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On the isospectral problem of the Camassa–Holm equation

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Abstract. In this paper, we study an isospectral problem of a weighted Sturm–Liouville equation with the Dirichlet boundary condition, which lies at the basis of the integrability of the Camassa–Holm equation. We will choose the general setting of the so-called measure differential equations to solve the optimization problem on eigenvalues. It should be noticed that our technique in this paper can be used to deal with other self-adjoint boundary conditions.

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Keywords. Eigenvalue, Optimization problem, Camassa–Holm equations, Isospectral problem.

1. Introduction. This paper is concerned with the spectral problem

$$y'' = \frac{1}{4}y + \lambda m(t)y, \tag{1.1}$$

with the Dirichlet boundary condition

$$y(0) = y(1) = 0, \tag{1.2}$$

where $m \in \mathcal{C}^- := \{f \in \mathcal{C}[0,1] : f(t) \le 0,\ f(t) \not\equiv 0\}$. It is well known that problem (1.1)–(1.2) has a sequence of simple eigenvalues $0 < \lambda_1(m) < \lambda_2(m) < \cdots < \lambda_k(m) \to +\infty$. See [6].

The weighted Sturm–Liouville problem (1.1) lies at the basis of the integrability of a celebrated recent model for shallow water waves, which is the following Camassa–Holm equation:

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{1.3}$$

where u is the fluid velocity in the x-direction. See [2,3]. To study the integrability of the Camassa–Holm equation (1.3), a key point is to understand the associated spectral problem (1.1) with a certain boundary condition, where $m = u - u_{xx}$ is the weight function. During the last two decades, there are many important and interesting results obtained for the Camassa–Holm equation. See [1,2,5,7,8,14,16,17] and the references therein. For results on the inverse spectral problems and the corresponding isospectral problems of the Camassa–Holm equation, we can refer to [9,11–13].

The purpose of this paper is to investigate an isospectral problem as follows:

$$I(h) := \inf\left\{-\int_0^1 m(t)\,\mathrm{d}t : m \in E_h\right\}, \tag{1.4}$$

where $h \in (0, +\infty)$ is fixed and

$$E_h := \{m \in \mathcal{C}^- : \lambda_1(m) = h\}. \tag{1.5}$$

In order to solve the minimization problem (1.4), we will choose the general setting of the so-called measure differential equations (MDEs) to understand the eigenvalues, eigenfunctions, and the minimization. Based on the relationship between ODEs and MDEs, we will obtain the conclusion about the minimization problem (1.4) as follows.

Theorem 1.1. *It holds that*

$$I(h) = \frac{\coth\frac{1}{4}}{h} \qquad \forall h \in (0 + \infty). \tag{1.6}$$

Moreover, the infimum $I(h)$ is never attained for any function of E_h.

It should be remarked that our technique in this paper can be used to deal with other self-adjoint boundary conditions, such as the Neumann boundary condition.

This paper is organized as follows. In Section 2, we will recall some basic facts on measure differential equations, which are used in the proof of the main results. In Section 3, based on the relationship between ODEs and MDEs, we will prove Theorem 1.1.

2. Preliminaries. In order to solve the minimization problem (1.4), we will extend in this section the lowest eigenvalues $\lambda_1(m)$ of (1.1) to the case of MDEs.

Let $\mathcal{C} = \mathcal{C}[0,1]$ be the space of continuous functions on $[0,1]$. Hence, the dual space of $\mathcal{C}$ is the space of Radon measures on $[0,1]$:

$$\mathcal{M}_0 = \mathcal{M}_0([0,1],\mathbb{R}) := (\mathcal{C}, \|\cdot\|_\infty)^*,$$

where measures $\nu \in \mathcal{M}_0$ are normalized as $\nu(0+) = 0$. See [4] and also [19] for more details on measures.

