Arch. Math. 120 (2023), 403–416

0003-889X/23/040403-14
published online February 20, 2023
https://doi.org/10.1007/s00013-023-01830-9

Archiv der Mathematik

Check for updates

Existence of solutions to the nonlinear Schrödinger equation on locally finite graphs

Zidong Qiu and Yang Liu

Abstract. Let $G = (V, E)$ be a locally finite connected graph and Δ be the usual graph Laplacian operator. According to Lin and Yang (Rev. Mat. Complut., 2022), using calculus of variations from local to global, we establish the existence of solutions to the nonlinear Schrödinger equation on locally finite graphs, say $-\Delta u + hu = fe^u$, $x \in V$. In particular, we suppose that there exist positive constants μ_0 and ω_0 such that the measure $\mu(x) \geq \mu_0$ for $x \in V$ and symmetric weight $\omega_{xy} \geq \omega_0$ for all $xy \in E$, if h and f satisfy distinct certain assumptions, we prove that the above-mentioned equation has a strictly negative solution by three cases.

Mathematics Subject Classification. 34B45, 35A15, 35R02.

Keywords. Calculus of variations on graph, Sobolev embedding theorem, Nonlinear Schrödinger equation, Locally finite graphs.

1. Introduction. As a graph is a discrete generalization of Euclidean space or Riemann manifold, there have been increasingly more studies on partial differential equations on graphs. For the discrete Laplacian case, in a series of works [5–7], by variational methods, Grigor'yan, Lin, and Yang solved several elliptic differential equations on graphs, say the Kazdan–Warner equation, the Yamabe equation, and the Schrödinger equation. Since then, Huang et al. [10] studied the existence of solutions to the mean field equation with Dirac delta mass on finite graphs, Hou and Sun [12] discussed the Chern–Simons–Higgs equation on graphs, Han et al. [8] investigated the nonlinear biharmonic equations on graphs. Using the new method of Brouwer degree, Sun and Wang [20] proved the existence of solutions to the Kazdan–Warner equation on a connected finite graph, a similar topic was studied by Liu [19] on the mean field equation on graphs. The Kazdan–Warner equation was generalized by Ge and Jiang [4] to certain infinite graphs. Recently, many results also have been obtained for parabolic equations on graphs. The blow-up phenomenon of the semilinear heat equation was studied by Lin and Wu [13,21] on locally finite graphs. Lin and Yang [16] proposed a heat flow for the mean field equation on finite graphs. For other related works, we refer the reader to [1–3,9,11,14,17, 18,23] and the references therein.

Now we recall some definitions on a graph. Let $G = (V, E)$ be a connected graph, where V denotes the vertex set and E denotes the edge set. Throughout this paper, we always assume that G satisfies the following conditions $(a)-(d)$.

(a) (Locally finite) For any $x \in V$, there exist only finitely many vertices $y \in V$ such that $xy \in E$.

(b) (Connected) For any $x, y \in V$, there exist only finitely many edges connecting x and y.

(c) (Symmetric weight) For any $x, y \in V$, let $\omega : V \times V \to \mathbb{R}$ be a positive symmetric weight, i.e., $\omega_{xy} > 0$ and $\omega_{xy} = \omega_{yx}$.

(d) (Positive finite measure) $\mu : V \to \mathbb{R}^+$ defines a positive finite measure on the graph G.

For any function $u : V \to \mathbb{R}$, the Laplacian of u is defined as

$$\Delta u(x) = \frac{1}{\mu(x)} \sum_{y\sim x} \omega_{xy}(u(y) - u(x)), \tag{1.1}$$

where $y \sim x$ means $xy \in E$. The associated gradient form of two functions u and v reads

$$\Gamma(u, v)(x) = \frac{1}{2\mu(x)} \sum_{y\sim x} \omega_{xy}(u(y) - u(x))(v(y) - v(x)).$$

Denote $\Gamma(u) \triangleq \Gamma(u, u)$, and the length of the gradient of u is represented by

$$|\nabla u|(x) = \sqrt{\Gamma(u)(x)} = \left(\frac{1}{2\mu(x)} \sum_{y\sim x} \omega_{xy}(u(y) - u(x))^2 \right)^{\frac{1}{2}}. \tag{1.2}$$

The integral of a function f on V is denoted by

$$\int_V f d\mu = \sum_{x\in V} \mu(x)f(x).$$

For any $p \geq 1$, the Lebesgue space $L^p(V)$ on the graph G is

$$L^p(V) = \{u : V \to \mathbb{R}, \quad \|u\|_{L^p(V)} < +\infty\}, \quad 1 \leq p \leq \infty,$$

where the norm of $u \in L^p(V)$ is given as

$$\|u\|_{L^p(V)} = \begin{cases} \left(\sum_{x\in V} \mu(x)|u(x)|^p\right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \sup_{x\in V} |u(x)|, & p = \infty. \end{cases}$$

For any $x, y \in V$, since the graph G is connected, there exists a shortest path γ connecting x and y. The distance between x and y is defined by $\rho(x, y)$, which means the number of edges belonging to the shortest path γ. That is, if $xy \in E$, then $\rho(x, y) = 1$, if $xy \notin E$, without loss of generality, we may choose a shortest path $\gamma = \{x_1, x_2, \ldots, x_{k+1}\}$ connecting x and y, then $\rho(x, y) = k$.

