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On the maximal singular integral with Riesz potentials

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Abstract. We investigate the maximal singular integral with Riesz potentials and give a short proof of its estimates through a result from Duoandikoetxea and Rubio de Francia in dimension $n > 1$ and the estimates of Fourier transforms of the measures in dimension $n = 1$. In particular, this enables us to drop the Dini condition when $n > 1$. In the appendix, we provide a proof of a counterexample for Corollary 4.1 in [Invent. Math. (1986)] by Duoandikoetxea and Rubio de Francia in dimension $n = 1$.

Mathematics Subject Classification. 42B20.

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1. Introdution. In the 1950 s, Calderón and Zygmund [2,3] introduced the following homogeneous singular integral operator:

$$I(f)(x) := \lim_{\varepsilon \to 0^+} \int\limits_{|y|>\varepsilon} \frac{\Omega(y)}{|y|^n} f(x-y)dy = \lim_{\varepsilon \to 0^+} I_\varepsilon(f)(x),$$

where Ω is homogeneous of degree 0, with mean value zero over the unit sphere $\mathbb{S}^{n-1}$. It was shown that the homogeneous singular integral operator I and the associated maximal operator $I^*(f)(x) := \sup_{\varepsilon>0} |I_\varepsilon(f)(x)|$ is bounded on $L^p(\mathbb{R}^n)$ for all $0 < p < \infty$, provided $\Omega \in L^q(\mathbb{S}^{n-1})$ for some $q > 1$. The original proof of this boundedness used the method of rotations (see [3,10,20]). In 1984, Duoandikoetxea and Rubio de Francia [9] offered a new proof of this result through the estimates of the corresponding Fourier transforms of measures, which had become a powerful tool in modern harmonic analysis (see [13,15]). Moreover, it was proven further in [1,17] that when $n > 1$, the singular integral operator I and the associated maximal operator I^* are both bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$, just provided $\Omega \in H^1(\mathbb{S}^{n-1})$, where $H^1(\mathbb{S}^{n-1})$ is the Hardy space on the unit sphere studied in [4] (see also [5]).

For $p = 1$, Seeger [19] showed that the operator I is of weak type $(1, 1)$ on $L^1(\mathbb{R}^n)$ if $\Omega \in L\log L(\mathbb{S}^{n-1})$, based on a microlocal decomposition of the kernel. However, it is still an open problem whether the weak type $(1, 1)$ boundedness of I and I^* hold just assuming $\Omega \in H^1(\mathbb{S}^{n-1})$. For the recent progress in the investigations of the weak type $(1, 1)$ bound criterions and the applications including the Calderón commutators and their variants, we refer the readers to [14] and the comprehensive work of Ding and Lai [7].

Recently, Yu et al. [21] established an estimate of an extension of the classical Calderón–Zygmund singular integral operator as follows:

$$T_\varepsilon(f)(x) := \int\limits_{|y|>\varepsilon} \frac{\Omega(y)}{|y|^{n-\beta}} f(x-y)dy,$$

where Ω is homogeneous of degree 0, with mean value zero over the unit sphere $\mathbb{S}^{n-1}$, and $0 < \beta < n$. This kind of integral operator appears in the approximation of the surface quasi-geostrophic systems (SQG) from the generalized SQG systems which cover incompressible Euler systems and have been widely studied in the past years by many experts (see [22] and the references therein).

