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Submanifolds with parallel weighted mean curvature vector in the Gaussian space

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Abstract. We establish a Nishikawa type maximum principle for the drift Laplacian and, under a suitable boundedness of the second fundamental form, we apply it to prove that the hyperplanes are the only complete n-dimensional submanifolds immersed with either parallel weighted mean curvature vector, for codimension $p \geq 2$, or constant weighted mean curvature, for codimension $p = 1$, in the $(n+p)$-dimensional Gaussian space $\mathbb{G}^{n+p}$, which corresponds to the Euclidean space $\mathbb{R}^{n+p}$ endowed with the Gaussian probability measure $d\mu = e^{-|x|^2/4}d\sigma$, where $d\sigma$ is the standard Lebesgue measure of $\mathbb{R}^{n+p}$. Furthermore, we also use a maximum principle at infinity to get additional rigidity results, as well as a nonexistence result related to nonminimal submanifolds immersed with parallel weighted mean curvature vector in $\mathbb{G}^{n+p}$.

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1. Introduction and statements of the main results. Given an $(n+p)$-dimensional Riemannian manifold $(\overline{M}^{n+p}, \bar{g})$ and a smooth function $f \in C^\infty(\overline{M})$, we recall that the *weighted manifold* associated to $\overline{M}^{n+p}$ and f is just the triple $(\overline{M}^{n+p}, \bar{g}, d\mu = e^{-f}d\overline{M})$, where $d\overline{M}$ denotes the standard volume element of $\overline{M}^{n+p}$. For a throughout discussion of weighted manifolds, we suggest the articles [10, 15, 19] and references therein.

An important example of a weighted Riemannian manifold is the so-called *Gaussian space* $\mathbb{G}^{n+p}$, which corresponds to the Euclidean space $\mathbb{R}^{n+p}$ endowed with the Gaussian probability measure

$$d\mu = e^{-\frac{|x|^2}{4}}d\sigma, \tag{1.1}$$

where $d\sigma$ is the standard Lebesgue measure of $\mathbb{R}^{n+p}$. In this context, the *f-mean curvature vector* of an immersed n-dimensional submanifold $X : M^n \looparrowright \mathbb{G}^{n+p}$ is defined by

$$\vec{H}_f = \vec{H} + (\overline{\nabla} f)^{\perp} = \vec{H} + \frac{1}{2} X^{\perp}. \tag{1.2}$$

Here, $\vec{H}$ stands for the standard (nonnormalized) mean curvature vector of the immersion $X : M^n \looparrowright \mathbb{R}^{n+p}$ and $(\,)^{\perp}$ denotes the normal part of a vector field on $\mathbb{R}^{n+p}$. When $\vec{H}_f$ vanishes identically, M^n is called a *self-shrinker* of the mean curvature flow, which plays an important role in the study of the mean curvature flow because it describes all possible blow ups at a given singularity of such a flow and, as it was pointed out by Colding and Minicozzi in [7], self-shrinkers are critical submanifolds for the entropy functional.

There exist in the literature many characterizations and rigidity results of self-shrinkers under appropriate hypothesis. For instance; Ecker and Huisken [9] proved that if a self-shrinker is an entire graph with polynomial volume growth, then it is a hyperplane. Later on, the condition of polynomial volume growth was removed by Wang [18]. In [12], Le and Sesum showed that any smooth self-shrinker with polynomial volume growth and satisfying $|A|^2 < \frac{1}{2}$ is a hyperplane, where A denotes the second fundamental form of an immersion. Afterwards, Cao and Li [3] generalized this result to arbitrary codimension proving that any smooth complete self-shrinker with polynomial volume growth and $|A|^2 \leq \frac{1}{2}$ is either a round sphere, a circular cylinder, or a hyperplane.

In [6], Cheng and Peng gave estimates on supremum and infimum of the squared norm of the second fundamental form of self-shrinkers without assumption on polynomial volume growth which, in particular, enabled them to obtain the rigidity theorem of [3] without assumption on polynomial volume growth (see also [5, Theorem 1.1] concerning the 2-dimensional case). More recently, Wang et al. [17] proved a rigidity theorem for complete n-dimensional submanifolds with parallel f-mean curvature vector in the Gaussian space $\mathbb{G}^{n+p}$, under an integral curvature pinching condition, generalizing a previous rigidity result for self-shrinkers due to Ding and Xin [8].