Throughout this paper, we fix a vertex $O \in V$, and denote the distance function between x and O by

$$\rho(x) = \rho(x,\, O).$$

For any integer $k > 0$, an open ball centered at O with radius k is denoted by

$$B_k = \{x \in V:\ \rho(x) < k\},$$

and the boundary of B_k is written as

$$\partial B_k = \{x \in V:\ \rho(x) = k\}.$$

For any fixed k, Grigor'yan et al. [6] defined the Sobolev space $W_0^{1,2}(B_k)$ and its norm by

$$W_0^{1,2}(B_k) = \left\{u:\ B_k \cup \partial B_k \to \mathbb{R} \;\middle|\; u|_{\partial B_k} = 0, \int\limits_{B_k} |\nabla u|^2 d\mu < +\infty\right\}$$

and

$$\|u\|_{W_0^{1,2}(B_k)} = \left(\int\limits_{B_k} |\nabla u|^2 d\mu\right)^{\frac{1}{2}}. \tag{1.3}$$

In fact, $W_0^{1,2}(B_k)$ is exactly a finite-dimensional linear space since the bounded domain B_k only contains finitely many vertices. Hence $W_0^{1,2}(B_k) = \mathbb{R}^{|B_k|}$, where $|B_k|$ is the number of points in B_k. Then $W_0^{1,2}(B_k)$ is pre-compact, precisely, if $\{u_j\}$ is a bounded sequence in ${W_0}^{1,2}(B_k)$, then there exists some $u \in W_0^{1,2}(B_k)$ such that up to a subsequence $\{u_j\}$ converges to u in $W_0^{1,2}(B_k)$.

Let us recall another important Sobolev space $W^{1,2}(V)$ and its norm, which are defined by

$$W^{1,2}(V) = \left\{u:\ V \to \mathbb{R} \;\middle|\; \int\limits_{V} (|\nabla u|^2 + u^2) d\mu < +\infty\right\}$$

and

$$\|u\|_{W^{1,2}(V)} = \left(\int\limits_{V} (|\nabla u|^2 + u^2) d\mu\right)^{\frac{1}{2}}. \tag{1.4}$$

Let $C_c(V) = \{u:\ V \to \mathbb{R} \mid \text{supp}\, u \subset V \text{ is a finite vertex set}\}$, and ${W_0}^{1,2}(V)$ be the completion of $C_c(V)$ under the norm as in (1.4). Obviously, $W^{1,2}(V)$ and $W_0^{1,2}(V)$ are Hilbert spaces with the inner product $\langle u,\ v\rangle = \int_V (\Gamma(u,v) + uv)\, d\mu$. Let $h(x) \geq h_0 > 0$ for all $x \in V$, we define the space of functions

$$\mathscr{H} = \left\{u \in W_0^{1,2}(V):\ \int_V (|\nabla u|^2 + hu^2) d\mu < \infty\right\} \tag{1.5}$$

with the norm

$$\|u\|_{\mathscr{H}} = \left(\int\limits_{V} (|\nabla u|^2 + hu^2) d\mu\right)^{\frac{1}{2}}. \tag{1.6}$$

It is clear that $\mathscr{H}$ is a Hilbert space with the inner product

$$\langle u,\ v\rangle_{\mathscr{H}} = \int_V (\Gamma(u,v) + huv)\, d\mu, \quad \forall\ u,\ v \in \mathscr{H}.$$

Unlike $W_0^{1,2}(B_k)$, $W^{1,2}(V)$ and $\mathscr{H}$ are infinite-dimensional spaces.

In 2017, using variational methods, Grigor'yan, Lin, and Yang [7] solved the Schrödinger equation $-\Delta u + hu = f(x,\ u)$ on locally finite graphs. They proposed an exact assumption on the locally finite graph G, namely, there exists a constant $\mu_0 > 0$ satisfying

$$\mu(x) \geq \mu_0, \quad \forall\ x \in V, \tag{1.7}$$

which makes the Sobolev embedding theorems hold on the graph. Moreover, if h and f satisfy certain distinct assumptions, they proved that the equation has a strictly positive solution. After that, Zhang and Zhao [22] studied a certain nonlinear Schrödinger equation $-\Delta u + (\lambda a(x) + 1)u = |u|^{p-1}u$ on locally finite graphs. Via the Nehari method, if $a(x)$ satisfies certain assumptions, for any $\lambda > 1$, the above equation admits a ground state solution.

More recently, Lin and Yang [15] proposed another assumption for a locally finite graph G, under which they obtained a broader Sobolev embedding theorem on a locally finite graph G. Instead of (1.7), they assumed that there exists a constant $\omega_0 > 0$ such that

$$\omega_{xy} \geq \omega_0, \quad \forall\ xy \in E. \tag{1.8}$$

They established a method of calculus of variations from local to global and solved the linear Schrödinger equation $-\Delta u + hu = f$, where $f : V \to \mathbb{R}$ is a function on G. Following the lines of them, in this paper, we consider the existence of solutions to the following nonlinear Schrödinger equation on locally finite graphs, say

$$\begin{cases} -\Delta u + hu = fe^u, \text{ in } V, \\ u \in \mathscr{H}, \end{cases} \tag{1.9}$$

where Δ is the Laplacian operator given as in (1.1), and $\mathscr{H}$ is defined as in (1.5). Now we are ready to state our main result.

