Arch. Math. 121 (2023), 329–342

0003-889X/23/030329-14
published online August 19, 2023
https://doi.org/10.1007/s00013-023-01893-8

Archiv der Mathematik

Rigidity of Willmore submanifolds and extremal submanifolds in the unit sphere

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Abstract. Let M be an n-dimensional ($n \geq 4$) compact Willmore (or extremal) submanifold in the unit sphere S^{n+p}. Denote by Ric and H the Ricci curvature and the mean curvature of M, respectively. It is proved that if $(\int_M (\mathrm{Ric}_-^\lambda)^{\frac{n}{2}})^{\frac{2}{n}} < A(n,\lambda,H,\rho)$ (or $B(n,\lambda,H,\rho)$), then M is a totally umbilical sphere, where $A(n,\lambda,H,\rho)$ and $B(n,\lambda,H,\rho)$ are two explicit positive constants depending on n, λ, H, and ρ. This extends parts of the results from a pointwise Ricci curvature lower bound to an integral Ricci curvature lower bound.

Mathematics Subject Classification. Primary 53C40; Secondary 53C24.

Keywords. Willmore submanifold, Sobolev inequality, Integral Ricci curvature, Extremal submanifold.

1. Introduction. Let $x : M \to S^{n+p}$ be an n-dimensional submanifold in an $(n+p)$-dimensional unit sphere S^{n+p}. Choose a local orthonormal frame field $\{e_1, \ldots, e_{n+p}\}$ in S^{n+p} such that, restricted to M, the $\{e_1, \ldots, e_n\}$ are tangent to M. We will make the following convention on the range of indices:

$$1 \leq i, j, k, \ldots \leq n; \qquad n+1 \leq \alpha, \beta, \gamma, \ldots \leq n+p.$$

Denote by H and S the mean curvature and the squared length of the second fundamental form of M, respectively. Then we have

$$S = \sum_{\alpha,i,j} (h_{ij}^\alpha)^2, \quad \mathbf{H} = \sum_\alpha H^\alpha e_\alpha, \quad H^\alpha = \frac{1}{n}\sum_k h_{kk}^\alpha, \quad H = |\mathbf{H}|,$$

where h_{ij}^α are the components of the second fundamental form of M:

Supported by the National Natural Science Foundation of China #12061036, #11761049, and the Jiangxi Province Natural Science Foundation of China #20202ACB201001.

We define the following non-negative function on M:

$$\rho^2 = S - nH^2, \tag{1.1}$$

which vanishes exactly at the umbilic points of M, the Willmore functional is

$$W(x) = \int_M \rho^n dv = \int_M (S - nH^2)^{\frac{n}{2}} dv. \tag{1.2}$$

It was shown in [4] that the Willmore functional is an invariant under the Möbius transformation of S^{n+p}. The Willmore submanifold was defined by [10].

Definition 1.1. $x : M \to S^{n+p}$ is called a Willmore submanifold if it is a critical point of the Willmore functional $W(x)$.

In particular, when $n = 2$, the functional essentially coincides with the well-known Willmore functional $W(x)$ and its critical points are the Willmore surfaces. The Euler–Lagrange equation (i.e., Willmore equation) can be found in [10, (1.2)].

Li [10] also proved the following pointwise pinching theorem for compact Willmore submanifolds.

Theorem A ([10]). *Let M be an n-dimensional ($n \geq 2$) compact Willmore submanifold in S^{n+p}. If $\rho^2 \leq \frac{n}{2-1/p}$, then either $\rho^2 \equiv 0$ and M is totally umbilical, or $\rho^2 \equiv \frac{n}{2-1/p}$. In the latter case, either $p = 1$ and M is a Willmore torus $W_{m,n-m} = S^m(\sqrt{\frac{n-m}{n}}) \times S^{n-m}(\sqrt{\frac{m}{n}})$; or $n = 2$, $p = 2$, and M is the Veronese surface.*

Define

$$F(x) = \int_M \rho^2 dv = \int_M (S - nH^2) dv, \tag{1.3}$$

which vanishes if and only if M is a totally umbilical submanifold. So the function $F(x)$ measures the extent to which $x(M)$ is a totally umbilical submanifold. Obviously, when $n = 2$, $F(x)$ reduces to the well-known Willmore functional $W(x)$.

Definition 1.2. $x : M \to S^{n+p}$ is called an extremal submanifold if it is a critical point of the functional $F(x)$.

Guo and Li [7] calculated the Euler–Lagrange equation of $F(x)$ and proved the following rigidity theorem.

Theorem B ([7]). *Let M be an n-dimensional ($n \geq 2$) compact extremal submanifold in S^{n+p}. If $\rho^2 \leq \frac{n}{2-1/p}$, then either $\rho^2 \equiv 0$ and M is totally umbilical, or $\rho^2 \equiv \frac{n}{2-1/p}$. In the latter case, either $p = 1$, $n = 2m$, and M is a Clifford torus $C_{m,m} = S^m(\sqrt{\frac{1}{2}}) \times S^m(\sqrt{\frac{1}{2}})$; or $n = 2$, $p = 2$, and M is the Veronese surface.*

Both of the above results are pointwise pinching theorems. It seems to be interesting to study the L^q-pinching condition. In [18] and [19], the authors obtain the following global pinching theorems for compact Willmore submanifolds and extremal submanifolds in a sphere.

