Arch. Math. 117 (2021), 423–432

0003-889X/21/040423-10
published online July 15, 2021
https://doi.org/10.1007/s00013-021-01633-w

Archiv der Mathematik

On the generic Conley conjecture

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Abstract. In this paper, we treat an open problem related to the number of periodic orbits of Hamiltonian diffeomorphisms on closed symplectic manifolds, the so-called (generic) Conley conjecture. The generic Conley conjecture states that generically Hamiltonian diffeomorphisms have infinitely many simple contractible periodic orbits. We prove the generic Conley conjecture for very wide classes of symplectic manifolds. Our proof is based on applications of the Birkhoff–Moser fixed point theorem and Floer homology theory.

Mathematics Subject Classification. 53D05, 37J12.

Keywords. Hamiltonian dynamics, Periodic orbit, Conley conjecture.

1. Introduction and main results. In this section, we briefly explain the main theme of this paper. Precise definitions and notations are given in the next section. The information of periodic orbits of Hamiltonian diffeomorphisms is very important in Hamiltonian dynamics. The Conley conjecture was originally stated for Hamiltonian diffeomorphisms on the standard torus $(\mathbb{T}^{2n}, \omega_0)$ [3]. It states that any Hamiltonian diffeomorphism on $(\mathbb{T}^{2n}, \omega_0)$ has infinitely many simple contractible periodic orbits (simple means that it is not an iterated periodic orbit of the lower period). It is easy to see that this conjecture can not be generalized to any closed symplectic manifold. For example, an irrational rotation on the standard sphere $S^2 \subset \mathbb{R}^3$ has only two contractible periodic orbits, the north pole and the south pole.

However, the Conley conjecture was proved for wide classes of closed symplectic manifolds. For example, the Conley conjecture holds on symplectically aspherical manifolds, negatively monotone symplectic manifolds, and symplectic manifolds with vanishing spherical Chern class [5,7–9,11,16]. So, today's Conley conjecture is a conjecture which states that every Hamiltonian diffeomorphism has infinitely many simple contractible periodic orbits on "almost all" closed symplectic manifolds.

Another variant of the above Conley conjecture is the so-called generic Conley conjecture [6,8,9]. The generic Conley conjecture states that "almost all" Hamiltonian diffeomorphisms have infinitely many simple contractible periodic orbits on every closed symplectic manifold. The Conley conjecture and the generic Conley conjecture state that Hamiltonian diffeomorphisms with finitely many simple periodic orbits (like the irrational rotation on the sphere S^2) are very rare. In summary, we have the following two conjectures.

Conjecture 1 ((Generic) Conley conjecture).

(1) *On "almost all" closed symplectic manifolds, every Hamiltonian diffeomorphism has infinitely many simple contractible periodic orbits.*
(2) *On every closed symplectic manifolds, "almost all" Hamiltonian diffeomorphisms have infinitely many simple contractible periodic orbits.*

In this paper, we study Conjecture 1 (2), the generic Conley conjecture. The statement of our main result is as follows.

Theorem 1.1. *Let (M, ω) be a $2n$-dimensional closed symplectic manifold and let $N \in \mathbb{N} \cup \{\infty\}$ be the minimum Chern number of (M, ω). Assume that (M, ω) satisfies at least one of the following conditions.*

(1) *n is odd.*
(2) *$H_{odd}(M : \mathbb{Q}) \neq 0$.*
(3) *$N > 1$.*

Then there is a C^∞-dense and C^∞-residual (=contains a countable intersection of C^∞-open dense subsets) subset $\mathcal{U} \subset Ham(M, \omega)$ such that any element of $\mathcal{U}$ has infinitely many simple contractible periodic orbits.

Note that the above conditions (1), (2), and (3) cover almost all closed symplectic manifolds.

Remark 1.1. The case (2) of Theorem 1.1 was also proved in [6, Proposition 1.6]. The case (3) is a generalization of [6, Theorem 1.2] where Ginzburg and Gürel proved the generic Conley conjecture for $N \geq n+1$. The proof of [6, Proposition 1.6] was an application of the Birkhoff–Moser fixed point theorem and the proof of [6, Theorem 1.2] was an application of "resonance relation" proved in [10]. Our proof of Theorem 1.1 is a modification of the former proof.