Formally, T_ε becomes I_ε if $\beta = 0$, thus T_ε can be viewed as an extension of the classical Calderón–Zygmund singular integral operator. Indeed, they proved the following result:

Theorem 1 ([21]). *Let $0 < \beta_0 < 1/2$ and Ω be a homogeneous function of degree 0, with mean value zero over the unit sphere $\mathbb{S}^{n-1}$. If there exist positive numbers B_1, B_2 such that $|\Omega(x)| \le B_1$ and $\int_0^1 \frac{\omega(\delta)}{\delta} d\delta = B_2$, where $\omega(\delta) := \sup\{|\Omega(x) - \Omega(x')| : \ |x - x'| \le \delta, \ x, x' \in \mathbb{S}^{n-1}\}$, then for any $f \in L^1(\mathbb{R}^n) \bigcap L^q(\mathbb{R}^n)$ with $1 < q < \infty$, there exists an absolute constant C depending on n, q, B_1, B_2, and β_0 such that*

$$\|T_\varepsilon f\|_q \le C \left(\|f\|_q + \frac{\beta^{(q-1)n/q}}{((q-1)n - q\beta)^{1/q}} \|f\|_1 \right)$$

holds uniformly for $\varepsilon > 0$ and $0 < \beta < \min\{1 - \beta_0\,,\ (q-1)n/q\}$.

It is well known that for any $0 < \beta < n$, it holds that

$$\|T_\varepsilon f\|_q \le C(n, q, \beta)\|f\|_p \quad \text{if } \frac{1}{p} - \frac{1}{q} = \frac{\beta}{n},$$
$$\|T_\varepsilon f\|_{q,\infty} \le C(n, \beta)\|f\|_1 \qquad \text{if } 1 - \frac{1}{q} = \frac{\beta}{n}$$

through the Riesz potentials (see [11, Theorem 1.2.3]). Then the estimate in Theorem 1 can be achieved by interpolations. Nevertheless, the constant $C(n, q, \beta)$ will be infinitely large when $\beta \to 0^+$. Therefore, the main point of interest in Theorem 1 is that the constant C is independent of β, and when $\beta \to 0^+$, the estimates of the classical truncated Calderón–Zygmund singular integral I_ε are formally recovered from Theorem 1. It is in this sense that T_ε can be viewed as an extension of the classical truncated Calderón–Zygmund singular integral and the corresponding estimates here are more accurate than the classical ones through Riesz potentials.

The proof of their estimate in Theorem 1 was via the so called "geometric approach", which originated from [16]. Although their method of proof is enlightening, the procedure of the proof is a little long and it requires the imposition of the Dini condition.

In this paper, we investigate further the maximal singular integral with Riesz potentials:

$$(T^* f)(x) := \sup_{\varepsilon>0} |T_\varepsilon f(x)|,$$

and give a short proof of its estimate (independent of β and hence better than the classical estimates of Riesz potentials) through a result from Duoandikoetxea and Rubio de Francia ([9], see also [12,18]) in dimension $n > 1$ and the estimates of Fourier transforms of the measures in dimension $n = 1$. In particular, we improve the results in [21] by dropping the Dini condition, when $n > 1$, which is imposed on Theorem 1. Indeed, we obtain the following result:

Theorem 2. *Let $1 \le p < q < \infty$ and Ω be a homogeneous function of degree 0, with mean value zero over the unit sphere $\mathbb{S}^{n-1}$.*

(1) For $n > 1$, if $\|\Omega\|_{L^r(\mathbb{S}^{n-1})} < \infty$, where $\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}$, then there exists a positive constant C depending on n, p, q, and $\|\Omega\|_{L^r(\mathbb{S}^{n-1})}$ such that for any $f \in L^p(\mathbb{R}^n) \bigcap L^q(\mathbb{R}^n)$,

$$\|T^* f\|_q \le C \left(\|f\|_q + \frac{\beta^{(r-1)n/r}}{((r-1)n - r\beta)^{1/r}} \|f\|_p \right)$$

holds for $0 < \beta < (r-1)n/r$.

(2) For $n = 1$, if there exists a positive number B_1 such that $|\Omega(x)| \le B_1$, then there exists a positive constant C depending only on p, q, and B_1 such that for any $f \in L^p(\mathbb{R}^n) \bigcap L^q(\mathbb{R}^n)$,

$$\|T^* f\|_q \le C \left(\|f\|_q + \frac{\beta^{(r-1)/r}}{((r-1) - r\beta)^{1/r}} \|f\|_p \right)$$

holds for $0 < \beta < (r-1)/r$, where $\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}$.