Proceeding with this picture and considering initially the case that the codimension $p \geq 2$, we obtain a sort of extension of [6, Theorem 1.1].

Theorem 1.1. *Let $X : M^n \looparrowright \mathbb{G}^{n+p}$ be a complete submanifold immersed with parallel f-mean curvature vector $\vec{H}_f$ in the $(n+p)$-dimensional Gaussian space $\mathbb{G}^{n+p}$, with $p \geq 2$. If the second fundamental form A of M^n satisfies*

$$\sup_M |A| < \gamma_1, \tag{1.3}$$

where

$$\gamma_1 = \frac{\sqrt{|\vec{H}_f|^2 + 3} - |\vec{H}_f|}{3}, \tag{1.4}$$

then M^n is a hyperplane of $\mathbb{G}^{n+p}$.

When the codimension is $p = 1$, the *f-mean curvature* H_f of $X : M^n \looparrowright \mathbb{G}^{n+1}$ is defined by

$$H_f = H + \langle \overline{\nabla} f, N \rangle = H + \frac{1}{2}\langle X, N \rangle, \tag{1.5}$$

where $H = \mathrm{tr}(A)$ corresponds to the standard mean curvature of $X : M^n \looparrowright \mathbb{R}^{n+1}$ with respect to its orientation N. When $H_f \equiv \lambda$ for some constant $\lambda \in \mathbb{R}$, M^n is also called a *λ-hypersurface*.

In [11], Guang proved a classification theorem for complete hypersurfaces with polynomial volume growth and constant f-mean curvature, under a suitable boundedness on the norm of the second fundamental form. More recently, Miranda and Vieira [14] replaced the assumption of polynomial volume growth in Guang's result by an assumption on the integral of the second fundamental form. They also generalized Cheng-Peng's result in codimension 1 for the case of constant f-mean curvature. In our second rigidity result, we obtain a slight improvement of [14, Corollary 11] in the sense that we are not requiring that the hypersurface M^n is embedded, but just immersed in $\mathbb{G}^{n+1}$.

Theorem 1.2. *Let $X : M^n \looparrowright \mathbb{G}^{n+1}$ be a complete hypersurface immersed with constant f-mean curvature H_f in the $(n+1)$-dimensional Gaussian space $\mathbb{G}^{n+1}$. If the second fundamental form A of M^n satisfies*

$$\sup_M |A| < \gamma_2, \tag{1.6}$$

where

$$\gamma_2 = \frac{\sqrt{H_f^2 + 2} - |H_f|}{2}, \tag{1.7}$$

then M^n is a hyperplane of $\mathbb{G}^{n+1}$.

The proofs of Theorems 1.1 and 1.2 are presented in Section 4. Our approach is based on a Nishikawa type maximum principle for the drift Laplacian, which is proved in Section 3 using a more general version of the Omori-Yau maximum principle due to Chen and Qiu [4] (see Proposition 3.1). Furthermore, in Section 5, we use a maximum principle at infinity due to Alías, Caminha, and Nascimento [1] to get additional rigidity results, as well as a nonexistence result related to nonminimal submanifolds immersed with parallel weighted mean curvature vector in $\mathbb{G}^{n+p}$ (see Theorems 5.2, 5.3, and 5.4). Before, in Section 2, we recall some basic facts related to submanifolds immersed in the Euclidean space.