Theorem C ([18]). *Let M be a compact Willmore surface in the unit sphere S^{2+p}. There exists an explicit positive constant $C = \frac{(\sqrt{2}-1)\sqrt{\pi}}{12\sqrt{3}}B$ such that if $(\int_M \rho^4)^{\frac{1}{2}} < C$, then $\rho^2 = 0$ and M is a totally umbilical sphere, where B is a constant.*

Theorem D ([19]). *Let M be an n-dimensional $(n \geq 3)$ compact extremal submanifold in S^{n+p}. There exists an explicit positive constant A_n depending only on n such that if $(\int_M \rho^n)^{\frac{2}{n}} < A_n$, then M is a totally umbilical submanifold.*

For each $x \in M$, let $R_m(x)$ be the smallest eigenvalue of the Ricci tensor at x, and $\mathrm{Ric}_-^\lambda(x) = \max\{0, (n-1)\lambda - R_m(x)\}$ for $\lambda \in \mathbf{R}$. Define

$$\| \mathrm{Ric}_-^\lambda \|_q = \left(\int\limits_M (\mathrm{Ric}_-^\lambda)^q \right)^{\frac{1}{q}}.$$

It is obvious that $\| \mathrm{Ric}_-^\lambda \|_q = 0$ if and only if $\mathrm{Ric} \geq (n-1)\lambda$.

Chen and Wei proved the following rigidity theorem.

Theorem E ([5]). *Let M be an n-dimensional $(n \geq 4)$ closed submanifold in a space form M_c^{n+p} with parallel mean curvature. Denote by H the norm of the parallel mean curvature of M. Assume $c + H^2 > 0$. Given λ satisfying $(n-2)(c+H^2) < (n-1)\lambda \leq (n-1)(c+H^2)$, if $\| \mathrm{Ric}_-^\lambda \|_{n/2} < \epsilon(n, c, \lambda, H)$, then M is a totally umbilical sphere $S^n(\frac{1}{\sqrt{c+H^2}})$. Here*

$$\epsilon(n, c, \lambda, H) = \frac{P_n}{1 + \frac{c_+ + H^2}{(n-1)\lambda - (n-2)(c+H^2)} P_n} \frac{1}{C^2(n)},$$

where $P_n = \frac{(n+2)(n-2)^2}{4n^2(n-1)^2}$.

In [14], Shu studied the rigidity of Willmore submanifolds in terms of Ricci curvatures and obtained the following theorem.

Theorem F ([14]). *Let M be an n-dimensional $(n \geq 5)$ compact Willmore submanifold in the unit sphere S^{n+p}. If the Ricci curvature* Ric*, H, and ρ of M satisfy*

$$\mathrm{Ric} \geq (n-2) + (n-2)H\rho + (n-1)H^2,$$

then either M is totally umbilic, or M is a Willmore torus $W_{m,m} = S^m(\sqrt{\frac{1}{2}}) \times S^m(\sqrt{\frac{1}{2}})$.

When $n = 2$, all minimal surfaces are Willmore surfaces (see [10, (1.3)]). But there are many compact non-minimal Willmore surface (see [1,2,6,11,13]). When $n \geq 3$, minimal submanifolds are not Willmore submanifolds in general, for example, Clifford minimal tori $S^m(\sqrt{\frac{m}{n}}) \times S^{n-m}(\sqrt{\frac{n-m}{n}})$ are not Willmore submanifolds when $n \neq 2m$. In [8], the authors proved that all n-dimensional minimal Einstein submanifolds in a sphere are Willmore submanifolds. Motivated by everything above, we shall prove the following global pinching theorems for compact Willmore and extremal submanifolds in S^{n+p}.

The PhD thesis [20] of the first author studied the rigidity of extremal submanifolds in terms of the Ricci curvature.

Theorem G ([20]). *Let M be an n-dimensional ($n \geq 4$) compact extremal submanifold in the unit sphere S^{n+p}. If the Ricci curvature of M, H, and ρ satisfy*

$$\mathrm{Ric} \geq (n-2) + (n-2)H\rho + (n-1)H^2,$$

then M is either totally umbilic, a Clifford torus $S^m(\sqrt{\frac{1}{2}}) \times S^m(\sqrt{\frac{1}{2}})$ in S^{n+1}, or $CP^2(\frac{3}{4})$ in S^7. Here $CP^2(\frac{3}{4})$ denotes the 2-dimensional complex projective space minimally immersed in S^7 with constant holomorphic sectional curvature.

Now we extend parts of Theorems F and G from a pointwise Ricci curvature lower bound to an integral Ricci curvature lower bound. Our main result in this paper is the following:

Theorem 1.3. *Let M be an n-dimensional ($n \geq 4$) Willmore submanifold in the unit sphere S^{n+p}. Given λ satisfying*

$$(n-2) + (n-2)H\rho + (n-1)H^2 < (n-1)\lambda \leq (n-1)(1+H_0^2) + (n-2)H_0\rho_0, \quad (1.4)$$

where $H_0 = \max_{x\in M} H$ and $\rho_0 = \max_{x \in M} \rho$, if

$$\| \mathrm{Ric}_-^\lambda \|_{\frac{n}{2}} < A(n, \lambda, H, \rho),$$

then M is a totally umbilical sphere. Here

$$A(\lambda, n, H, \rho) = \frac{1}{\frac{n^3(n-1)}{(n-2)^2} + \frac{(1+H_0^2)}{(n-1)(\lambda - H_0^2) - (n-2)(1+H_0\rho_0)}} \frac{1}{C^2(n)}.$$

Corollary 1.4. *Let M be a 4-dimensional compact Willmore submanifold in the unit sphere S^{4+p}. If*

$$\mathrm{Ric} > 2 + 2H\rho + 3H^2,$$

then M is totally umbilic.