Remark 1. The proof of part (1) of Theorem 2 relies on [9, Corollary 4.1], however, it should be noted that [9, Corollary 4.1] does not hold in dimension $n = 1$ (see [18] or the proof in the appendix below). So in the one-dimensional case, we need to pay more attentions

As an immediate corollary of Theorem 2, we obtain the corresponding estimates (independent of β) for the singular integrals with Riesz potentials:

$$(Tf)(x) := \lim_{\varepsilon \to 0^+} \int_{|y|>\varepsilon} \frac{\Omega(y)}{|y|^{n-\beta}} f(x-y) dy = \lim_{\varepsilon \to 0^+} T_\varepsilon(f)(x),$$

where Ω is homogeneous of degree 0, with mean value zero over the unit sphere $\mathbb{S}^{n-1}$. Indeed, we have

Corollary 1. *Let $1 \leq p < q < \infty$ and Ω be a homogeneous function of degree 0, with mean value zero over the unit sphere $\mathbb{S}^{n-1}$.*

(1) For $n > 1$, if $\|\Omega\|_{L^r(\mathbb{S}^{n-1})} < \infty$, where $\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}$, then there exists a positive constant C depending on n, p, q, and $\|\Omega\|_{L^r(\mathbb{S}^{n-1})}$ such that for any $f \in L^p(\mathbb{R}^n) \bigcap L^q(\mathbb{R}^n)$,

$$\|Tf\|_q \leq C \left(\|f\|_q + \frac{\beta^{(r-1)n/r}}{((r-1)n - r\beta)^{1/r}} \|f\|_p \right)$$

holds for $0 < \beta < (r-1)n/r$.

(2) For $n = 1$, if there exists a positive number B_1 such that $|\Omega(x)| \leq B_1$, then there exists a positive constant C depending only on p, q, and B_1 such that for any $f \in L^p(\mathbb{R}^n) \bigcap L^q(\mathbb{R}^n)$,

$$\|Tf\|_q \leq C \left(\|f\|_q + \frac{\beta^{(r-1)/r}}{((r-1) - r\beta)^{1/r}} \|f\|_p \right)$$

holds for $0 < \beta < (r-1)/r$, where $\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}$.

2. The proof of Theorem 2. In the same way as [21], we split the singular integral T_ε into two parts: the one near the origin and the other one apart from the origin, depending on the parameter β. More precisely, let ψ be a positive radial Schwartz function supported in the ball $\{x \in \mathbb{R}^n : \ |x| \leq 2\}$ and be equal to 1 on the unit ball. Let $\psi_\beta(x) = \psi(\beta x)$, then we have $T_\varepsilon = T_{1,\varepsilon} + T_{2,\varepsilon}$, where

$$T_{1,\varepsilon}(f)(x) := \int\limits_{|y|>\varepsilon} \frac{\Omega(y)}{|y|^{n-\beta}} \psi_\beta(y) f(x-y) dy,$$

$$T_{2,\varepsilon}(f)(x) := \int\limits_{|y|>\varepsilon} \frac{\Omega(y)}{|y|^{n-\beta}} (1 - \psi_\beta(y)) f(x-y) dy.$$

First, by resorting to Young's inequality (see [10, Theorem 1.2.12]), we obtain the following estimate on $\|T_2^* f\|_q$:

Lemma 1. *Let $1 \leq p < q < \infty$.*

(1) For $n > 1$, if $\|\Omega\|_{L^r(\mathbb{S}^{n-1})} < \infty$, where $\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}$, then there exists a positive constant C depending on n, p, q, and $\|\Omega\|_{L^r(\mathbb{S}^{n-1})}$ such that for any $f \in L^p(\mathbb{R}^n)$,

$$\|T_2^* f\|_q \leq C \frac{\beta^{(r-1)n/r}}{((r-1)n - r\beta)^{1/r}} \|f\|_p$$

holds for $0 < \beta < (r-1)n/r$, where $\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}$.