2. Preliminaries. Let $X : M^n \looparrowright \mathbb{R}^{n+p}$ be an n-dimensional connected submanifold immersed in the $(n+p)$-dimensional Euclidean space $\mathbb{R}^{n+p}$. We choose a local field of orthonormal frame $\{e_1, \ldots, e_{n+p}\}$ in $\mathbb{R}^{n+p}$, with dual coframe $\{\omega_1, \ldots, \omega_{n+p}\}$, such that, at each point of M^n, $e_1, \ldots, e_n$ are tangent to M^n and $e_{n+1}, \ldots, e_{n+p}$ are normal to M^n. We will use the following convention for indices

$$1 \le A, B, C, \ldots \le n+p, \quad 1 \le i, j, k, \ldots \le n, \quad \text{and}$$

$$n+1 \leq \alpha, \beta, \gamma, \ldots \leq n+p.$$

With restricting on M^n, the second fundamental form A, the curvature tensor R of M^n, and the normal curvature tensor $R^\perp$ of M^n are given by

$$\omega_{i\alpha} = \sum_j h^\alpha_{ij}\omega_j, \quad A = \sum_{i,j,\alpha} h^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha,$$
$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l,$$
$$d\omega_{\alpha\beta} = \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\alpha} - \frac{1}{2}\sum_{k,l} R^\perp_{\alpha\beta kl}\omega_k \wedge \omega_l,$$

where ω_{BC} are the connection 1-forms on $\mathbb{R}^{n+p}$. Moreover, the Gauss equation is given by

$$R_{ijkl} = \sum_\alpha \left(h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}\right). \tag{2.1}$$

Hence, denoting by H^α the components of the mean curvature vector, that is,

$$\vec{H} = \sum_\alpha H^\alpha e_\alpha = \sum_\alpha \left(\sum_k h^\alpha_{kk}\right) e_\alpha,$$

it is not difficult to verify from (2.1) that the components of the Ricci tensor R_{ik} satisfy

$$R_{ik} = \sum_\alpha H^\alpha h^\alpha_{ik} - \sum_{\alpha,j} h^\alpha_{ij}h^\alpha_{jk}. \tag{2.2}$$

3. A Nishikawa type result for the drift Laplacian. According to [4], we define the *Bakry–Émery–Ricci tensor* $\overline{\mathrm{Ric}}_f$ of a weighted manifold $\overline{M}^{n+1}_f$ as being the following extension of the standard Ricci tensor $\overline{\mathrm{Ric}}$:

$$\overline{\mathrm{Ric}}_f = \overline{\mathrm{Ric}} - \overline{\mathrm{Hess}} f. \tag{3.1}$$

Furthermore, given a hypersurface M^n immersed in $\overline{M}^{n+1}_f$, the *f-divergence operator* on M^n is defined by

$$\mathrm{div}_f(X) = e^f \mathrm{div}(e^{-f}X) \tag{3.2}$$

for all tangent vector fields X on M^n and, for a smooth function $u : M^n \to \mathbb{R}$, its *drift Laplacian* (or *f-Laplacian*) is given by

$$\Delta_f u = \mathrm{div}_f(\nabla u) = \Delta u - \langle \nabla u, \nabla f\rangle. \tag{3.3}$$

In our next result, we apply a version of the Omori-Yau maximum principle due to Chen and Qiu [4, Theorem 1] to get a sort of extension of Nishikawa's result in [16].

Proposition 3.1. *Let $(M^n, \langle\, ,\rangle)$ be an n-dimensional complete Riemannian manifold and let f be a smooth function on M^n such that* $\mathrm{Ric}_f \geq -G(r)\langle\, ,\rangle$*, where*

r is the distance function on M^n from a fixed point of it, $G : \mathbb{R} \to \mathbb{R}$ is a positive continuous function satisfying

$$\varphi(t) := \int_{\rho_0+1}^{t} \frac{dr}{\int_{\rho_0}^{r} G(s)ds + 1} \longrightarrow +\infty \quad (t \to +\infty)$$

for some positive constant $\rho_0 \in \mathbb{R}$. If $u \in \mathcal{C}^2(M)$ is a nonnegative function on M^n such that

$$\Delta_f u \geq \beta u^{1+\alpha} \tag{3.4}$$

for some positive constants $\alpha, \beta \in \mathbb{R}$, then u is identically zero on M^n.

Proof. Let $u \in \mathcal{C}^2(M)$ be a nonnegative function. We consider on M^n the function F given by

$$F = \frac{1}{(1+u)^\lambda} \tag{3.5}$$

for some constant $\lambda > 0$ which will be chosen later. We have that $0 < F \leq 1$ and, in particular, $\inf F \geq 0$.