For a fixed measure

$$\mu \in \mathcal{M}_0^- := \{\nu \in \mathcal{M}_0 : \nu \neq 0 \text{ and } \nu(t) \text{ is decreasing on } [0,1]\},$$

we study the following second order measure differential equation

$$\mathrm{d}y^\bullet = \frac{1}{4} y\,\mathrm{d}t + \lambda y\,\mathrm{d}\mu(t), \tag{2.1}$$

with the Dirichlet boundary condition (1.2). Here we are adopting the notation from [20]. It should be remarked that the measure differential equation (2.1) reduces to the ordinary differential equation (1.1) when the measure $\mu \in \mathcal{M}_0^-$ is continuously differentiable with the derivative $m(t) = \mu'(t) \in \mathcal{C}^-$.

As for the lowest eigenvalue for the equation (2.1) with the boundary condition (1.2), we can establish the following minimization characterization by similar arguments as those in [19].

Lemma 2.1. *Given $\mu \in \mathcal{M}_0^-$, we have the following minimization characterization of the lowest eigenvalue $\lambda_1(\mu)$ for the problems (2.1) and (1.2):*

$$\lambda_1(\mu) = \min_{u \in \mathcal{W}_0^{1,2} \setminus \{0\}} \frac{\int_{[0,1]} u'^2 \, \mathrm{d}t + \frac{1}{4} \int_{[0,1]} u^2 \, \mathrm{d}t}{-\int_{[0,1]} u^2 \, \mathrm{d}\mu(t)}, \tag{2.2}$$

where

$$\mathcal{W}_0^{1,2} := \{z \in W^{1,2}[0,1] : z(0) = z(1) = 0\}.$$

Notice that $\mathcal{M}_0$ is a Banach space equipped with the norm

$$\|\mu\|_{\mathbf{V}} = \sup \left\{ \sum_{i=0}^{k-1} |\mu(t_{i+1}) - \mu(t_i)| : 0 = t_0 < t_1 < \cdots < t_k = 1, \ k \in \mathbb{N} \right\},$$

the total variation of $\mu(t)$ on $[0,1]$. Meanwhile, as a dual space, $\mathcal{M}_0$ can be equipped with the weak* topology w^*. We say that μ_n is weakly* convergent to μ_0 if and only if it holds that

$$\lim_{n \to \infty} \int_{[0,1]} f \, \mathrm{d}\mu_n = \int_{[0,1]} f \, \mathrm{d}\mu_0$$

for any $f \in \mathcal{C}$. Generally, a measure can not be a limit of smooth measures in the norm $\|\cdot\|_{\mathbf{V}}$. However, we have the following fact about the weak* topology.

Lemma 2.2 ([18]). *For fixed $\mu_0 \in \mathcal{M}_0^-$, there is a sequence of smooth measures $\{\mu_n\} \subset \mathcal{C}^\infty \cap \mathcal{M}_0^-$ satisfying*

$$\mu_n \to \mu_0 \ \text{in} \ (\mathcal{M}_0^-, w^*)$$

and $\|\mu_n\|_{\mathbf{V}} = \|\mu_0\|_{\mathbf{V}}$.

Based on the continuous dependence of solutions of initial value problems of measure differential equations in measures [20] and the characterization (2.2), we can prove the continuity of the eigenvalue $\lambda_1(\mu)$ in measures $\mu \in \mathcal{M}_0^-$ with respect to w^* topologies.

Lemma 2.3. *The following nonlinear functional is continuous:*

$$(\mathcal{M}_0^-, w^*) \to \mathbb{R}, \qquad \mu \to \lambda_1(\mu).$$

Meanwhile, the normalized eigenfunctions $y_1(t,\mu)$, corresponding to $\lambda_1(\mu)$ with

$$y_1(t,\mu) \geq 0 \ \text{and} \ ||y_1(\cdot,\mu)||_2 = 1,$$

are continuous in μ with respect to w^* topologies, i.e.,

$$(\mathcal{M}_0^-, w^*) \to (\mathcal{C}, \|\cdot\|_\infty), \qquad \mu \to y_1(\cdot, \mu),$$

and

$$(\mathcal{M}_0^-, w^*) \to (\mathcal{W}^{1,2}, w), \qquad \mu \to y_1(\cdot, \mu).$$