(2) For $n = 1$, if there exists a positive number B_1 such that $|\Omega(x)| \leq B_1$, then there exists a positive constant C depending only on p, q, and B_1 such that for any $f \in L^p(\mathbb{R}^n)$,

$$\|T_2^* f\|_q \leq C \frac{\beta^{(r-1)/r}}{((r-1) - r\beta)^{1/r}} \|f\|_p$$

holds for $0 < \beta < (r-1)/r$, where $\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}$.

Proof. (1) For $n>1$, by Young's inequality, we have

$$\begin{aligned}\|T_2^* f\|_q &\le B_1 \left\| \int_{\mathbb{R}^n} \frac{\chi_{\{y:\ |y|\ge 1/\beta\}}}{|y|^{n-\beta}} |f(x-y)| dy \right\|_q \\ &\le B_1 \left(\int_{|y|\ge 1/\beta} \frac{|\Omega(y)|^r}{|y|^{(n-\beta)r}} dy \right)^{1/r} \|f\|_p \\ &\le C \frac{\beta^{(r-1)n/r}}{((r-1)n - r\beta)^{1/r}} \|f\|_p,\end{aligned}$$

which holds uniformly for $\varepsilon>0$ and $0<\beta<(r-1)n/r$, where $\frac{1}{q}+1=\frac{1}{p}+\frac{1}{r}$.

(2) For $n=1$, the proof is similar to that of (1). □

According to Lemma 1, we just need to estimate $\|T_1^*(f)\|_q$ and this is the key part of the whole proof. For this, we first generalize [9, Theorem A] to be more suitable for our proof of the estimate of $\|T_1^*(f)\|_q$ below.

Lemma 2. *Suppose that $\{\mu_k\}_{k\in\mathbb{Z}}$ is a sequence of Borel measures in $\mathbb{R}^n$ such that $\mu_k \ge 0$, $\sup\limits_{k\in\mathbb{Z}} \hat{\mu}_k(0) \le C$, and*

$$\begin{aligned}|\hat{\mu}_k(\xi) - \hat{\mu}_k(0)| &\le C|a_{k+1}\xi|^{\alpha}, \\ |\hat{\mu}_k(\xi)| &\le C|a_k\xi|^{-\alpha},\end{aligned}$$

for some lacunary sequence of positive numbers $\{a_k\}_{k\in\mathbb{Z}}$ and some $\alpha>0$. Then the maximal operator

$$M_\mu(f)(x) := \sup_{k\in\mathbb{Z}} |(\mu_k * f)(x)|$$

is bounded on $L^p(\mathbb{R}^n)$ for all $1<p<\infty$.

Proof. We can assume that $\alpha \le 1$ and define a sequence of functions $\{\Phi_k\}_{k\in\mathbb{Z}}$ by $\widehat{\Phi}_k(\xi) = \hat{\mu}_k(0)\widehat{\Phi}(a_k\xi)$, where Φ is a positive Schwartz function such that $\widehat{\Phi}(0)=1$. Then, due to the fact that $\sup_{k\in\mathbb{Z}} \hat{\mu}_k(0) \le C$, it is easy to see that the same Fourier transform estimates for μ_k are also satisfied by the sequence of measures $\{\Phi_k(x)dx\}_{k\in\mathbb{Z}}$. It follows that the Fourier transforms of the measures $d\sigma_k(x) := d\mu_k(x) - \Phi_k(x)dx$ satisfy the following estimates:

$$\begin{aligned}|\hat{\sigma}_k(\xi)| &\le C|a_{k+1}\xi|^{\alpha}, \\ |\hat{\sigma}_k(\xi)| &\le C|a_k\xi|^{-\alpha}.\end{aligned}$$