Theorem 1.1. *Let $G = (V, E)$ be a graph satisfying conditions $(a) - (d)$. Suppose that there exists a constant $h_0 > 0$ such that $h(x) \geq h_0$ for all $x \in V$. Let f be a negative function on G, i.e., $f(x) < 0$ for all $x \in V$. If any of the following three assumptions is satisfied:*

(i) $f \in L^1(V) \cap L^2(V)$;
(ii) $\mu(x) \geq \mu_0 > 0$ *for all* $x \in V$, $f \in L^1(V)$;
(iii) $\omega_{xy} \geq \omega_0 > 0$ *for all* $xy \in E$, *for some* $p \geq 1$ *and any fixed vertex* $O \in V$, *the distance function* $\rho \in L^p(V)$, *and* $f \in L^1(V) \cap L^{p/(p-1)}(V)$; *then the equation* (1.9) *has a strictly negative solution.*

The remaining parts of this paper are organized as follows: In Section 2, we introduce the Sobolev embedding theorems on locally finite graphs, which come from [7,15] directly. Furthermore, we give a specific proof of the Sobolev embedding theorem on $W_0^{1,2}(B_k)$. In Section 1.1, we use the variational methods to prove Theorem 1.1, then we deduce an interesting special corollary. Throughout this paper, we do not distinguish a sequence and its subsequence, and use C to denote absolute constants without distinguishing them even in the same line.

2. Sobolev embedding theorem. For any integer $k > 0$, we define another norm for $W_0^{1,2}(B_k)$ by

$$\|u\|_{W_0^{1,2}(B_k)} = \left(\int\limits_{B_k} (|\nabla u|^2 + hu^2) d\mu \right)^{\frac{1}{2}}, \tag{2.1}$$

which is different from the norm as in (1.3). And we have the following lemma.

Lemma 2.1. *For any fixed integer $k > 0$, the norm (2.1) is equivalent to that in (1.3).*

Proof. For any fixed integer $k > 0$, denote the norm (1.3) as $\|u\|^*_{W_0{}^{1,2}(B_k)}$. Since $h(x)$ is a known coefficient function and B_k contains only finitely many vertices, $h(x)$ has a maximum value on B_k, which is recorded as $h_M^{(k)}$. In [24], Zhu proved the Poincaré inequality on finite graph

$$\int\limits_{B_k} u^2 d\mu \le C_k \int\limits_{B_k} |\nabla u|^2 d\mu, \quad \forall\ u \in W_0{}^{1,2}(B_k), \tag{2.2}$$

where C_k is a constant depending on k. Inserting (2.2) into (2.1), we have

$$\begin{aligned} \|u\|^2_{W_0^{1,2}(B_k)} = \int\limits_{B_k} (|\nabla u|^2 + hu^2) d\mu &\le \int\limits_{B_k} (|\nabla u|^2 + h_M^{(k)} u^2) d\mu \\ &\le (1 + h_M^{(k)} C_k) \int\limits_{B_k} |\nabla u|^2 d\mu \\ &= (1 + h_M^{(k)} C_k) \|u\|^{*2}_{W_0{}^{1,2}(B_k)}. \end{aligned} \tag{2.3}$$

On the other hand, noting that $hu^2 \ge h_0 u^2 \ge 0$ and $\mu(x) > 0$ for all $x \in B_k$, one has

$$\begin{aligned} \|u\|^{*2}_{W_0{}^{1,2}(B_k)} &= \sum_{x\in B_k} \mu(x) |\nabla u|^2(x) \le \sum_{x \in B_k} \mu(x) \left(|\nabla u|^2(x) + h(x) u^2(x) \right) \\ &= \|u\|^2_{W_0^{1,2}(B_k)}. \end{aligned} \tag{2.4}$$

Combining (2.3) and (2.4), the lemma is proved. □

Remark 2.2. *What is interesting about the graph is that $W_0^{1,2}(B_k)$ is the Euclidean space $\mathbb{R}^{|B_k|}$, and the norms of Euclidean space are equivalent, which is consistent with Lemma 2.1 on graphs.*

For our convenience, we use the norm (2.1) to prove the Sobolev embedding theorem on $W_0^{1,2}(B_k)$, note that Lin and Yang [15] did not give a specific proof.