Remark 1.5. It is easy to see that $\| \mathrm{Ric}_-^\lambda \|_{\frac{n}{2}} = 0$ if and only if $\mathrm{Ric} \geq (n-1)\lambda$. From (1.4), we know this means $\mathrm{Ric} > (n-2) + (n-2)H\rho + (n-1)H^2$. When $n \geq 5$, this generalizes Theorem F in the sense of strict inequality. Theorem 1.3 extends Theorem F to $n = 4$.

Theorem 1.6. *Let M be an n-dimensional ($n \geq 4$) extremal submanifold in the unit sphere S^{n+p}. If*

$$\| \mathrm{Ric}_-^\lambda \|_{\frac{n}{2}} < B(\lambda, n, H, \rho),$$

then M is a totally umbilical sphere, where

$$B(\lambda,n,H,\rho)=\frac{1}{\frac{4n(n-1)^2}{(n-2)^2}+\frac{1}{(n-1)(\lambda-H_0^2)-(n-2)(1+H_0\rho_0)}}\frac{1}{C^2(n)}.$$

Remark 1.7. It is easy to see that this generalizes Theorem G in the sense of strict inequality.

2. Preliminaries. In this section, we review some fundamental formulas for submanifolds. Let M be an n-dimensional compact submanifold in S^{n+p}. Thus the Gauss equations are as follows

$$R_{ijkl}=(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})+\sum_\alpha(h^\alpha_{ik}h^\alpha_{jl}-h^\alpha_{il}h^\alpha_{jk}). \tag{2.1}$$

Thus we have

$$R_{ij}=(n-1)\delta_{ij}+n\sum_\alpha H^\alpha h^\alpha_{ij}-\sum_{\alpha,k}h^\alpha_{ik}h^\alpha_{kj}, \tag{2.2}$$

$$R=n(n-1)+n^2H^2-S=n(n-1)(1+H^2)-\rho^2. \tag{2.3}$$

The Codazzi and Ricci equations are given by

$$h^\alpha_{ijk}=h^\alpha_{ikj}, \tag{2.4}$$

$$R_{\alpha\beta ij}=\sum_k h^\alpha_{ik}h^\beta_{kj}-\sum_k h^\beta_{ik}h^\alpha_{kj}. \tag{2.5}$$

The Ricci identity shows that

$$h^\alpha_{ijkl}-h^\alpha_{ijlk}=\sum_m h^\alpha_{mj}R_{mikl}+\sum_m h^\alpha_{im}R_{mjkl}+\sum_\beta h^\beta_{ij}R_{\beta\alpha kl}. \tag{2.6}$$

Denote by h^α_{ij} the components of the second fundamental form of M. Define the following tensors:

$$\tilde h^\alpha_{ij}=h^\alpha_{ij}-H^\alpha\delta_{ij},\ \sigma_{\alpha\beta}=\sum_{i,j}h^\alpha_{ij}h^\beta_{ij},\ \tilde\sigma_{\alpha\beta}=\sum_{i,j}\tilde h^\alpha_{ij}\tilde h^\beta_{ij}. \tag{2.7}$$

Then the $(p\times p)$-matrix $(\tilde\sigma_{\alpha\beta})$ is symmetric and can be assumed to be diagonalized for a suitable choice of $e_{n+1},\dots,e_{n+p}$. Set

$$\tilde\sigma_{\alpha\beta}=\tilde\sigma_\alpha\delta_{\alpha\beta}, \tag{2.8}$$

we have by a direct calculation

$$\sum_k\tilde h^\alpha_{kk}=0,\ \sigma_{\alpha\beta}=\tilde\sigma_{\alpha\beta}+nH^\alpha H^\beta,\ \rho^2=\sum_\alpha\tilde\sigma_\alpha=S-nH^2, \tag{2.9}$$

$$\sum_{i,j,k,\alpha}h^\beta_{kj}h^\alpha_{ij}h^\alpha_{ik}=\sum_{i,j,k,\alpha}\tilde h^\beta_{kj}\tilde h^\alpha_{ij}\tilde h^\alpha_{ik}+2\sum_{i,j,\alpha}H^\alpha\tilde h^\alpha_{ij}\tilde h^\beta_{ij}+H^\beta\rho^2+nH^2H^\beta, \tag{2.10}$$

and

$$\sum_{\alpha,\beta}\tilde\sigma^2_{\alpha\beta}=\sum_\alpha\tilde\sigma^2_\alpha\le\left(\sum_\alpha\tilde\sigma_\alpha\right)^2=\rho^4. \tag{2.11}$$

The above symbols and formulas are quoted from [10]. For the convenience of narration and the following proof, let us repeat it here.