Then by Plancherel's theorem, the operator

$$G(f) := \left(\sum_{-\infty}^{\infty} |\sigma_k * f|^2 \right)^{1/2}$$

is bounded on $L^2(\mathbb{R}^n)$. Now we observe that

$$M_\mu f \le \sup_{k\in\mathbb{Z}} |f*\Phi_k| + \sup_{k\in\mathbb{Z}} |f*\sigma_k| \le \sup_{k\in\mathbb{Z}} |f*\Phi_k| + G(f).$$

Since

$$\sup_{k\in\mathbb{Z}} \widehat{\Phi}_k(0) = \sup_{k\in\mathbb{Z}} \hat{\mu}_k(0)\widehat{\Phi}(0) = \sup_{k\in\mathbb{Z}} \hat{\mu}_k(0) \leq C,$$

by Minkowski's integral inequality, the operator

$$\sup_{k\in\mathbb{Z}} |f * \Phi_k|$$

is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$. Thus, we obtain the boundedness of the maximal operator M_μ on $L^2(\mathbb{R}^n)$. However, since $d\sigma_k(x) = d\mu_k(x) - \Phi_k(x)dx$, the maximal operator

$$\sigma_*(f) := \sup_{k\in\mathbb{Z}} ||\sigma_k| * f|$$

satisfies the following inequality:

$$\sigma_*(f) \leq M_\mu f + \sup_{k\in\mathbb{Z}} |f * \Phi_k|$$

and thereby, is bounded on $L^2(\mathbb{R}^n)$. Therefore, according to [9, Theorem B], the operator $G(\cdot)$ is bounded on $L^p(\mathbb{R}^n)$ for all $|1/p - 1/2| < 1/4$, i.e., $4/3 < p < 4$. If we follow the procedure above again and again, then for any $p \in (1, \infty)$, we can obtain the the boundedness of the maximal operator M_μ on $L^p(\mathbb{R}^n)$ in finite steps. The proof is complete. □

By Lemma 2, we are now able to estimate $\|T_1^*(f)\|_q$. Indeed, we prove the following result:

Lemma 3. *Let* $1 < q < \infty$.

(1) For $n > 1$, *if* $\|\Omega\|_{L^r(\mathbb{S}^{n-1})} < \infty$ *for some* $r > 0$, *then there exists a positive constant* C *depending on* n, q, *and* $\|\Omega\|_{L^r(\mathbb{S}^{n-1})}$ *such that for any* $f \in L^q(\mathbb{R}^n)$,

$$\|T_1^* f\|_q \leq C\|f\|_q$$

holds for $0 < \beta < n$.

(2) For $n = 1$, *if there exists a positive number* B_1 *such that* $|\Omega(x)| \leq B_1$, *then there exists a positive constant* C *depending on* q, B_1 *such that for any* $f \in L^q(\mathbb{R}^n)$,

$$\|T_1^* f\|_q \leq C\|f\|_q$$

holds for $0 < \beta < 1$.

Proof. For $n > 1$, we see that the kernel corresponding to $\lim_{\varepsilon\to 0^+} T_{1,\varepsilon}$ is

$$K(x) := \frac{\Omega(x)}{|x|^{n-\beta}}\psi_\beta(x) = h(|x|)\frac{\Omega(x)}{|x|^n}, \quad x \in \mathbb{R}^n\backslash\{0\},$$

where $h(r) := r^\beta \psi_{\beta,1}(r)$ and $\psi_{\beta,1}(|x|) := \psi_\beta(x)$. Then there exists a positive constant C depending on n such that

$$\int_0^R |h(r)|^2 dr \leq CR, \quad \forall R > 0.$$

Indeed, to prove the above inequality, we just need to consider the case $R < 2/\beta$. Then

$$\int_0^R |h(r)|^2 dr = \int_0^R r^{2\beta} dr = \frac{R^{2\beta+1}}{2\beta+1} \leq R^{2\beta+1} \leq \frac{2^{2\beta}}{(\beta^\beta)^2} R.$$