3. Main results. At first, we will extend problem (1.4) to the measure case. More precisely, let

$$\tilde{E}_h := \{\mu \in \mathcal{M}_0^- : \lambda_1(\mu) = h\}. \tag{3.1}$$

We will study the following minimization problem on measure differential equations

$$\tilde{I}(h) := \inf \left\{ -\int_{[0,1]} d\mu(t) : \mu \in \tilde{E}_h \right\}. \tag{3.2}$$

Using the continuity in Lemma 2.3, we will obtain that the minimal value $\tilde{I}(h)$ defined in (3.2) can be attained by some measure in $\tilde{E}_h$.

Lemma 3.1. *Given $h \in (0, +\infty)$, there exists some measure $\mu_h \in \tilde{E}_h$ such that*

$$-\int_{[0,1]} d\mu_h(t) = \tilde{I}(h).$$

Proof. Since $\tilde{I}(h) > 0$, one can take a minimizing sequence $\{\mu_n\}_{n=1}^{+\infty} \subset \tilde{E}_h$ such that

$$-\int_{[0,1]} d\mu_n(t) \to \tilde{I}(h)$$

as $n \to +\infty$. Since $\mu_n \in \mathcal{M}_0^-$, then $\|\mu_n\|_{\mathbf{V}} = -\int_{[0,1]} d\mu_n(t) \le C$ for all $n \ge 1$ for some constant $C > 0$. According to the Banach-Alaoglu theorem, there is a subsequence $\{\mu_{n_k}\}_{k=1}^{+\infty} \subset \{\mu_n\}_{n=1}^{+\infty}$ such that $\mu_{n_k} \to \mu_h$ in $(\mathcal{M}_0, w^*)$ for some $\mu_h \in \mathcal{M}_0$. Then, we have

$$-\int_{[0,1]} d\mu_h(t) = -\lim_{k \to +\infty} \int_{[0,1]} d\mu_{n_k}(t) = \tilde{I}(h).$$

Furthermore, it holds that $\mu_h \in \mathcal{M}_0^-$ and $\lambda_1(\mu_h) = \lim_{k \to +\infty} \lambda_1(\mu_{n_k}) = h$ by Lemma 2.3, which implies that $\mu_h \in \tilde{E}_h$. □

In order to describe the minimizing measures, we denote the (unit) Dirac measure δ_a located at $a \in (0, 1)$ as follows:

$$\delta_a(t) = \begin{cases} 0 & \text{for } t \in [0, a), \\ 1 & \text{for } t \in [a, 1]. \end{cases}$$

Notice that $-r\delta_a \in \mathcal{M}_0^-$ for $r > 0$. To find the solution of problem (3.2), we solve the following measure differential equation

$$\mathrm{d}y^\bullet = \frac{1}{4} y \,\mathrm{d}t + \lambda y \,\mathrm{d}(-r\delta_a(t)), \qquad t \in [0, 1], \tag{3.3}$$

with the initial condition

$$(y(0), y^\bullet(0)) = (y_0, z_0) \in \mathbb{R}^2. \tag{3.4}$$

Denote

$$\omega = \frac{1}{2} i \in \mathbb{C}.$$

The solution $(y(t), y^\bullet(t))$ of equation (3.3) is

$$\begin{pmatrix} y(t) \\ y^\bullet(t) \end{pmatrix} = \begin{cases} \begin{pmatrix} y_0 \cos\omega t + z_0 \frac{\sin\omega t}{\omega} \\ -\omega y_0 \sin\omega t + z_0 \cos\omega t \end{pmatrix} & \text{for } t \in [0, a), \\ \begin{pmatrix} \hat{y}_0 \cos\omega(t-a) + \hat{z}_0 \frac{\sin\omega(t-a)}{\omega} \\ -\omega\hat{y}_0 \sin\omega(t-a) + \hat{z}_0 \cos\omega(t-a) \end{pmatrix} & \text{for } t \in [a, 1], \end{cases}$$

with

$$(\hat{y}_0, \hat{z}_0) = \left(y_0 \cos\omega a + z_0 \frac{\sin\omega a}{\omega}, \ -\omega y_0 \sin\omega a + z_0 \cos\omega a - \lambda r \hat{y}_0 \right).$$

