+1}| = 1/(2N+1)$ for all $N \in \mathbb{N}$). □

Proof of Theorem 1.1. Firstly, we show that $\frac{1}{2}$ is the minimum coefficient.

Assuming $C < \frac{1}{2}$, then

$$\sum_{j=1}^{\infty} |a_j|^p \preceq \int_1^{\infty} C^{2p} j^{-p/2}\, dj = \frac{2C^{2p}}{p-2} \quad \text{for } p > 2. \tag{1.6}$$

It can be seen that $2C^{2p} < 1/2$ for $p = 2 + \varepsilon$ with small $\varepsilon > 0$.

Meanwhile, it follows from Proposition 1.2 that the sum of the p–th power equi-partitions is

$$\sum_{j=1}^{\infty} |a_j|^p = \tau(p) \geq (1-2^{1-p})\zeta(p-1) \quad \text{for } p > 2. \tag{1.7}$$

We then recall a property of the Riemann zeta function $\zeta(x)$ (e.g., see [3]):

$$\lim_{p \to 2+0} \left(\zeta(p-1) - \frac{1}{p-2} \right) = \gamma, \tag{1.8}$$

where γ is Euler's constant. Thus, we have

$$\lim_{p \to 2+0} \left(\left(1-2^{1-p}\right) \zeta(p-1) - \frac{1}{2(p-2)} \right) = C \tag{1.9}$$

for some constant $C(= \frac{1}{2}(\log 2 + \gamma))$. Thus, it follows from (1.7) that

$$\lim_{p \to 2+0} \left(\sum_{j=1}^{\infty} |a_j|^p - \frac{1}{2(p-1)} \right) \geq \lim_{p \to 2+0} \left(\left(1-2^{1-p}\right) \zeta(p-1) - \frac{1}{2(p-1)} \right) = C$$

and from (1.6) that

$$\lim_{p \to 2+0} \sum_{j=1}^{\infty} |a_j|^p - \frac{1}{2(p-1)} = -\infty.$$

This is a contradiction, as desired.

To prove the existence of suitable partitions satisfying $a_j \sim Cj^{-1/2}$ for $C \geq 1/2$, we use the spectral properties of the Neumann–Poincaré (NP) operator, which is a boundary integral operator, defined on boundaries of a region in $\mathbb{R}^3$ (e.g., see [2] and references therein for details). The NP operators on $L^2(\partial\Omega)$ are compact if $\partial\Omega$ is in $C^{1,\alpha}$, that is, the corresponding *non-zero* spectrum consists only of eigenvalues. We emphasize that corresponding eigenvalues on prolate ellipsoids $\partial\Omega$ satisfy all properties of lengths for odd partitions of the interval $[0, 1/2]$ (see [1,9,12]):

$$\begin{array}{ccccc} & & M_{1,1} & & \\ & M_{2,1} & M_{2,2} & M_{2,3} & \\ M_{3,1} & M_{3,2} & M_{3,3} & M_{3,4} & M_{3,5} \\ & \vdots & \vdots & \vdots & \end{array}$$

Here, each row represents a partition of $[0, 1/2]$ and consists of an odd number of *non-zero* subintervals. The sum $\sum_{k=1}^{2N+1} |M_{N,k}| = 1/2$ for each $N \in \mathbb{N}_{\geq 0}$. These novel facts are not elementary, but the results are available here. Furthermore, it was recently proven [10,11] that NP eigenvalues satisfy the so-called Weyl law, namely,

$$a_j \sim \widetilde{C} j^{-1/2}$$

for $\widetilde{C} \geq 1/4$. Here, the coefficient $\widetilde{C}$ is explicitly calculated using the Willmore energy $W(\partial\Omega)$ and the Euler characteristic $\chi(\partial\Omega)$ of the boundary surface $\partial\Omega$. It follows that the coefficient $\widetilde{C}$ can take arbitrary real values larger than 1/4 (see [10, 11] for details). As a result, there exist odd partitions of a half interval $[0, 1/2]$ such that the enumerated sequence satisfies

$$a_j \sim \widetilde{C} j^{-1/2}$$

for all $\widetilde{C} \geq 1/4$. When we consider the interval $[0, 1]$ instead of the half interval $[0, 1/2]$, $C = 2\widetilde{C} \geq 1/2$ automatically. □

2. Discussion. We proved the decay rates on odd partitions of the unit interval. Theorem 1.1 is proven by a fundamental property of the Riemann zeta function and the spectral theory of NP operators. If the partitions are permitted to have *zero*-length sets, this fact is proved in an elementary manner. Can one give an elementary proof of Theorem 1.1 as it is? To the best of my knowledge, we do not know alternative proofs other than spectral theory.

The partitions of the unit interval have been considered for a number of years from various viewpoints (e.g., see [4, 7]). For more general partitions, can one prove the existence of a decay sequence for suitable orders?

The difference sequence of the Farey sequence, for instance, is the partitions of the unit interval [7]:

$$\begin{array}{ccccccccc} &&&& 1 &&&& \\ &&& 1/2 && 1/2 &&& \\ && 1/3 & 1/6 && 1/6 & 1/3 && \\ & 1/4 & 1/12 & 1/6 && 1/6 & 1/12 & 1/4 & \\ 1/5 & 1/20 & 1/12 & 1/15 & 1/10 & 1/15 & 1/12 & 1/20 & 1/5 \\ &&& \vdots & \vdots & \vdots &&& \end{array}$$

Thus the enumerated decreasing sequence $\{a_j\}$ is denoted as

$$1, 1/2, 1/2, 1/3, 1/3, 1/4, 1/4, 1/5, 1/5, 1/6, 1/6, 1/6, 1/6, 1/6, 1/6, \ldots, \quad (2.1)$$

and its corresponding behavior seems to be $a_j \sim C j^{-1/3}$, as it is known that the N-th row consists of N^2 numbers asymptotically. The decreasing order $j^{-1/3}$ depends on the number of partitions. For general partitions of the unit interval, can one prove analogous results to Theorem 1.1? The minimum decay can be easily guessed, and it should be given by equi-partitions.

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