Moreover, given any tangent vector field X on M, from (3.5), we obtain

$$\begin{aligned}\langle \nabla F, X\rangle &= X\left(\frac{1}{(1+u)^\lambda}\right) = -\lambda(1+u)^{-\lambda-1}X(u)\\ &= -\frac{\lambda}{(1+u)^{\lambda+1}}\langle \nabla u, X\rangle = \langle -\lambda F^{\frac{\lambda+1}{\lambda}}\nabla u, X\rangle,\end{aligned}$$

that is,

$$\nabla F = -\lambda F^{\frac{\lambda+1}{\lambda}}\nabla u. \tag{3.6}$$

Consequently, using (3.6), we get

$$\begin{aligned}\Delta F &= \operatorname{div}(-\lambda F^{\frac{\lambda+1}{\lambda}}\nabla u) = -\lambda \nabla u(F^{\frac{\lambda+1}{\lambda}}) - \lambda F^{\frac{\lambda+1}{\lambda}}\Delta u\\ &= -(\lambda+1)F^{\frac{1}{\lambda}}\nabla u(F) - \lambda F^{\frac{\lambda+1}{\lambda}}\Delta u\\ &= \lambda(\lambda+1)F^{\frac{2+\lambda}{\lambda}}|\nabla u|^2 - \lambda F^{\frac{\lambda+1}{\lambda}}\Delta u. \end{aligned} \tag{3.7}$$

Thus, from (3.3) and (3.7), we have

$$\begin{aligned}\Delta_f F &= \lambda(\lambda+1)F^{\frac{2+\lambda}{\lambda}}|\nabla u|^2 - \lambda F^{\frac{\lambda+1}{\lambda}}\Delta u + \lambda F^{\frac{\lambda+1}{\lambda}}\langle \nabla f, \nabla u\rangle\\ &= -\lambda F^{\frac{\lambda+1}{\lambda}}(\Delta u - \langle \nabla f, \nabla u\rangle) + \lambda(\lambda+1)F^{\frac{2+\lambda}{\lambda}}|\nabla u|^2. \end{aligned} \tag{3.8}$$

Hence, from (3.6) and (3.8), we reach at the following relation

$$\lambda F \Delta_f F = -\lambda^2 F^{\frac{2\lambda+1}{\lambda}}\Delta_f u + (\lambda+1)|\nabla F|^2. \tag{3.9}$$

On the other hand, since F is bounded on M^n, we have

$$\lim_{x\to\infty} \frac{F(x)}{\varphi(r(x))} = 0$$

since $\frac{1}{\varphi(r(x))} \longrightarrow 0$ when $r(x) \longrightarrow +\infty$. Thus, taking into account our constraint on Ric_f, we can apply [4, Theorem 1] to guarantee the existence of a sequence $\{x_m\}_{m\in\mathbb{N}} \subset M^n$ such that

$$\begin{cases} 0 \le \inf_M F \le F(x_m) < \inf_M F + \frac{1}{m}, \\ |\nabla F|(x_m) < \frac{1}{m}, \\ \Delta_f F(x_m) > -\frac{1}{m}. \end{cases} \tag{3.10}$$

Combining (3.4), (3.9), and (3.10), we obtain

$$\begin{aligned} -\frac{1}{m}\lambda F(x_m) &< \lambda F(x_m)\Delta_f F(x_m) \\ &= -\lambda^2 F^{\frac{2\lambda+1}{\lambda}}(x_m)\Delta_f u(x_m) + (\lambda+1)|\nabla F|^2(x_m) \\ &\le -\lambda^2\beta u^{1+\alpha}(x_m)F^{\frac{2\lambda+1}{\lambda}}(x_m) + (\lambda+1)|\nabla F|^2(x_m) \\ &\le (\lambda+1)|\nabla F|^2(x_m). \end{aligned} \tag{3.11}$$