From (2.2), we get

$$R_{ij} = (n-1)\delta_{ij} + (n-2)\sum_{\alpha} H^{\alpha}\tilde{h}^{\alpha}_{ij} + (n-1)H^2\delta_{ij} - \sum_{\alpha,k}\tilde{h}^{\alpha}_{ik}\tilde{h}^{\alpha}_{kj}. \quad (2.12)$$

Let R_m be the smallest eigenvalue of the Ricci tensor. By using the Cauchy–Schwarz inequality $\sum_{\alpha} H^{\alpha}\tilde{h}^{\alpha}_{ij} \le H\rho$ and (2.12), we have

$$\frac{\rho^2}{n} \le (n-1)(1+H^2) + (n-2)H\rho - R_m. \quad (2.13)$$

Given λ satisfying

$$(n-2) + (n-2)H\rho + (n-1)H^2 < (n-1)\lambda \le (n-1)(1+H_0^2) + (n-2)H_0\rho_0, \quad (2.14)$$

we can set

$$\Lambda := (n-1)\lambda = (n-2) + (n-2)H_0\rho_0 + (n-1)H_0^2 + \delta \quad (2.15)$$

for some $\delta > 0$. Put $\mathrm{Ric}_{-}^{\lambda} = \max\{0, (n-1)\lambda - R_m\}$. By definition,

$$(n-2) + (n-2)H\rho + (n-1)H^2 - R_m \le -\delta + \Lambda - R_m \le -\delta + \mathrm{Ric}_{-}^{\lambda}. \quad (2.16)$$

Lemma 2.1 ([5,17]). *Let M^n $(n \ge 3)$ be a closed submanifold in S^{n+p}. Then for all $t > 0$ and $f \in C^1(M)$, $f \ge 0$, we have*

$$\int_M |\nabla f|^2 dv \ge k_1 \left(\int_M f^{\frac{2n}{n-2}} dv\right)^{\frac{n-2}{n}} - k_2 \int_M (1+H^2) f^2 dv, \quad (2.17)$$

where

$$k_1 = \frac{(n-2)^2}{4(n-1)^2(1+t)}\frac{1}{C^2(n)},\ k_2 = \frac{(n-2)^2}{4(n-1)^2 t},\ C(n) = 2^n\frac{(n+1)^{1+1/n}}{(n-1)\omega_n^{1/n}},$$

and ω_n is the volume of the unit ball in R^n.

Lemma 2.2 ([19]). *Let M be an n-dimensional compact Riemannian submanifold in S^{n+p}. Then*

$$|\nabla\tilde{h}|^2 \ge |\nabla\rho_{\varepsilon}|^2, \quad (2.18)$$

where $|\nabla\tilde{h}|^2 = \sum_{\alpha,i,j,k}(\tilde{h}^{\alpha}_{ijk})^2$, $\rho_{\varepsilon} = \sqrt{\sum_{\alpha}\sum_{i,j}(\tilde{h}^{\alpha}_{ij} + \varepsilon\delta_{ij})^2} > 0$, $\varepsilon > 0$.

From (2.17) and (2.18), we have

$$\int_M |\nabla\tilde{h}|^2 dv \ge \int_M |\nabla\rho_{\varepsilon}|^2 dv \ge k_1\left(\int_M \rho_{\varepsilon}^{\frac{2n}{n-2}} dv\right)^{\frac{n-2}{n}} - k_2\int_M (1+H^2)\rho_{\varepsilon}^2 dv. \quad (2.19)$$

Letting $\varepsilon \to 0$ in (2.19), we obtain

$$\int_M |\nabla\tilde{h}|^2 dv \ge k_1\left(\int_M \rho^{\frac{2n}{n-2}} dv\right)^{\frac{n-2}{n}} - k_2\int_M (1+H^2)\rho^2 dv. \quad (2.20)$$

An argument similar to (2.20) shows that

$$\frac{n^2}{4}\int_M \rho^{n-2}|\nabla\tilde{h}|^2 dv \geq k_1\left(\int_M \rho^{\frac{n^2}{n-2}}\right)^{\frac{n-2}{n}} dv - k_2\int_M (1+H^2)\rho^n dv. \quad (2.21)$$

3. Proof of Theorem 1.3. By use of (2.4), (2.6), and the definition of Δ and ρ^2, we have

$$\begin{aligned}\frac{1}{2}\Delta\rho^2 &= \frac{1}{2}\Delta\left(\sum_{\alpha,i,j}(h_{ij}^{\alpha})^2\right) - \frac{1}{2}\Delta(nH^2)\\ &= \sum_{\alpha,i,j,k}(h_{ijk}^{\alpha})^2 + \sum_{\alpha,i,j,k} h_{ij}^{\alpha}h_{kijk}^{\alpha} - \frac{1}{2}\Delta(nH^2)\\ &= |\nabla h|^2 - n^2|\nabla^{\perp}\mathbf{H}|^2 + \sum_{\alpha,i,j,k}(h_{ij}^{\alpha}h_{kki}^{\alpha})_j + \sum_{\alpha,i,j,k,m} h_{ij}^{\alpha}h_{mk}^{\alpha}R_{mijk}\\ &\quad + \sum_{\alpha,i,j,m} h_{ij}^{\alpha}h_{im}^{\alpha}R_{mj} + \sum_{\alpha,\beta,i,j,k} h_{ij}^{\alpha}h_{ik}^{\beta}R_{\beta\alpha jk} - \frac{1}{2}\Delta(nH^2). \end{aligned} \quad (3.1)$$