To obtain eigenvalues of problems (2.1)–(1.2), we only need to study the solution of the equaiton (2.1) with the initial value $(y_0, z_0) = (0, 1)$. In this case,

$$(\hat{y}_0, \hat{z}_0) = \left(\frac{\sin\omega a}{\omega}, \ \cos\omega a - \lambda r \frac{\sin\omega a}{\omega} \right) \qquad \text{for } a \in (0, 1).$$

Denote

$$\Theta(a, \lambda) := y(1).$$

By the above formulas, we get that

$$\begin{aligned} \Theta(a, \lambda) &= \frac{\sin\omega a}{\omega} \cos\omega(1-a) + \left(\cos\omega a - \lambda r \frac{\sin\omega a}{\omega} \right) \frac{\sin\omega(1-a)}{\omega} \\ &= \frac{\sin\omega}{\omega} - \lambda r \frac{\sin\omega a}{\omega} \frac{\sin\omega(1-a)}{\omega}. \end{aligned}$$

Now $\lambda \in \mathbb{C}$ is an eigenvalue of (2.1)–(1.2) if and only if λ satisfies

$$\Theta(a, \lambda) = 0,$$

which implies that

$$\lambda_1(-r\delta_a) = \frac{\omega \sin\omega}{r \sin\omega a \sin\omega(1-a)}. \tag{3.5}$$

Especially, when $a = 1/2$, we have that

$$\lambda_1(-r\delta_{1/2}) = \frac{\omega \sin\omega}{r \sin^2 \frac{\omega}{2}} = \frac{\coth \frac{1}{4}}{r}.$$

Moreover, we have the following properties of the function $\lambda_1(-r\delta_a)$ by considering $a \in (0, 1)$ as a variable.

Lemma 3.2. *It holds that*

$$\inf_{a \in (0,1)} \lambda_1(-r\delta_a) = \lambda_1(-r\delta_{1/2}) = \frac{\coth \frac{1}{4}}{r}.$$

Proof. By (3.5), it is easy to check that $\lambda_1(-r\delta_a)$ is strictly decreasing in $a \in (0, 1/2]$ and $\lambda_1(-r\delta_{1-a}) = \lambda_1(-r\delta_a)$ for all $a \in (0, 1)$. □

Lemma 3.3. *For fixed $\mu \in \mathcal{M}_0^-$ with $-\int_{[0,1]} d\mu(t) = r$, there exists $a \in (0, 1)$ such that*

$$\lambda_1(\mu) \geq \lambda_1(-r\delta_a).$$

Proof. Assuming $\mu \in \mathcal{M}_0^-$ with $-\int_{[0,1]} d\mu(t) = r$, we have $\|\mu\|_{\mathbf{V}} = r$ since $\mu(t)$ is decreasing. Taking an eigenfunction $y(t)$ corresponding to $\lambda_1(\mu)$, it holds that

$$\lambda_1(\mu) = \frac{\int_{[0,1]} (y')^2\, \mathrm{d}t + \frac{1}{4}\int_{[0,1]} y^2\, \mathrm{d}t}{-\int_{[0,1]} y^2\, \mathrm{d}\mu(t)}. \tag{3.6}$$

Notice that there exists $a \in (0, 1)$ such that

$$\|y\|_\infty = \max_{t\in[0,1]} |y(t)| = |y(a)|.$$

Then, the denominator in (3.6) can be estimated as

$$0 < -\int\limits_{[0,1]} y^2\, \mathrm{d}\mu(t) \leq (y(a))^2\|\mu\|_{\mathbf{V}} = (y(a))^2 r = -\int\limits_{[0,1]} y^2\, \mathrm{d}(-r\delta_a(t)). \tag{3.7}$$