Using (3.5) and making $m \longrightarrow +\infty$ in (3.11), we get

$$0 = \lim_{m\to+\infty} F^{\frac{2\lambda+1}{\lambda}}(x_m)u^{1+\alpha}(x_m) = \lim_{m\to+\infty} \frac{u^{1+\alpha}(x_m)}{(1+u(x_m))^{2\lambda+1}}. \tag{3.12}$$

At this point, taking $2\lambda = \alpha$, from (3.12), we obtain

$$\lim_{m\to\infty}\left(\frac{u(x_m)}{1+u(x_m)}\right)^{1+\alpha} = 0.$$

But, from (3.5), we also have that

$$\lim_{m\to+\infty} F(x_m) = \inf_M F \quad \text{if and only if} \quad \lim_{m\to+\infty} u(x_m) = \frac{1-(\inf_M F)^{\frac{2}{\alpha}}}{(\inf_M F)^{\frac{2}{\alpha}}}.$$

Consequently, we get $\frac{1-(\inf_M F)^{\frac{2}{\alpha}}}{(\inf_M F)^{\frac{2}{\alpha}}} = 0$ and, hence, $\inf_M F = 1$. Therefore, since $F \le 1$, we conclude that u must be identically zero on M^n. □

4. Proofs of Theorems 1.1 and 1.2. We start presenting the proof of Theorem 1.1.

Proof of Theorem 1.1. Since we are assuming that $\vec{H}_f$ is parallel in the normal bundle, from [17, Lemma 1, Eq. (4)], we have that

$$\begin{aligned} \Delta_f|A|^2 = 2|\nabla A|^2 + |A|^2 + 2\sum_{i,j,k,\alpha,\beta} H_f^\beta h_{jk}^\beta h_{ij}^\alpha h_{ik}^\alpha \\ -2\sum_{\alpha,\beta}\left(\sum_{i,j} h_{ij}^\alpha h_{i,j}^\beta\right)^2 - 2\sum_{i,j,\alpha,\beta}\left(\sum_p (h_{ip}^\alpha h_{pj}^\beta - h_{jp}^\alpha h_{pi}^\beta)\right)^2, \end{aligned} \tag{4.1}$$

where $H_f^\alpha = H^\alpha + \frac{1}{2}\langle X, e_\alpha\rangle$ and $H^\alpha = \sum_i h_{ii}^\alpha$.

On the other hand, taking into account that $p \geq 2$, from [13, Theorem 1], we get the following inequality

$$\sum_{\alpha,\beta}\left(\sum_{i,j} h_{ij}^{\alpha} h_{i,j}^{\beta}\right)^2 + \sum_{i,j,\alpha,\beta}\left(\sum_{p}(h_{ip}^{\alpha} h_{pj}^{\beta} - h_{jp}^{\alpha} h_{pi}^{\beta})\right)^2 \leq \frac{3}{2}|A|^4. \tag{4.2}$$

Thus, considering (4.2) into (4.1), we obtain

$$\begin{aligned}\Delta_f|A|^2 &\geq 2|\nabla A|^2 + |A|^2 + 2\sum_{i,j,k,\alpha,\beta} H_f^{\beta} h_{jk}^{\beta} h_{ij}^{\alpha} h_{ik}^{\alpha} - 3|A|^4 \\ &\geq \left(1 - 2|\vec{H}_f||A| - 3|A|^2\right)|A|^2. \end{aligned} \tag{4.3}$$

At this point, we note that the constant γ_1 defined in (1.4) is the positive root of the function $\zeta(t) = 1 - 2|\vec{H}_f|t - 3t^2$. So, from hypothesis (1.3), we can take a positive constant γ such that $\sup_M |A| < \gamma < \gamma_1$ and, considering the behavior of $\zeta(t)$ for $0 \leq t \leq \gamma_1$, we get

$$1 - 2|\vec{H}_f||A| - 3|A|^2 \geq \frac{1 - 2|\vec{H}_f|\gamma - 3\gamma^2}{\gamma}|A|. \tag{4.4}$$

Hence, from (4.3) and (4.4), we arrive at the following estimate

$$\Delta_f|A|^2 \geq \beta\left(|A|^2\right)^{1,5}, \tag{4.5}$$

where $\beta = \frac{1-2|\vec{H}_f|\gamma - 3\gamma^2}{\gamma}$.