From (2.1), (2.2), and (2.7), we have

$$\begin{aligned}&\sum_{\alpha,i,j,k,m} h_{ij}^{\alpha}h_{mk}^{\alpha}R_{mijk} + \sum_{\alpha,i,j,m} h_{ij}^{\alpha}h_{im}^{\alpha}R_{mj}\\ &\quad = \sum_{\alpha,i,j,m} h_{ij}^{\alpha}\left(\sum_k h_{mk}^{\alpha}R_{mijk} + h_{im}^{\alpha}R_{mj}\right)\\ &\quad = n\rho^2 - \sum_{\alpha,\beta}\tilde{\sigma}_{\alpha\beta}^2 + nH^2\rho^2 + n\sum_{\alpha,\beta,i,j,k} H^{\beta}\tilde{h}_{kj}^{\beta}\tilde{h}_{ij}^{\alpha}\tilde{h}_{ik}^{\alpha}\\ &\quad\quad -\frac{1}{2}\sum_{\alpha,\beta} N(\tilde{A}_{\alpha}\tilde{A}_{\beta} - \tilde{A}_{\beta}\tilde{A}_{\alpha}), \end{aligned} \quad (3.2)$$

where $N(\tilde{A}_{\alpha}\tilde{A}_{\beta} - \tilde{A}_{\beta}\tilde{A}_{\alpha}) = \mathrm{tr}[(\tilde{A}_{\alpha}\tilde{A}_{\beta} - \tilde{A}_{\beta}\tilde{A}_{\alpha})^2]$, $\tilde{A}_{\alpha} = (\tilde{h}_{ij}^{\alpha}) = (h_{ij}^{\alpha} - H^{\alpha}\delta_{ij})$. By use of (2.5) and (2.7), we get

$$\sum_{\alpha,i,j,k,} h_{ij}^{\alpha}h_{ki}^{\beta}R_{\beta\alpha jk} = -\frac{1}{2}\sum_{\alpha,\beta} N(\tilde{A}_{\alpha}\tilde{A}_{\beta} - \tilde{A}_{\beta}\tilde{A}_{\alpha}). \quad (3.3)$$

Putting (3.2) and (3.3) into (3.1), we obtain

$$\begin{aligned}\frac{1}{2}\Delta\rho^2 &= |\nabla h|^2 - n^2|\nabla^{\perp}\mathbf{H}|^2 + \sum_{\alpha,i,j,k}(h_{ij}^{\alpha}h_{kki}^{\alpha})_j\\ &\quad + n\rho^2 - \sum_{\alpha,\beta}\tilde{\sigma}_{\alpha\beta}^2 + nH^2\rho^2 + n\sum_{\alpha,\beta,i,j,k} H^{\beta}\tilde{h}_{kj}^{\beta}\tilde{h}_{ij}^{\alpha}\tilde{h}_{ik}^{\alpha}\\ &\quad - \sum_{\alpha,\beta} N(\tilde{A}_{\alpha}\tilde{A}_{\beta} - \tilde{A}_{\beta}\tilde{A}_{\alpha}) - \frac{1}{2}\Delta(nH^2). \end{aligned} \quad (3.4)$$

Hence

$$\frac{1}{2}\rho^{n-2}\Delta\rho^2 = \rho^{n-2}(|\nabla h|^2 - n|\nabla^{\perp}\mathbf{H}|^2) + \rho^{n-2}\sum_{\alpha,i,j,k}(h^{\alpha}_{ij}h^{\alpha}_{kki})_j - \frac{1}{2}\rho^{n-2}\Delta(nH^2)$$
$$-n(n-1)\rho^{n-2}|\nabla^{\perp}\mathbf{H}|^2 + n\rho^{n-2}\sum_{\alpha,\beta,i,j,k}H^{\beta}\tilde{h}^{\beta}_{kj}\tilde{h}^{\alpha}_{ij}\tilde{h}^{\alpha}_{ik}$$
$$+\rho^{n-2}\left[n\rho^2 + nH^2\rho^2 - \sum_{\alpha,\beta}N(\tilde{A}_{\alpha}\tilde{A}_{\beta} - \tilde{A}_{\beta}\tilde{A}_{\alpha}) - \sum_{\alpha,\beta}\tilde{\sigma}^2_{\alpha\beta}\right]. \tag{3.5}$$

On the other hand, Li [10] has given a characterization of Willmore submanifolds in the following Lemma 3.1.

Lemma 3.1 ([10, Lemma 4.3]). *Let M be an n-dimensional submanifold in the unit sphere S^{n+p}. Then M is a Willmore submanifold if and only if for $n+1 \le \alpha \le n+p$,*

$$(n-1)\rho^{n-2}\Delta^{\perp}H^{\alpha}$$
$$= -2(n-1)\sum_i(\rho^{n-2})_i H^{\alpha}_{,i} - (n-1)H^{\alpha}\Delta(\rho^{n-2})$$
$$-\rho^{n-2}\left(\sum_{\beta}H^{\beta}\tilde{\sigma}_{\alpha\beta} + \sum_{\beta,i,j,k}\tilde{h}^{\alpha}_{ij}\tilde{h}^{\beta}_{ik}\tilde{h}^{\beta}_{kj}\right) + \sum_{i,j}(\rho^{n-2})_{i,j}(nH^{\alpha}\delta_{ij} - h^{\alpha}_{ij}),$$

where $\Delta(\rho^{n-2}) = \sum_i(\rho^{n-2})_{i,i}$, $\Delta^{\perp}H^{\alpha} = \sum_i H^{\alpha}_{,ii}$, and $(\rho^{n-2})_{i,j}$ is the Hessian of ρ^{n-2} with respect to the induced met Ric, *$H^{\alpha}_{,i}$ and $H^{\alpha}_{,ij}$ are the components of the first and second covariant derivative of the mean curvature vector field* $\mathbf{H}$.