Now (3.6) and (3.7) imply that $\lambda_1(\mu)$ satisfies

$$\lambda_1(\mu) \geq \frac{\int_{[0,1]} (y')^2\, \mathrm{d}t + \frac{1}{4}\int_{[0,1]} y^2\, \mathrm{d}t}{-\int_{[0,1]} y^2\, \mathrm{d}(-r\delta_a(t))} \geq \lambda_1(-r\delta_a),$$

in which the last inequality holds by the minimization characterization (2.2). □

As for the minimizer of the minimization problem (3.2), we have the following conclusion.

Lemma 3.4. *Let $\mu_h \in \tilde{E}_h$ be a minimizer as in Lemma 3.1 and $\|\mu_h\|_{\mathbf{V}} = r_h$. It holds that*

$$h = \lambda_1(\mu_h) = \lambda_1(-r_h\delta_{1/2}) = \frac{\coth\frac{1}{4}}{r_h}.$$

Furthermore, we have $\mu_h = -r_h\delta_{1/2}$.

Proof. Because of Lemmas 3.2 and 3.3, we have

$$\frac{\coth\frac{1}{4}}{r_h} = \lambda_1(-r_h\delta_{1/2}) \leq \lambda_1(\mu_h) = h.$$

It remains to show this is in fact an equality. Otherwise, let us suppose the strict inequality

$$\frac{\coth\frac{1}{4}}{r_h} < h.$$

Then there exists some $0 < r < r_h$ such that

$$\lambda_1(-r\delta_{1/2}) = \frac{\coth\frac{1}{4}}{r} = h.$$

Hence, it holds that $-r\delta_{1/2} \in \tilde{E}_h$. However, $\|-r\delta_{1/2}\|_{\mathbf{V}} = r < r_h = \|\mu_h\|_{\mathbf{V}}$, which contradicts the assumption that μ_h is a minimizer.

Furthermore, assume that $y(t;\mu_h)$ is an eigenfunction with respect to $\lambda_1(\mu_h)$ and $|y(a;\mu_h)| = \max_{t\in[0,1]} |y(t;\mu_h)|$ for some $a \in (0,1)$. So, we have

$$\begin{aligned}
\lambda_1(\mu_h) &= \frac{\int_{[0,1]}(y'(t;\mu_h))^2\,\mathrm{d}t + \frac{1}{4}\int_{[0,1]} y^2(t;\mu_h)\,\mathrm{d}t}{-\int_{[0,1]} y^2(t;\mu_h)\,\mathrm{d}\mu_h(t)} \\
&\geq \frac{\int_{[0,1]}(y'(t;\mu_h))^2\,\mathrm{d}t + \frac{1}{4}\int_{[0,1]} y^2(t;\mu_h)\,\mathrm{d}t}{-\int_{[0,1]} y^2(t;\mu_h)\,\mathrm{d}(-r_h\delta_a(t))} \\
&\geq \frac{\int_{[0,1]}(y'(t;\mu_h))^2\,\mathrm{d}t + \frac{1}{4}\int_{[0,1]} y^2(t;\mu_h)\,\mathrm{d}t}{-\int_{[0,1]} y^2(t;\mu_h)\,\mathrm{d}(-r_h\delta_{1/2}(t))} \geq \lambda_1(-r_h\delta_{1/2}),
\end{aligned}$$

which implies the identity

$$\frac{\int_{[0,1]}(y'(t;\mu_h))^2\,\mathrm{d}t + \frac{1}{4}\int_{[0,1]} y^2(t;\mu_h)\,\mathrm{d}t}{-\int_{[0,1]} y^2(t;\mu_h)\,\mathrm{d}(-r_h\delta_{1/2}(t))} = \lambda_1(-r_h\delta_{1/2})$$

since $\lambda_1(\mu_h) = \lambda_1(-r_h\delta_{1/2})$.