Theorem 2.3. *Let $G = (V, E)$ be a graph satisfying conditions $(a) - (d)$. For any $u \in W_0^{1,2}(B_k)$ and any $1 \le q \le \infty$, there exists a positive constant C depending only on q, h_0, and B_k such that*

$$\|u\|_{L^q(V)} \le C\|u\|_{W_0^{1,2}(B_k)}. \tag{2.5}$$

Proof. For any $x \in B_k$ and $u \in W_0^{1,2}(B_k)$, noting that B_k contains only finitely many vertices, by (1.2) and (2.1), we obtain

$$\begin{aligned} h_0 \min_{x\in B_k} \mu(x)u^2(x) &\le \sum_{x\in B_k} \mu(x)h(x)u^2(x) \\ &\le \frac{1}{2}\sum_{x\in B_k}\sum_{y\sim x} \omega_{xy}(u(y)-u(x))^2 + \sum_{x\in B_k}\mu(x)h(x)u^2(x) \\ &= \|u\|^2_{W_0^{1,2}(B_k)}, \end{aligned}$$

which implies

$$\|u\|_{L^\infty(B_k)} \le \big(h_0 \min_{x\in B_k}\mu(x)\big)^{-\frac{1}{2}}\|u\|_{W_0^{1,2}(B_k)}.$$

Thus, for any $1 \le q < +\infty$, we have

$$\left(\sum_{x\in B_k}\mu(x)|u(x)|^q\right)^{\frac{1}{q}} \le \big(h_0 \min_{x\in B_k}\mu(x)\big)^{-\frac{1}{2}} V(B_k)^{\frac{1}{q}}\|u\|_{W_0^{1,2}(B_k)},$$

where $V(B_k) = \sum_{x\in B_k}\mu(x)$ denotes the volume of B_k. Therefore (2.5) holds. □

Along the lines of [7,15], we introduce the following Sobolev embedding theorems on $W^{1,2}(V)$. The proof process of Theorem 2.4 and 2.5 is given based on the locally finite graph case, and the theorems are also correct for $W_0^{1,2}(B_k)$, so we will not give the finite-dimensional form of them.

Theorem 2.4 (Sobolev embedding theorem 1, [7]). *Let $G = (V, E)$ be a graph satisfying conditions $(a) - (d)$. If (1.7) is satisfied, then for any $u \in W^{1,2}(V)$ and any $2 \le q \le \infty$, there exists a positive constant C depending only on q and μ_0 satisfying $\|u\|_{L^q(V)} \le C\|u\|_{W^{1,2}(V)}$. In particular,*

$$\|u\|_{L^\infty(V)} \le \frac{1}{\sqrt{\mu_0}}\|u\|_{W^{1,2}(V)}.$$

Theorem 2.5 (Sobolev embedding theorem 2, [15]). *Let $G = (V, E)$ be a graph satisfying conditions $(a) - (d)$. If (1.8) is satisfied, and the distance function $\rho(x) \in L^p(V)$ for some $p > 0$, then for any $u \in W^{1,2}(V)$, there exists a positive constant C depending only on ω_0, p, and $\mu(O)$ such that*

$$\|u\|_{L^p(V)} \le C(\|\rho\|_{L^p(V)} + 1)\|u\|_{W^{1,2}(V)}.$$

3. Proof of Theorem 1.1. In this section, we shall prove Theorem 1.1 by using a direct variational method from local to global. Fix a point $O \in V$, denote the distance between x and O by $\rho(x)$. Then we can define the open ball on the graph centered at O with radius $k \in \mathbb{Z}^+$, we write $B_k = \{x \in V : \ \rho(x) < k\}$. For fixed k, let $W_0^{1,2}(B_k)$ be the Sobolev space with the norm (2.1).

Define a functional $J_k \ : \ W_0^{1,2}(B_k) \to \mathbb{R}$ by

$$J_k(u) = \frac{1}{2}\int\limits_{B_k}(|\nabla u|^2 + hu^2)d\mu - \int\limits_{B_k} fe^u d\mu, \quad \forall\ u \in {W_0}^{1,2}(B_k), \tag{3.1}$$

which is the variational functional corresponding to the equation (1.9) on $W_0^{1,2}(B_k)$.

Case (i). $f \in L^1(V) \cap L^2(V)$.

To begin with, we show that the functional (3.1) is bounded from below, and then we give the result that its infimum can be achieved. Noting that $e^u > u + 1$ holds for all $u \in \mathbb{R}$, and $f(x) < 0$ for all $x \in V$, we thus have $fe^u < f(u+1) \leq |fu| + |f|$. Using the Hölder inequality, we obtain

$$\|fu\|_{L^1(B_k)} \leq \|f\|_{L^2(B_k)}\|u\|_{L^2(B_k)} = \Big(\int\limits_{B_k} f^2 d\mu\Big)^{\frac{1}{2}}\Big(\int\limits_{B_k} u^2 d\mu\Big)^{\frac{1}{2}}.$$

Due to $f \in L^2(V)$ and $f^2 > 0$, one has

$$\int\limits_{B_k} |fu| d\mu \leq \Big(\int\limits_{V} f^2 d\mu\Big)^{\frac{1}{2}}\Big(\int\limits_{B_k} u^2 d\mu\Big)^{\frac{1}{2}}. \tag{3.2}$$

In view of $h(x) \geq h_0 > 0$ for all $x \in V$, by the Young inequality and (3.2), we deduce that

$$\begin{aligned}\int\limits_{B_k} |fu| d\mu &\leq \frac{1}{\sqrt{h_0}}\Big(\int\limits_{V} f^2 d\mu\Big)^{\frac{1}{2}}\Big(\int\limits_{B_k} hu^2 d\mu\Big)^{\frac{1}{2}} \\ &\leq \frac{1}{\sqrt{h_0}}\Big(\int\limits_{V} f^2 d\mu\Big)^{\frac{1}{2}}\Big(\int\limits_{B_k} (hu^2 + |\nabla u|^2) d\mu\Big)^{\frac{1}{2}} \\ &\leq \frac{\epsilon}{h_0}\int\limits_{V} f^2 d\mu + \frac{1}{4\epsilon}\int\limits_{B_k}(hu^2 + |\nabla u|^2)d\mu. \end{aligned} \tag{3.3}$$