Using Stokes' formula and Lemma 3.1, we have (see [10])

$$-n(n-1)\int_M \rho^{n-2}|\nabla^{\perp}\mathbf{H}|^2 dv + n\int_M \rho^{n-2}\left(\sum_{\alpha,\beta,i,j,m}H^{\beta}\tilde{h}^{\beta}_{mj}\tilde{h}^{\alpha}_{ji}\tilde{h}^{\alpha}_{im}\right)dv$$
$$= -\frac{1}{2}n(n+1)\int_M\sum_i(\rho^{n-2})_i(H^2)_i dv - n\int_M \rho^{n-2}\sum_{\alpha,\beta}H^{\alpha}H^{\beta}\tilde{\sigma}_{\alpha\beta}dv$$
$$-n\int_M\sum_{\alpha,i,j}H^{\alpha}h^{\alpha}_{ij}(\rho^{n-2})_{i,j}dv. \tag{3.6}$$

By a direct computation, we get (see [10])

$$\int_M \rho^{n-2}\sum_{\alpha,i,j,k}(h^{\alpha}_{ij}h^{\alpha}_{kki})_j dv$$
$$= n\int_M\left(\sum_{\alpha,i,j}H^{\alpha}h^{\alpha}_{ij}(\rho^{n-2})_{i,j}\right)dv + \frac{n^2}{2}\int_M\sum_i(\rho^{n-2})_i(H^2)_i dv, \tag{3.7}$$

and

$$-\frac{1}{2}\int\limits_M \rho^{n-2}\Delta(nH^2)dv = \frac{n}{2}\int\limits_M \sum_i (\rho^{n-2})_i (H^2)_i dv. \tag{3.8}$$

By use of (3.6), (3.7), and (3.8), integrating (3.5) over M, we have

$$\begin{aligned}\frac{1}{2}\int\limits_M \rho^{n-2}\Delta\rho^2 dv = &\int\limits_M \rho^{n-2}(|\nabla h|^2 - n|\nabla^\perp \mathbf{H}|^2)dv \\ &+n\int\limits_M \rho^{n-2}\left(H^2\rho^2 - \sum_{\alpha,\beta} H^\alpha H^\beta \tilde{\sigma}_{\alpha\beta}\right)dv + n\int\limits_M \rho^n dv \\ &-\int\limits_M \rho^{n-2}\sum_{\alpha,\beta}(N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) + \tilde{\sigma}_{\alpha\beta}^2)dv. \end{aligned}\tag{3.9}$$

We have by a direct calculation

$$H^2\rho^2 - \sum_{\alpha,\beta} H^\alpha H^\beta \tilde{\sigma}_{\alpha\beta} \geq 0, \tag{3.10}$$

$$|\nabla h|^2 - n|\nabla^\perp \mathbf{H}|^2 = |\nabla \tilde{h}|^2, \tag{3.11}$$

where $|\nabla^\perp \mathbf{H}|^2 = \sum_{\alpha,i}(H^\alpha_{,i})^2$, $|\nabla\tilde{h}|^2 = \sum_{\alpha,i,j,k}(\tilde{h}^\alpha_{ijk})^2 = \sum_{\alpha,i,j,k}(h^\alpha_{ijk} - H^\alpha_{,k}\delta_{ij})^2$. Thus

$$\begin{aligned}\frac{1}{2}\int\limits_M \rho^{n-2}\Delta\rho^2 dv \geq &\int \rho^{n-2}|\nabla\tilde{h}|^2 dv \\ &+\int\limits_M \rho^{n-2}\sum_{\alpha,\beta}[n\rho^2 - N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) - \tilde{\sigma}_{\alpha\beta}^2]dv.\end{aligned} \tag{3.12}$$

It is easy to see that

$$\frac{1}{2}\int\limits_M \rho^{n-2}\Delta\rho^2 dv = -\frac{4(n-2)}{n^2}\int\limits_M |\nabla \rho^{\frac{n}{2}}|^2 dv. \tag{3.13}$$

Putting (3.13) into (3.12) yields

$$\begin{aligned}0 \geq &\frac{4(n-2)}{n^2}\int\limits_M |\nabla\rho^{\frac{n}{2}}|^2 dv + \int \rho^{n-2}|\nabla\tilde{h}|^2 dv \\ &+\int\limits_M \rho^{n-2}\sum_{\alpha,\beta}[n\rho^2 - N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) - \tilde{\sigma}_{\alpha\beta}^2]dv.\end{aligned} \tag{3.14}$$