Moreover, from (2.2), we get that the boundedness of $|A| = \sqrt{\sum_{\alpha,i,j}(h_{ij}^{\alpha})^2}$ implies, in particular, that the Ricci tensor of M^n is bounded from below. But, from (1.1), we have that $\mathrm{Hess} f = \frac{1}{2}$. Consequently, from (3.1), we have that the Bakry–Émery–Ricci tensor of M^n is also bounded from below.

Therefore, we are in position to apply Proposition 3.1 to conclude that $|A|$ vanishes identically on M^n, which means that M^n is an n-dimensional hyperplane of $\mathbb{G}^{n+p}$. □

We close this section with the proof of Theorem 1.2.

Proof of Theorem 1.2. Since we are assuming that H_f is constant, from [11, Lemma 2.1, Equation (2.3)], we get

$$\begin{aligned}\Delta_f|A|^2 &\geq 2|\nabla A|^2 + 2\left(\frac{1}{2} - |A|^2\right)|A|^2 - 2|H_f||\langle A^2, A\rangle| \\ &\geq 2\left(\frac{1}{2} - |A|^2\right)|A|^2 - 2|H_f|^2|A|^3. \end{aligned} \tag{4.6}$$

Thus, from (4.6), we have that

$$\Delta_f|A|^2 \geq \left(1 - 2|H_f||A| - 2|A|^2\right)|A|^2. \tag{4.7}$$

Now, we observe that the constant γ_2 defined in (1.7) is the positive root of the function $\varsigma(t) = 1 - 2|H_f|t - 2t^2$. Consequently, from hypothesis (1.6), we can choose a positive constant $\tilde{\gamma}$ such that $\sup_M |A| < \tilde{\gamma} < \gamma_2$ and, considering the behavior of $\varsigma(t)$ for $0 \leq t \leq \gamma_2$, we obtain

$$1 - 2|H_f||A| - 2|A|^2 \geq \frac{1 - 2|H_f|\tilde{\gamma} - 2\tilde{\gamma}^2}{\tilde{\gamma}}|A|. \tag{4.8}$$

Hence, from (4.7) and (4.8), we deduce that

$$\Delta_f |A|^2 \geq \tilde{\beta} \left(|A|^2\right)^{1,5}, \tag{4.9}$$

where $\tilde{\beta} = \frac{1-2|H_f|\tilde{\gamma}-2\tilde{\gamma}^2}{\tilde{\gamma}}$.

On the other hand, as in the proof of Theorem 1.1, we have that the boundedness of $|A|$ guarantees that the Bakry–Émery–Ricci tensor of M^n is bounded from below.

Therefore, we can apply once more Proposition 3.1 to infer that $|A|$ is identically zero on M^n, that is, M^n is a hyperplane of $\mathbb{G}^{n+1}$. □

5. Further results. Let M^n be a complete noncompact Riemannian manifold and let $d(\cdot, o) : M^n \to [0, +\infty)$ denote the Riemannian distance of M^n, measured from a fixed point $o \in M^n$. We say that a smooth function $u \in C^\infty(M)$ *converges to zero at infinity* when it satisfies the following condition

$$\lim_{d(x,o)\to+\infty} u(x) = 0. \tag{5.1}$$

Keeping in mind this previous concept, the following maximum principle at infinity corresponds to [1, Theorem 2.2, item (a)].

Lemma 5.1. *Let M^n be a complete noncompact Riemannian manifold and let $X \in \mathfrak{X}(M)$ be a vector field on M^n. Assume that there exists a nonnegative, non-identically vanishing function $u \in C^\infty(M)$ which converges to zero at infinity and such that $\langle \nabla u, X\rangle \geq 0$. If $\operatorname{div} X \geq 0$ on M^n, then $\langle \nabla u, X\rangle \equiv 0$ on M^n.*

Now, we are in position to present our next rigidity result.