Hence, $y(t;\mu_h)$ is also an eigenfunction with respect to $\lambda_1(-r_h\delta_{1/2})$ and then $\mu_h = -r_h\delta_{1/2}$. □

Now, we can solve the minimization problem (3.2) on measure differential equations.

Theorem 3.1. *It holds that*

$$\tilde{I}(h) = \frac{\coth\frac{1}{4}}{h} \qquad \forall h \in (0+\infty).$$

Moreover, the minimal value $\tilde{I}(h)$ can be attained and only attained by $\mu = -\tilde{I}(h)\delta_{1/2}$.

Proof. It follows from Lemma 3.4 that the minimizer $\mu_h = -r_h\delta_{1/2}$ with $r_h = \frac{\coth\frac{1}{4}}{h}$. So, we have that

$$\tilde{I}(h) = \|-r_h\delta_{1/2}\|_{\mathbf{V}} = r_h = \frac{\coth\frac{1}{4}}{h}.$$

□

Finally, we will show the main conclusion of this paper.

Proof of Theorem 1.1. Notice that $m \in E_h$ induces an absolutely continuous measure

$$\mu_m(t) := \int_{[0,t]} m(s)\,\mathrm{d}s \in \mathcal{M}_0^-.$$

Furthermore, we have $-\int_{[0,1]} d\mu_m = -\int_{[0,1]} m(s)\,\mathrm{d}s$ and $\lambda_1(\mu_m) = \lambda_1(m) = h$. Hence, it holds that

$$\tilde{I}(h) \le -\int\limits_{[0,1]} d\mu_m = -\int\limits_{[0,1]} m(s)\,\mathrm{d}s,$$

which yields

$$\tilde{I}(h) \le I(h).$$

On the other hand, we know that the minimizer $\mu_h = -\frac{\coth\frac{1}{4}}{h}\delta_{1/2} \in \tilde{E}_h$ and

$$\tilde{I}(h) = -\int\limits_{[0,1]} \mathrm{d}\mu_h(s).$$

By Lemma 2.2, there exists a sequence of smooth measures $\{\mu_n\} \subset \mathcal{C}^\infty \cap \mathcal{M}_0^-$ satisfying

$$\mu_n \to \mu_h \text{ in } (\mathcal{M}_0^-, w^*),$$

and $\|\mu_n\|_{\mathbf{V}} = \|\mu_n'\|_1 = r_h$. Let $\varphi_n = \frac{\lambda_1(\mu_n)}{\lambda_1(\mu_h)}\mu_n$. Then $\lambda_1(\varphi_n) = \lambda_1(\mu_h) = h$.

Therefore, by Lemma 2.3, we obtain

$$\begin{aligned}\tilde{I}(h) = -\int\limits_{[0,1]} d\mu_h(s) &= \lim_{n\to\infty} -\int\limits_{[0,1]} d\mu_n(s) = \lim_{n\to\infty} -\frac{\lambda_1(\mu_h)}{\lambda_1(\mu_n)}\int\limits_{[0,1]} \frac{\lambda_1(\mu_n)}{\lambda_1(\mu_h)}\mu_n'(s)\,\mathrm{d}s \\ &= \lim_{n\to\infty} -\frac{\lambda_1(\mu_h)}{\lambda_1(\mu_n)}\int\limits_{[0,1]} \varphi_n'(s)\,\mathrm{d}s \ge \lim_{n\to\infty} \frac{\lambda_1(\mu_h)}{\lambda_1(\mu_n)} I(h) = I(h).\end{aligned}$$

We have thus proved the equality $\tilde{I}(h) = I(h)$. Hence, the proof is complete by Theorem 3.1. □

Remark 3.1. Notice that for every $h > 0$, one has $m(t) \in E_h$ if and only if $hm(t) \in E_1$. Hence, it follows that for every $h > 0$,

$$I(h) = \frac{I(1)}{h},$$

which is consistent with the formula (1.6) in Theorem 1.1.

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