Taking $\varepsilon = 1$ and inserting (3.3) into (3.1), we have

$$J_k(u) \geq \frac{1}{4}\|u\|^2_{{W_0}^{1,2}(B_k)} - \frac{1}{h_0}\|f\|^2_{L^2(V)} - \|f\|_{L^1(V)}, \quad \forall\ u \in {W_0}^{1,2}(B_k), \tag{3.4}$$

which implies

$$J_k(u) \geq -\frac{1}{h_0}\|f\|^2_{L^2(V)} - \|f\|_{L^1(V)},$$

where we recall that $f \in L^1(V) \cap L^2(V)$ is a known coefficient function. Accordingly, the right side of the inequality is a constant which does not depend on the radius k, that is, for any open ball centered at O, the variational functional has the same lower bound. Denote

$$\Lambda_k = \inf_{u \in {W_0}^{1,2}(B_k)} J_k(u),$$

obviously, $0 \in {W_0}^{1,2}(B_k)$, and $J_k(0) = -\int_{B_k} f d\mu \leq \|f\|_{L^1(V)} < +\infty$, thus it follows that

$$-\frac{1}{h_0}\|f\|^2_{L^2(V)} - \|f\|_{L^1(V)} \leq \Lambda_k \leq \|f\|_{L^1(V)}. \tag{3.5}$$

Next we claim that the infimum Λ_k is reachable in ${W_0}^{1,2}(B_k)$. Noting that $\{\Lambda_k\}$ is a bounded sequence of numbers, for any fixed positive integer k, we can take a sequence of functions $\{u_j^{(k)}\}$ in ${W_0}^{1,2}(B_k)$ such that $J_k(u_j^{(k)}) \to \Lambda_k$ as $j \to \infty$. For any $\varepsilon > 0$, it follows from (3.4) and (3.5) that

$$\frac{1}{4}\|u_j^{(k)}\|^2_{{W_0}^{1,2}(B_k)} - \frac{1}{h_0}\|f\|^2_{L^2(V)} - \|f\|_{L^1(V)} \leq \Lambda_k + \varepsilon \leq \|f\|_{L^1(V)} + \varepsilon,$$

which implies that $\{u_j^{(k)}\}$ is bounded in ${W_0}^{1,2}(B_k)$. By taking into account that ${W_0}^{1,2}(B_k)$ is pre-compact, there exists some $u_k \in W_0^{1,2}(B_k)$ such that up to a subsequence $\{u_j^{(k)}\}$ converges to u_k in $W_0^{1,2}(B_k)$ under the norm (2.1). Consequently, $\Lambda_k = J_k(u_k)$, and the critical function u_k satisfies the Euler-Lagrange equation

$$\begin{cases} -\Delta u_k + h u_k = f e^{u_k}, \text{ in } B_k, \\ u_k = 0, \qquad\qquad\quad \text{ on } \partial B_k, \end{cases} \tag{3.6}$$

then our claim is proved.

Now we are in position to characterize $\{u_k\}$ from local to global. Let $u = u_k$ in (3.4), together with (3.5), we have

$$\begin{aligned} \|u_k\|^2_{{W_0}^{1,2}(B_k)} &\leq 4\left(\Lambda_k + \frac{1}{h_0}\|f\|^2_{L^2(V)} + \|f\|_{L^1(V)}\right) \\ &\leq 4\left(\frac{1}{h_0}\|f\|^2_{L^2(V)} + 2\|f\|_{L^1(V)}\right), \end{aligned} \tag{3.7}$$

which yields that $\|u_k\|_{{W_0}^{1,2}(B_k)} \leq C$ for a constant C independent of k. Therefore, the critical functions $\{u_k\}$ all have the same upper bound. For any finite set $K \subset V$, we can always find a sufficiently large $k \in \mathbb{Z}^+$ such that $K \subset B_k$. According to Theorem 2.3, it follows from (3.7) that

$$\|u_k\|_{L^\infty(K)} \leq \left(h_0 \min_{x \in K} \mu(x)\right)^{-\frac{1}{2}} \|u_k\|_{W_0^{1,2}(B_k)} \leq C. \tag{3.8}$$

Noting that $\{u_k\}$ is a sequence of functions defined on $B_k \bigcup \partial B_k$, we extend the domain $B_k \bigcup \partial B_k$ to V and obtain

$$u_k(x) = \begin{cases} u_k(x), & x \in B_k, \\ 0, & x \notin B_k. \end{cases} \tag{3.9}$$