Set $Q = n\rho^2 - \sum_{\alpha,\beta}(N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) + \tilde{\sigma}^2_{\alpha\beta})$. Now we estimate the lower bound of Q based on ρ^2 and Ric^λ_-. For a fixed α, we choose $\{e_i\}$ such that A_α is diagonalized, $A_\alpha = \mathrm{diag}\{\lambda^\alpha_1, \ldots, \lambda^\alpha_n\}$, then (2.12) gives

$$\sum_j \sum_{\beta\neq\alpha}(\tilde{h}^\beta_{ij})^2 = (n-1)(1+H^2) + (n-2)\sum_\gamma H^\gamma \tilde{h}^\gamma_{ii} - (\tilde{\lambda}^\alpha_i)^2 - R_{ii}, \tag{3.15}$$

and

$$\sum_{\beta} N(\tilde{A}_\alpha \tilde{A}_\beta - \tilde{A}_\beta \tilde{A}_\alpha) = \sum_{\beta \neq \alpha} N(\tilde{A}_\alpha \tilde{A}_\beta - \tilde{A}_\beta \tilde{A}_\alpha)$$
$$= \sum_{\beta \neq \alpha} \sum_{ij} (\tilde{h}_{ij}^\beta)^2 (\tilde{\lambda}_i^\alpha - \tilde{\lambda}_j^\alpha)^2 \leq 4 \sum_{\beta \neq \alpha} \sum_{ij} (\tilde{h}_{ij}^\beta)^2 (\tilde{\lambda}_i^\alpha)^2$$
$$\leq 4 \sum_i [(n-1)(1+H^2) + (n-2) \sum_\gamma H^\gamma \tilde{h}_{ii}^\gamma - (\tilde{\lambda}_i^\alpha)^2 - R_m] (\tilde{\lambda}_i^\alpha)^2$$
$$\leq 4[(n-1)(1+H^2) + (n-2)H\rho - R_m] N(\tilde{A}_\alpha) - 4N(\tilde{A}_\alpha^2). \tag{3.16}$$

Summing over α, we get

$$\sum_{\alpha,\beta} N(\tilde{A}_\alpha \tilde{A}_\beta - \tilde{A}_\beta \tilde{A}_\alpha)$$
$$\leq 4[(n-1)(1+H^2) + (n-2)H\rho - R_m]\rho^2 - 4 \sum_\alpha N(\tilde{A}_\alpha^2). \tag{3.17}$$

Obviously

$$\sum_\alpha N(\tilde{A}_\alpha^2) \geq \frac{1}{n} \sum_\alpha (N(\tilde{A}_\alpha))^2 = \frac{1}{n} \sum_\alpha \tilde{\sigma}_\alpha^2. \tag{3.18}$$

Substituting (3.18) into (3.17), we have

$$\sum_{\alpha,\beta} N(\tilde{A}_\alpha \tilde{A}_\beta - \tilde{A}_\beta \tilde{A}_\alpha) \leq 4[(n-1)(1+H^2) + (n-2)H\rho - R_m]\rho^2 - \frac{4}{n} \sum_\alpha \tilde{\sigma}_\alpha^2. \tag{3.19}$$

Combining (2.11) with (3.19), we obtain

$$Q \geq n\rho^2 - 4[(n-1)(1+H^2) + (n-2)H\rho - R_m]\rho^2 + \frac{4-n}{n}\rho^4$$
$$= (4-n)(\frac{\rho^2}{n} - 1)\rho^2 - 4[(n-2) + (n-1)H^2 + (n-2)H\rho - R_m]\rho^2. \tag{3.20}$$

From (2.13), (2.16), and (3.20), by using $n \geq 4$, we get

$$Q \geq -n[(n-2) + (n-1)H^2 + (n-2)H\rho - R_m]\rho^2$$
$$\geq -n[-\delta + \Lambda - R_m]\rho^2, \tag{3.21}$$

i.e.,

$$Q \geq -n(-\delta + \mathrm{Ric}_-^\lambda)\rho^2. \tag{3.22}$$

By Lemma 2.1, (2.21), (3.22), and (3.14), we get

$$0 \geq \frac{4(n-1)}{n^2} \left[k_1 \left(\int_M \rho^{\frac{n^2}{n-2}} dv \right)^{\frac{n-2}{n}} - k_2 \int_M (1+H^2)\rho^n dv \right]$$
$$+ n\delta \int_M \rho^n dv - n \int_M (\mathrm{Ric}_-^\lambda)\rho^n dv. \tag{3.23}$$

Applying the Hölder inequality, we obtain

$$0 \geq \frac{4(n-1)}{n^2}[k_1\|\rho^n\|_{\frac{n}{n-2}} - k_2(1+H_0^2)\int_M \rho^n dv] + n\delta\int_M \rho^n dv - n\| \operatorname{Ric}_-^\lambda \|_{\frac{n}{2}}||\rho^n\|_{\frac{n}{n-2}}, \tag{3.24}$$

i.e.,

$$0 \geq \frac{4(n-1)}{n^2}k_1\|\rho^n\|_{\frac{n}{n-2}} - n\| \operatorname{Ric}_-^\lambda \|_{\frac{n}{2}}\|\rho^n\|_{\frac{n}{n-2}} + \left[n\delta - \frac{4(n-1)}{n^2}k_2(1+H_0^2)\right]\int_M \rho^n dv. \tag{3.25}$$

By taking

$$t = \frac{(n-2)^2(1+H_0^2)}{\delta n^3(n-1)}$$

such that $[n\delta - \frac{4(n-1)}{n^2}k_2(1+H_0^2)] = 0$, we have

$$0 \geq \left\{\frac{4(n-1)}{n^2}k_1 - n\| \operatorname{Ric}_-^\lambda \|_{\frac{n}{2}}\right\}\|\rho^n\|_{\frac{n}{n-2}}.$$

Therefore, under the assumption $\| \operatorname{Ric}_-^\lambda \|_{\frac{n}{2}} < \frac{4(n-1)}{n^3}k_1$, it is easy to see from the above that $\rho^2 = 0$ and M is totally umbilical.