An easy calculation shows that β^β obtains its minimum value when $\beta = e^{-1}$. Thus, $\frac{1}{(\beta^\beta)^2} \leq e^{2e^{-1}}$. Therefore,

$$\int_0^R |h(r)|^2 dr \leq 2^{2n} e^{2e^{-1}} R = CR, \quad \forall R > 0.$$

Now we resort to [9, Corollary 4.1] to obtain that, when $n > 1$, there exists a positive constant C depending on n, q, B_1 such that for any $f \in L^q(\mathbb{R}^n)$,

$$\|T_1^* f\|_q \leq C\|f\|_q$$

holds for $0 < \beta < n$.

It remains to prove the case for $n = 1$. To this end, we define the sequence of measures

$$d\sigma_k(r) := |r|^\beta \psi_\beta(r) \frac{\Omega(r)}{|r|} \chi_{[2^k, 2^{k+1}]}(|r|) dr,$$

where $r \in \mathbb{R}$ and $k \in \mathbb{Z}$. We have to prove the following estimates of the Fourier transforms of the measures σ_k: there exist an $\alpha > 0$ and a positive constant C depending on B_1 such that for any $\xi \in \mathbb{R}$, it holds that

$$|\hat{\sigma}_k(\xi)| \leq C \min(|2^k \xi|, |2^k \xi|^{-1}).$$

Due to the definitions of the measures σ_k, we just need to prove the above estimates for $2^k \leq 2/\beta$, and this should be kept in mind in the following proof.

(1) Firstly, we prove that $|\hat{\sigma}_k(\xi)| \leq C|2^k \xi|$. Since

$$\|\sigma_k\| \leq C \int_{2^k}^{2^{k+1}} \frac{1}{r^{1-\beta}} dr \leq C 2^\beta 2^{k\beta} \int_{2^k}^{2^{k+1}} \frac{1}{r} dr \leq C 2^2 e^{e^{-1}} \log 2 \leq C,$$

and the measure σ_k is supported in the interval $\{r: |r| \leq 2^{k+1}\}$ and $\hat{\sigma}_k(0) = 0$, it follows that $|\hat{\sigma}_k(\xi)| \leq C|2^k \xi|$.

(2) Secondly, we prove that $|\hat{\sigma}_k(\xi)| \leq C|2^k \xi|^{-1}$. For this, we have

$$\begin{aligned} \hat{\sigma}_k(\xi) &= \int_{2^k}^{2^{k+1}} \left(r^\beta \psi_\beta(r) \frac{\Omega(1)}{r} e^{-2\pi i r \xi} + r^\beta \psi_\beta(r) \frac{\Omega(-1)}{r} e^{2\pi i r \xi} \right) dr \\ &=: J_1 + J_2. \end{aligned}$$

We just have to estimate $|J_1| := |\int_{2^k}^{2^{k+1}} r^\beta \psi_\beta(r) \frac{\Omega(1)}{r} e^{-2\pi i r \xi} dr|$ while the estimate of $|J_2| := |\int_{2^k}^{2^{k+1}} r^\beta \psi_\beta(r) \frac{\Omega(-1)}{r} e^{2\pi i r \xi} dr|$ follows similarly. To this end, we have