Then (3.8) ensures that we can take the convergent subsequence of $\{u_k\}$ point by point, and there exists a function u^* on V such that $\{u_k\}$ converges to u^*

locally uniformly in V. Namely, for any fixed positive integer l,

$$\lim_{k\to\infty} u_k(x) = u^*(x), \quad \forall\ x \in B_l.$$

Next we claim that $u^* \in \mathscr{H}$. In order to prove this, it suffices to show that u^* is equal to some function which is in $\mathscr{H}$. We prove this by verifying that the weak convergent limit of $\{u_k\}$ in $\mathscr{H}$ is equal to u^*. Using the norm (1.6), one has

$$\begin{aligned}
\|u_k\|^2_{\mathscr{H}} &= \int_V (|\nabla u_k|^2 + hu_k^2)d\mu \\
&= \frac{1}{2}\sum_{x\in V}\sum_{y\sim x}\omega_{xy}(u_k(y)-u_k(x))^2 + \sum_{x\in V}\mu(x)h(x)u_k^2(x) \\
&= \frac{1}{2}\sum_{y\sim x,\ x\in B_k}\omega_{xy}(u_k(y)-u_k(x))^2 \\
&\quad + \frac{1}{2}\sum_{y\sim x,\ x\in\partial B_k}\omega_{xy}(u_k(y)-u_k(x))^2 \\
&\quad + \sum_{x\in B_k}\mu(x)h(x)u_k^2(x) \\
&\le \sum_{y\sim x,\ x\in B_k}\omega_{xy}(u_k(y)-u_k(x))^2 + \sum_{x\in B_k}\mu(x)h(x)u_k^2(x) \\
&\le 2\|u_k\|^2_{W_0^{1,2}(B_k)}.
\end{aligned}$$

Combining with (3.7), we deduce that $\{u_k\}$ defined as in (3.9) belongs to $\mathscr{H}$ and is bounded in $\mathscr{H}$. By taking into account that $\mathscr{H}$ is a Hilbert space, there exists some $\widetilde{u} \in \mathscr{H}$ such that up to a subsequence, $\{u_k\}$ converges to $\widetilde{u}$ weakly in $\mathscr{H}$. That is to say, for any $\phi \in C_c(V)$,

$$\int_V u_k\phi d\mu \to \int_V \widetilde{u}\phi d\mu \tag{3.10}$$

as $k \to +\infty$. We select a special $\phi(x) \in C_c(V)$ to verify $u^* = \widetilde{u}$. For any fixed $x_1 \in V$, let

$$\phi(x) = \begin{cases} 1, & x = x_1, \\ 0, & x \neq x_1. \end{cases} \tag{3.11}$$

Obviously, $\phi(x) \in C_c(V)$. Inserting (3.11) into (3.10), it follows that $u_k(x_1) \to \widetilde{u}(x_1)$ as $k \to +\infty$. Since x_1 is arbitrary, convergence is true for all $x \in V$. So we have $u_k(x) \to \widetilde{u}(x)$ for all $x \in V$. By the uniqueness of limit, as a consequence, $u^*(x) = \widetilde{u}(x)$ for all $x \in V$. This leads to $u^* \in \mathscr{H}$.

Finally, we shall prove that u^* is a strictly negative solution of the equation (1.9). Indeed, it follows from (3.6) that

$$-\int_V \Delta u_k\phi d\mu + \int_V hu_k\phi d\mu = \int_V fe^{u_k}\phi d\mu, \quad \forall\ \phi \in C_c(V). \tag{3.12}$$

For any fixed $x_1 \in V$, let ϕ be as in (3.11) and $k \to +\infty$ in (3.12), we have

$$-\Delta u^*(x_1) + h(x_1)u^*(x_1) = f(x_1)e^{u^*(x_1)}.$$

Since x_1 is arbitrary, u^* is a solution of the equation (1.9). Next we prove that u^* is strictly negative satisfying $u^*(x) < 0$ for all $x \in V$, it suffices to show that $\max_{x\in V} u^*(x) < 0$. Obviously, $u^*(x) \not\equiv 0$, suppose not, we have $f(x) \equiv 0$, that would contradict the fact that $f(x) < 0$ for all $x \in V$. Therefore, we suppose that there exists some $x_0 \in V$ satisfying $u^*(x_0) = \max_{x\in V} u^*(x) \geq 0$. If $u^*(x_0) = \max_{x\in V} u^*(x) = 0$, then by $f(x_0) < 0$, we have

$$-\Delta u^*(x_0) = f(x_0) < 0.$$

This is impossible, according to $\mu(x_0) > 0$ and $\omega_{x_0y} > 0$, together with (1.1), we have

$$-\Delta u^*(x_0) = -\frac{1}{\mu(x_0)} \sum_{y\sim x_0} \omega_{x_0y} u^*(y) \geq 0.$$

If $u^*(x_0) = \max_{x\in V} u^*(x) > 0$, in view of $h(x_0) > 0$ and $f(x_0) < 0$, we then have

$$-\Delta u^*(x_0) = -h(x_0)u^*(x_0) + f(x_0)e^{u^*(x_0)} < 0.$$

This is also impossible, it is clear that

$$-\Delta u^*(x_0) = -\frac{1}{\mu(x_0)} \sum_{y\sim x_0} \omega_{x_0y}(u^*(y) - u^*(x_0)) \geq 0.$$

Hence $u^*(x) < 0$ for all $x \in V$. This gives the desired result.