Theorem 5.2. *Let $X : M^n \looparrowright \mathbb{G}^{n+p}$ be a complete noncompact submanifold immersed with parallel f-mean curvature vector $\vec{H}_f$ in the $(n+p)$-dimensional Gaussian space $\mathbb{G}^{n+p}$, with $p \geq 2$. If the second fundamental form A of M^n is such that $|A|$ converges to zero at infinity and $|A| \leq \gamma_1$, where γ_1 is the positive constant defined in (1.4), then M^n is a hyperplane of $\mathbb{G}^{n+p}$.*

Proof. Let us suppose by contradiction that M^n is not a hyperplane of $\mathbb{G}^{n+p}$ or, equivalently, that $|A|$ does not vanish identically on M^n. Taking the vector field $X = e^{-f}\nabla|A|^2$, since we are assuming that $|A| \leq \gamma_1$, from (3.2), (3.3), and (4.3), we obtain that

$$\operatorname{div} X = e^{-f}\Delta_f|A|^2 \geq 0.$$

Moreover, choosing the smooth function $u = |A|^2$, we also have that

$$\langle \nabla u, X\rangle = e^{-f}|\nabla|A|^2|^2 \geq 0. \tag{5.2}$$

Consequently, since we are also supposing that $|A|$ converges to zero at infinity, we can apply Lemma 5.1 to get that $\langle \nabla u, X\rangle \equiv 0$ on M^n. So, returning to

(5.2), we conclude that $|A|$ must be constant on M^n. Therefore, from (5.1), we have that $|A|$ is identically zero on M^n and, hence, we reach a contradiction. □

Taking into account inequality (4.7), it is not difficult to see that we can reason as in the proof of Theorem 5.2 to obtain the following rigidity result for codimension 1, which is in consonance with [6, Theorem 1.1].

Theorem 5.3. *Let $X : M^n \looparrowright \mathbb{G}^{n+1}$ be a complete noncompact hypersurface immersed with constant f-mean curvature H_f in the $(n+1)$-dimensional Gaussian space $\mathbb{G}^{n+1}$. If the second fundamental form A of M^n is such that $|A|$ converges to zero at infinity and $|A| \leq \gamma_2$, where γ_2 is the positive constant defined in (1.7), then M^n is a hyperplane of $\mathbb{G}^{n+1}$.*

Finally, we obtain the following nonexistence result.

Theorem 5.4. *There does not exist a complete noncompact nonminimal submanifold $X : M^n \looparrowright \mathbb{G}^{n+p}$ immersed with parallel f-mean curvature vector $\vec{H}_f$ in the $(n+p)$-dimensional Gaussian space $\mathbb{G}^{n+p}$ such that $|\vec{H}|$ converges to zero at infinity and, in the points $x \in M^n$ where $|\vec{H}(x)| \neq 0$, $|A|^2 \leq \frac{|\vec{H}|}{2(|\vec{H}|+|\vec{H}_f|)}$.*

Proof. Let us suppose by contradiction the existence of such a submanifold M^n. Taking the vector field $X = e^{-f}\nabla|H|^2$ and since we are assuming that $|A|^2 \leq \frac{|\vec{H}|}{2(|\vec{H}|+|\vec{H}_f|)}$ in the points where $|\vec{H}|$ does not vanish, from (3.2), (3.3), and [17, Lemma 1, equation (5)], we obtain that

$$\mathrm{div} X = e^{-f}\Delta_f|\vec{H}|^2 \geq |\vec{H}|\left(|\vec{H}| - 2(|\vec{H}| + |\vec{H}_f|)|A|^2\right) \geq 0.$$

Choosing the smooth function $u = |\vec{H}|^2$, we also have that

$$\langle \nabla u, X\rangle = e^{-f}|\nabla|\vec{H}|^2|^2 \geq 0. \tag{5.3}$$

Thus, since we are also supposing that $|\vec{H}|$ converges to zero at infinity, we can apply once more Lemma 5.1 to get that $\langle \nabla u, X\rangle \equiv 0$ on M^n. Consequently, returning to (5.3), we conclude that $|\vec{H}|$ must be constant on M^n. Therefore, from (5.1), we have that $|\vec{H}|$ is identically zero on M^n and, hence, we reach a contradiction with the hypothesis that M^n is not a minimal submanifold. □