$$
\begin{aligned}
|J_1| &= \left| \int_{2^k}^{2^{k+1}} r^\beta \psi_\beta(r) \frac{\Omega(1)}{r} e^{-2\pi i r\xi} dr \right| = \frac{|\Omega(1)|}{2\pi|\xi|} \left| \int_{2^k}^{2^{k+1}} \frac{\psi_\beta(r)}{r^{1-\beta}} de^{-2\pi i r\xi} \right| \\
&\leq \frac{B_1}{2\pi|\xi|} \left(\left| \frac{\psi_\beta(r)}{r^{1-\beta}} \Big|_{2^k}^{2^{k+1}} \right| + \left| \int_{2^k}^{2^{k+1}} e^{-2\pi i r\xi} \frac{\beta\psi'(\beta r) r^{1-\beta} - (1-\beta) r^{-\beta}\psi_\beta(r)}{r^{2(1-\beta)}} dr \right| \right) \\
&\leq \frac{C}{|\xi|} \left(\frac{C}{2^k} + \int_{2^k}^{2^{k+1}} \left(\frac{|\beta\psi'(\beta r) r^{1-\beta}|}{r^{2(1-\beta)}} + \frac{|(1-\beta) r^{-\beta}\psi_\beta(r)|}{r^{2(1-\beta)}} \right) dr \right) \\
&\leq \frac{C}{|\xi|} \left(\frac{C}{2^k} + \int_{2^k}^{2^{k+1}} \left(\frac{C\beta}{r^{1-\beta}} + \frac{C(1-\beta)}{r^{2-\beta}} \right) dr \right) \\
&\leq \frac{C}{|\xi|} \left(\frac{C}{2^k} + \frac{C}{2^k} \right) \\
&\leq \frac{C}{|2^k\xi|}.
\end{aligned}
$$

Thus, we obtain that $|\hat{\sigma}_k(\xi)| \leq \frac{C}{|2^k\xi|}$, and accordingly,

$$|\hat{\sigma}_k(\xi)| \leq C \min(|2^k\xi|, |2^k\xi|^{-1}).$$

Note that if we denote by $|\sigma_k|$ the total variation of σ_k, then the same reasoning as for σ_k shows that

$$
\begin{aligned}
\left| \widehat{|\sigma_k|}(\xi) - \widehat{|\sigma_k|}(0) \right| &\leq C|2^k\xi|, \\
\left| \widehat{|\sigma_k|}(\xi) \right| &\leq C|2^k\xi|^{-1}.
\end{aligned}
$$

According to Lemma 2, the maximal operator

$$\sigma_*(f) := \sup_{k\in\mathbb{Z}} ||\sigma_k| * f|$$

is bounded on $L^q(\mathbb{R})$ for all $1 < q < \infty$. Since the measure σ_k is supported in the interval $\{r : \ |r| \leq 2^{k+1}\}$, by [9, Theorem E], the maximal operator

$$T_*(f) := \sup_{k\in\mathbb{Z}} \left| \sum_{j=k}^{\infty} \sigma_j * f \right|$$

is bounded on $L^q(\mathbb{R})$ for all $1 < q < \infty$. Now if $2^{k-1} \leq \varepsilon < 2^k$, we have the following inequality:

$$|T_{1,\varepsilon} f| \leq \left| \sum_{j=k}^{\infty} \sigma_j * f \right| + \sigma_*(|f|).$$

Therefore,

$$T_1^* f \leq T_*(f) + \sigma_*(|f|).$$

It follows that there exists a positive constant C depending only on q, B_1 such that for any $f \in L^q(\mathbb{R})$,

$$\|T_1^* f\|_q \leq C\|f\|_q$$

holds for $0 < \beta < 1$. Thus, we have completed the proof of the case for $n = 1$. □

Combining Lemma 1 with Lemma 3, Theorem 2 is proven since

$$\|T^* f\|_q \leq \|T_1^* f\|_q + \|T_2^* f\|_q.$$

3. Further remarks.

Remark 2. From [9, Corollary 4.2] or [8], we note that when $n > 1$, Lemma 3 still holds if the space $L^q(\mathbb{R}^n)$ is replaced by the weighted space $L^q_\omega(\mathbb{R}^n)$, where ω is an A_q weight . It may be of interest to investigate whether there is the analogous weighted version of Lemma 1 and thus the analogous weighted version of Theorem 2. For recent remarkable developments of the sparse bounds for maximal rough singular integrals via the Fourier transform, we refer to the works of Di Plinio, Hytönen, and Li [6].