4. Proof of Theorem 1.6. First of all, Guo and Li calculated the Euler–Lagrange equation of $F(x)$ given by (1.3).

Theorem 4.1 ([7]). *Let M be an n-dimensional submanifold in the unit sphere S^{n+p}. Then M is an extremal submanifold if and only if it satisfies for $n+1 \leq \alpha \leq n+p$,*

$$\sum_{\beta,i,j,k} \tilde{h}_{ij}^\alpha \tilde{h}_{ik}^\beta \tilde{h}_{kj}^\beta = -(n-1)\Delta^\perp H^\alpha - \sum_\beta H^\beta \tilde{\sigma}_{\alpha\beta} - H^\alpha\rho^2 + \frac{n}{2}H^\alpha\rho^2. \tag{4.1}$$

From (4.1), we have

$$n\sum_{\alpha,\beta,i,j,k} H^\beta \tilde{h}_{kj}^\beta \tilde{h}_{ij}^\alpha \tilde{h}_{ik}^\alpha = -n(n-1)\sum_\beta H^\beta\Delta^\perp H^\beta - n\sum_{\alpha,\beta} H^\beta H^\alpha \tilde{\sigma}_{\alpha\beta} + \frac{n(n-2)}{2}H^2\rho^2. \tag{4.2}$$

Putting (4.2) into (3.4) yields

$$\frac{1}{2}\Delta\rho^2 = |\nabla h|^2 - n^2|\nabla^\perp \mathbf{H}|^2 + \sum_{\alpha,i,j,k}(h_{ij}^\alpha h_{kki}^\alpha)_j + n\rho^2 - \sum_{\alpha,\beta}\tilde{\sigma}_{\alpha\beta}^2 + nH^2\rho^2 - n(n-1)\sum_\beta H^\beta\Delta^\perp H^\beta - n\sum_{\alpha,\beta} H^\beta H^\alpha\tilde{\sigma}_{\alpha\beta} + \frac{n(n-2)}{2}H^2\rho^2 - \sum_{\alpha,\beta} N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) - \frac{1}{2}\Delta(nH^2). \tag{4.3}$$

Integrating (4.3) over M and using Stokes' formula, we have

$$\frac{1}{2}\int_M \Delta\rho^2 dv = \int_M (|\nabla h|^2 - n|\nabla^{\perp}\mathbf{H}|^2)dv + \int_M \left(n\rho^2 - \sum_{\alpha,\beta}\tilde{\sigma}^2_{\alpha\beta}\right)dv$$
$$+\int_M n\left(H^2\rho^2 - \sum_{\alpha,\beta} H^{\beta}H^{\alpha}\tilde{\sigma}_{\alpha\beta}\right)dv + \int_M \frac{n(n-2)}{2}H^2\rho^2 dv$$
$$-\int_M \sum_{\alpha,\beta} N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha)dv. \tag{4.4}$$

By (3.10) and (3.11),

$$0 \geq \int_M |\nabla\tilde{h}|^2 dv + \frac{n(n-2)}{2}\int_M H^2\rho^2 dv$$
$$+n\int_M \rho^2 dv - \int_M \sum_{\alpha,\beta}(N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) + \tilde{\sigma}^2_{\alpha\beta})dv. \tag{4.5}$$

Substituting (3.22) into (4.5), by the definition of Q, we have

$$0 \geq \int_M |\nabla\tilde{h}|^2 dv + \frac{n(n-2)}{2}\int_M H^2\rho^2 dv$$
$$-\int_M n(-\delta + \mathrm{Ric}^{\lambda}_{-})\rho^2 dv. \tag{4.6}$$

From (2.20), we get

$$0 \geq k_1\left(\int_M \rho^{\frac{2n}{n-2}}dv\right)^{\frac{n-2}{n}} - k_2\int_M (1+H^2)\rho^2 dv,$$
$$+\frac{n(n-2)}{2}\int_M H^2\rho^2 dv - n\int_M(-\delta + \mathrm{Ric}^{\lambda}_{-})\rho^2 dv, \tag{4.7}$$

i.e.,

$$0 \geq k_1\left(\int_M \rho^{\frac{2n}{n-2}}dv\right)^{\frac{n-2}{n}} - n\int_M (\mathrm{Ric}^{\lambda}_{-})\rho^2 dv$$
$$-\int_M (k_2 - n\delta)\rho^2 dv + \int_M \left[\frac{n(n-2)}{2} - k_2\right]H^2\rho^2 dv. \tag{4.8}$$

We choose $t = \frac{(n-2)^2}{4\delta n(n-1)^2}$ such that $k_2 = n\delta$. By using Hölder's inequality, from the above, we obtain

$$0 \geq k_1\|\rho^2\|_{\frac{n}{n-2}} - n\|\,\mathrm{Ric}^{\lambda}_{-}\,\|_{\frac{n}{2}}\|\rho^2\|_{\frac{n}{n-2}}. \tag{4.9}$$

Therefore, under the assumption

$$\| \operatorname{Ric}_{-}^{\lambda} \|_{\frac{n}{2}} < \frac{k_1}{n} = \frac{(n-2)^2\delta}{4n(n-1)^2\delta + (n-2)^2} \frac{1}{C^2(n)},$$

it is easy to see from (4.9) that $\rho^2 = 0$ and M is totally umbilical.

Acknowledgements. The authors thank Professor Chen Hang for his careful reading and helpful suggestions.

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Received: 16 January 2023

Accepted: 6 July 2023