Remark 5.5. Related to the nonexistence result obtained in Theorem 5.4, it is worth to observe that Angenent [2] proved the existence of embedded self-shrinkers from $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ into $\mathbb{R}^{n+1}$ satisfying the hypothesis $\inf H^2 = 0$. Furthermore, Cheng and Peng [6] obtained some classification theorems concerning complete self-shrinkers whose squared norm of the second fundamental form is constant and such that $|\vec{H}| > 0$ and the principal normal $\nu = \frac{\vec{H}}{|\vec{H}|}$ is parallel.

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References

[1] Alías, L.J., Caminha, A., do Nascimento, F.Y.: A maximum principle at infinity with applications to geometric vector fields. J. Math. Anal. Appl. **474**, 242–247 (2019)

[2] Angenent, S.: Shrinking doughnuts. In: Nonlinear Diffusion Equations and Their Equilibrium States, 3 (Gregynog, 1989), pp. 21–38. Progr. Nonlinear Differential Equations Appl., 7, Birkhäuser Boston, Boston, MA (1992)

[3] Cao, H.D., Li, H.: A gap theorem for self-shrinkers of the mean curvature flow in arbitrary codimension. Calc. Var. PDE **46**, 879–889 (2013)

[4] Chen, Q., Qiu, H.: Rigidity of self-shrinkers and translating solitons of mean curvature flows. Adv. Math. **294**, 517–531 (2016)

[5] Cheng, Qing-Ming., Ogata, Shiho: 2-Dimensional complete self-shrinkers in $\mathbb{R}^3$. Math. Z. **284**, 537–542 (2016)

[6] Cheng, Q.M., Peng, Y.: Complete self-shrinkers of the mean curvature flow. Calc. Var. PDE **52**, 497–506 (2015)

[7] Colding, T., Minicozzi, W.P., II.: Generic mean curvature flow I: generic singularities. Ann. Math. **175**, 755–833 (2012)

[8] Ding, Q., Xin, Y.L.: The rigidity theorems of self-shrinkers. Trans. Amer Math. Soc. **366**, 5067–5085 (2014)

[9] Ecker, K., Huisken, G.: Mean curvature evolution of entire graphs. Ann. Math. **130**, 453–471 (1989)

[10] Espinar, J.M.: Gradient Schrödinger operators, manifolds with density and applications. J. Math. Anal. Appl. **455**, 1505–1528 (2017)

[11] Guang, Q.: Gap and rigidity theorems of λ-hypersurfaces. Proc. Amer Math. Soc. **146**, 4459–4471 (2018)

[12] Le, N.Q., Sesum, N.: Blow-up rate of the mean curvature during the mean curvature flow and a gap theorem for self-shrinkers. Comm. Anal. Geom. **19**, 633–659 (2011)

[13] Li, A.M., Li, J.M.: An instrinsic rigidity theorem for minimal submanifolds in a sphere. Arch. Math. (Basel) **58**, 582–594 (1992)

[14] Miranda, I., Vieira, M.: Constant weighted mean curvature hypersurfaces in shrinking Ricci solitons. arXiv:2108.13601v1 (2021)

[15] Morgan, F.: Manifolds with density. Not. Amer Math. Soc. **52**, 853–858 (2005)

[16] Nishikawa, S.: On Maximal spacetime hypersurfaces in a Lorentzian manifold. Nagoya Math. J. **95**, 117–124 (1984)

[17] Wang, H., Xu, H., Zhao, E.: Submanifolds with parallel Gaussian mean curvature vector in Euclidean spaces. Manuscr. Math. **161**, 439–465 (2020)

[18] Wang, L.: A Benstein type theorem for self-similar shrinkers. Geom. Dedicata **15**, 297–303 (2011)

[19] Wei, G., Wylie, W.: Comparison geometry for the Bakry–Émery–Ricci tensor. J. Differential Geom. **83**, 377–405 (2009)

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