Case (*ii*). $\mu(x) \geq \mu_0 > 0$ for all $x \in V$, $f \in L^1(V)$.

By Theorem 2.4, using the Sobolev embedding theorem, we have for any $u \in W_0^{1,2}(B_k)$,

$$\|u\|_{L^\infty(B_k)} \leq \frac{1}{\sqrt{\mu_0}} \|u\|_{W_0^{1,2}(B_k)}.$$

Similar to *Case* (*i*), by the Hölder inequality and the Young inequality, one has

$$\begin{aligned} \int_{B_k} f e^u d\mu &\leq \int_{B_k} |fu| d\mu + \int_{B_k} |f| d\mu \\ &\leq \|u\|_{L^\infty(B_k)} \|f\|_{L^1(B_k)} + \|f\|_{L^1(V)} \\ &\leq \frac{1}{\sqrt{\mu_0}} \|u\|_{W_0^{1,2}(B_k)} \|f\|_{L^1(V)} + \|f\|_{L^1(V)} \\ &\leq \frac{1}{4} \|u\|^2_{{W_0}^{1,2}(B_k)} + \frac{1}{\mu_0} \|f\|^2_{L^1(V)} + \|f\|_{L^1(V)}. \end{aligned} \tag{3.13}$$

Hence it follows from (3.1) and (3.13) that

$$J_k(u) \geq \frac{1}{4} \|u\|^2_{{W_0}^{1,2}(B_k)} - \frac{1}{\mu_0} \|f\|^2_{L^1(V)} - \|f\|_{L^1(V)}, \quad \forall\, u \in {W_0}^{1,2}(B_k).$$

Then we obtain an analog of (3.5), that is,

$$-\frac{1}{\mu_0}\|f\|^2_{L^1(V)} - \|f\|_{L^1(V)} \leq \Lambda_k \leq \|f\|_{L^1(V)}. \tag{3.14}$$

Based on (3.14), our remaining part of the proof is a generalization of *Case* (i), and we omit this part.

Case (iii). $\omega_{xy} \geq \omega_0 > 0$ for all $xy \in E$, for some $p \geq 1$ and any fixed vertex $O \in V$, the distance function $\rho \in L^p(V)$, and $f \in L^1(V) \cap L^{p/(p-1)}(V)$. By Theorem 2.5, for any $u \in W_0^{1,2}(B_k)$, there exists a positive constant C depending only on ω_0, $\mu(O)$, and $\|\rho\|_{L^p(V)}$ such that

$$\|u\|_{L^p(B_k)} \leq C\|u\|_{W_0^{1,2}(B_k)}.$$

Then we obtain

$$\begin{aligned}\int\limits_{B_k} fe^u d\mu &\leq \int\limits_{B_k} |fu| d\mu + \int\limits_{B_k} |f| d\mu \\ &\leq \|u\|_{L^p(B_k)}\|f\|_{L^{\frac{p}{p-1}}(V)} + \|f\|_{L^1(V)} \\ &\leq C\|u\|_{W_0^{1,2}(B_k)}\|f\|_{L^{\frac{p}{p-1}}(V)} + \|f\|_{L^1(V)} \\ &\leq \frac{1}{4}\|u\|^2_{W_0{}^{1,2}(B_k)} + C\|f\|^2_{L^{\frac{p}{p-1}}(V)} + \|f\|_{L^1(V)}.\end{aligned}$$

Similar to *Case* (ii), for the variational functional $J_k(u)$, one has

$$J_k(u) \geq \frac{1}{4}\|u\|^2_{W_0{}^{1,2}(B_k)} - C\|f\|^2_{L^{\frac{p}{p-1}}(V)} - \|f\|_{L^1(V)}$$

and

$$-C\|f\|^2_{L^{\frac{p}{p-1}}(V)} - \|f\|_{L^1(V)} \leq \Lambda_k \leq \|f\|_{L^1(V)}.$$

The remaining part of the proof is totally analogous to that of *Case* (i), and we omit this part. This completely ends the proof of Theorem 1.1. □

There is an interesting special result of *Case* (iii) in Theorem 1.1 as follows.

Corollary 3.1. *Let $G = (V, E)$ be a finite graph satisfying conditions $(b) - (d)$. Suppose that $h(x) > 0$ and $f(x) < 0$ for all $x \in V$. Then the equation (1.9) on the finite graph G has a strictly negative solution.*

Proof. If G is a finite connected graph satisfying conditions $(b) - (d)$, then V only contains finitely many vertices. Therefore, there always exist constants $h_0 = \min_{x\in V} h(x) > 0$ and $\omega_0 = \min_{xy\in E} \omega_{xy} > 0$ such that $h(x) \geq h_0$ for all $x \in V$ and $\omega_{xy} \geq \omega_0 > 0$ for all $xy \in E$. For any fixed vertex $O \in V$, noting that G is a finite connected graph, concerning the distance function $\rho(x) = \rho(x, O)$, we deduce that

$$\|\rho(x)\|_{L^\infty} = \max_{x\in V} |\rho(x)| < +\infty,$$

which implies that $\rho \in L^\infty(V)$. Furthermore, it always holds that

$$\int_V |f| d\mu = \sum_{x\in V} \mu(x)|f(x)| < +\infty$$

and $f \in L^1(V)$.