Remark 3. Since the above proof of Lemma 3 is not applicable to the case of $q = 1$, it may be of interest to investigate whether the corresponding weak-type $(1, 1)$ boundedness of $T_{1,\varepsilon}$ holds uniformly for $\varepsilon > 0$ and $0 < \beta < n$, just provided the conditions imposed on Lemma 3 . In the course of writing this manuscript, we noticed that Chen et al. were investigating this case and employing the different method originated from Seeger [19].

Appendix. As mentioned in the beginning, [9, Corollary 4.1] does not hold in dimension $n = 1$. However, so far as we have the material, we can not find any proof of it in the literature. Thus, we provide our proof of this counterexample for the convenience of the readers.

Proposition 1. *In $\mathbb{R}^1$, if we define the function $K(x) := \frac{\sin(|x|)}{x}$ for any $x \neq 0$ and $K(0) := 1$, then the Fourier transform of it is not essentially bounded, i.e., $\widehat{K} \notin L^\infty(\mathbb{R}^1)$. Therefore, the maximal operator*

$$H^* f(x) := \sup_{\varepsilon > 0} \left| \int_{|x-y|>\varepsilon} K(y) f(x-y) \right|$$

and the corresponding singular integral operator

$$Hf(x) := \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} K(y) f(x-y)$$

are both not bounded on $L^2(\mathbb{R}^1)$, though the function K satisfies all the conditions (except that $n = 1$) required in [9, Corollary 4.1].

Proof. It is easy to see that $K \in L^2(\mathbb{R}^1)$, so

$$\widehat{K}(\xi) = \lim_{M\to\infty,\ N\to-\infty} \int_N^M \frac{\sin(|x|)}{x} e^{-ix\xi} dx.$$

Note that we just need to consider the imaginary part of it. What is more, due to the fact that $\frac{\sin(|x|)\sin(x\xi)}{x}$ is an even function, we can just consider the integral on the positive interval $(0, M)$. Define the function

$$I(\xi) := \lim_{M\to\infty,\ \varepsilon\to 0^+} \int_\varepsilon^M \frac{\sin(x)\sin(x\xi)}{x} dx.$$

It is obvious that $I(1) = \infty$. In what follows, we just consider $\xi \in (1/2, 1)$. We have

$$\begin{aligned}
\int_\varepsilon^M \frac{\sin(x)\sin(x\xi)}{x} dx &= \int_\varepsilon^M \frac{\cos((1-\xi)x) - \cos((1+\xi)x)}{2x} dx \\
&= \frac{1}{2} \int_\varepsilon^M \int_{1-\xi}^{1+\xi} \sin(tx) dt dx \\
&= \frac{1}{2} \int_{1-\xi}^{1+\xi} \int_\varepsilon^M \sin(tx) dx dt \\
&= \frac{1}{2} \int_{1-\xi}^{1+\xi} \frac{\cos(t\varepsilon) - \cos(M\varepsilon)}{t} dt \\
&= \frac{1}{2} \int_{1-\xi}^{1+\xi} \frac{\cos(t\varepsilon)}{t} dt - \frac{1}{2} \int_{(1-\xi)M}^{(1+\xi)M} \frac{\cos(t)}{t} dt.
\end{aligned}$$

On the one hand, by the Lebesgue dominated convergence theorem, we see that

$$\lim_{\varepsilon\to 0^+} \frac{1}{2} \int_{1-\xi}^{1+\xi} \frac{\cos(t\varepsilon)}{t} dt = \frac{1}{2} \log(\frac{1+\xi}{1-\xi}).$$

On the other hand, by integration by parts, we have

$$\lim_{M\to\infty} \frac{1}{2} \int_{(1-\xi)M}^{(1+\xi)M} \frac{\cos(t)}{t} dt = 0.$$

Accordingly, for $\xi \in (1/2, 1)$, we have $I(\xi) = \frac{1}{2}\log(\frac{1+\xi}{1-\xi})$, which implies our claim and the proof is complete. □

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