As a consequence, by taking into account *Case* (*iii*) in Theorem 1.1, we deduce that the equation (1.9) has a strictly negative solution on the finite graph G. □

Conflict of interest On behalf of all authors, the corresponding author declare that there are no conflicts of interests regarding the publication of this paper.

References

[1] Akduman, S., Pankov, A.: Nonlinear Schrödinger equation with growing potential on infinite metric graphs. Nonlinear Anal. **184**, 258–272 (2019)

[2] Bianchi, D., Setti, A., Wojciechowski, R.: The generalized porous medium equation on graphs: existence and uniqueness of solutions with ℓ^1 data. Calc. Var. Partial Differential Equations **61**(5), Paper No. 171, 42 pp. (2022)

[3] do Ó, J. M., Medeiros, E., Severo, U.: On a quasilinear nonhomogeneous elliptic equation with critical growth in $\mathbb{R}^N$. J. Differential Equations **246**, 1363–1386 (2009)

[4] Ge, H., Jiang, W.: Kazdan–Warner equation on infinite graphs. J. Korean Math. Soc. **55**, 1091–1101 (2018)

[5] Grigor'yan, A., Lin, Y., Yang, Y.: Yamabe type equations on graphs. J. Differential Equations **261**, 4924–4943 (2016)

[6] Grigor'yan, A., Lin, Y., Yang, Y.: Kazdan–Warner equation on graph. Calc. Var. Partial Differential Equations **55**, Paper No. 92, 13 pp. (2016)

[7] Grigor'yan, A., Lin, Y., Yang, Y.: Existence of positive solutions to some nonlinear equations on locally finite graphs. Sci. China Math. **60**, 1311–1324 (2017)

[8] Han, X., Shao, M., Zhao, L.: Existence and convergence of solutions for nonlinear biharmonic equations on graphs. J. Differential Equations **268**, 3936–3961 (2020)

[9] Han, X., Shao, M.: p-Laplacian equations on locally finite graphs. Acta Math. Sin. (Engl. Ser.) **37**(11), 1645–1678 (2021)

[10] Huang, A., Lin, Y., Yau, S.: Existence of solutions to mean field equations on graphs. Comm. Math. Phys. **377**, 613–621 (2020)

[11] Huang, H., Wang, J., Yang, W.: Mean field equation and relativistic Abelian Chern–Simons model on finite graphs. J. Funct. Anal. **281**(10), 109218, 36 pp. (2021)

[12] Hou, S., Sun, J.: Existence of solutions to Chern–Simons–Higgs equations on graphs. Calc. Var. Partial Differential Equations **61**(4), Paper No. 139, 13 pp. (2022)

[13] Lin, Y., Wu, Y.: The existence and nonexistence of global solutions for a semilinear heat equation on graphs. Calc. Var. Partial Differential Equations **56**, Paper No. 102, 22 pp. (2017)

[14] Lin, Y., Xie, Y.: Application of Rothe's method to a nonlinear wave equation on graphs. Bull. Korean Math. Soc. **59**(3), 745–756 (2022)

[15] Lin, Y., Yang, Y.: Calculus of variations on locally finite graphs. Rev. Mat. Complut. **35**, 791–813 (2022)

[16] Lin, Y., Yang, Y.: A heat flow for the mean field equation on a finite graph. Calc. Var. Partial. Differential Equations **60**, Paper No. 206, 15 pp. (2021)

[17] Liu, S., Yang, Y.: Multiple solutions of Kazdan–Warner equation on graphs in the negative case. Calc. Var. Partial Differential Equations **59**, Paper No. 164, 15 pp. (2020)

[18] Liu, Y.: Multiple solutions of a perturbed Yamabe-type equation on graph. J. Korean Math. Soc. **59**, 911–926 (2022)

[19] Liu, Y.: Brouwer degree for mean field equation on graph. Bull. Korean Math. Soc. **59**, 1305–1315 (2022)

[20] Sun, L., Wang, L.: Brouwer degree for Kazdan–Warner equations on a connected finite graph. Adv. Math. **404**, 108422 (2022)

[21] Wu, Y.: Blow-up for a semilinear heat equation with Fujita's critical exponent on locally finite graphs. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **115**, Paper No. 133, 16 pp. (2021)

[22] Zhang, N., Zhao, L.: Convergence of ground state solutions for nonlinear Schrödinger equations on graphs. Sci. China Math. **61**, 1481–1494 (2018)

[23] Zhang, X., Lin, A.: Positive solutions of p-th Yamabe type equations on infinite graphs. Proc. Amer. Math. Soc. **147**, 1421–1427 (2019)

[24] Zhu, X.: Mean field equations for the equilibrium turbulence and Toda systems on connected finite graphs. J. Partial Differential Equations **35**, 199–207 (2022)

Zidong Qiu and Yang Liu
School of Mathematics
Renmin University of China
Beijing 100872
China
e-mail: qiuzidong961002@ruc.edu.cn

Yang Liu
e-mail: dliuyang@ruc.edu.cn

Received: 17 November 2022

Accepted: 12 January 2023