Remark 1.2. The key part of Theorem 1.1 is the assertion that the set $\mathcal{U}$ is C^∞-(but not just C^1-)generic. The C^1-generic existence of infinitely many periodic points readily follows from the C^1 closing lemma in [15]. Furthermore, a stronger result holds in dimension two. Asaoka-Irie proved a C^∞ closing lemma for Hamiltonian diffeomorphisms of closed surfaces in [1]. In particular, periodic points are dense C^∞-generically in dimension two. The C^∞ closing problem for Hamiltonian diffeomorphisms in dimension ≥ 4 is an open problem. However, Herman's example in [12] indicates that there is little hope to extend the C^∞ closing lemma to higher dimensions.

2. Preliminaries. In this section, we explain notations and terminologies used in this paper.

2.1. Elementary notations. Let (M, ω) be a symplectic manifold, so M is a finite-dimensional C^∞-manifold and $\omega \in \Omega^2(M)$ is a symplectic form on M. In this paper, we always assume that M is a closed manifold.

For any C^∞-function $H \in C^\infty(M)$, we define the Hamiltonian vector field X_H by the following relation:

$$\omega(X_H, \cdot) = -dH.$$

We can also consider a S^1-dependent (=1-periodic) Hamiltonian function H and a Hamiltonian vector field X_H by the same formula. The time 1 flow of X_H is called a Hamiltonian diffeomorphism generated by H. We denote this flow by ϕ_H. The set of all Hamiltonian diffeomorphisms is called the Hamiltonian diffeomorphism group and we denote the Hamiltonian diffeomorphism group of (M, ω) by $\mathrm{Ham}(M, \omega)$, i.e.,

$$\mathrm{Ham}(M, \omega) = \{\phi_H \mid H \in C^\infty(S^1 \times M)\}.$$

We also consider "iterations" of H and ϕ_H. For any integer $k \in \mathbb{N}$, we define $H^{(k)}$ as

$$H^{(k)} = kH(kt, x).$$

It is straightforward to see that $\phi_{H^{(k)}} = (\phi_H)^k$. Let $P^l(H)$ be the space of l-periodic contractible periodic orbits of X_H, i.e.,

$$P^l(H) = \{x : S^1_l \to M \mid \dot{x}(t) = X_{H_t}(x(t)), x : \text{contractible}\},$$
$$S^1_l = \mathbb{R}/l \cdot \mathbb{Z}.$$

It is also straightforward to see that there is a one-to-one correspondence between $P^k(H)$ and $P^1(H^{(k)})$. We abbreviate $P^1(H)$ to $P(H)$. An l-periodic orbit $x \in P^l(H)$ is called simple if there is no l'-periodic orbit $y \in P^{l'}(H)$ which satisfies the following conditions:

$$l = l' \cdot m \quad (l', m \in \mathbb{N}),$$
$$x(t) = y(\pi_{l,l'}(t)).$$

Here $\pi_{l,l'} : S_l \to S_{l'}$ is the natural projection. So a periodic orbit is simple if and only if it is not an iterated periodic orbit of the lower period.

Next, we explain the definition of the minimum Chern number N. A symplectic manifold (M, ω) becomes an almost complex manifold, and its tangent bundle has the natural first Chern class $c_1(TM) \in H^2(M : \mathbb{Z})$. The minimum Chern number $N \in \mathbb{N} \cup \{+\infty\}$ is the positive generator of $c_1(TM)|_{\pi_2(M)}$. Note that if the image is zero, N is defined by $N = +\infty$.

2.2. Floer homology and degrees of periodic orbits. In this subsection, we explain basic notations of Floer homology theory and the Conley-Zehnder index of periodic orbits. Let H be a 1-periodic Hamiltonian function. We call H non-degenerate if the differential map $d\phi_H : TM_x \to TM_x$ does not have 1 as an eigenvalue for any fixed point $x \in \mathrm{Fix}(\phi_H)$. Note that H is non-degenerate if and only if $\mathrm{graph}(\phi_H) \subset M \times M$ is transverse to the diagonal $\Delta_M \subset M \times M$.

We construct the Novikov covering of $P(H)$ as

$$\widetilde{P(H)} = \{(u, x) \mid x \in P(H), u : D^2 \to M, \partial u = x\}/\sim$$

where D^2 is the two dimensional disc $D^2 \subset \mathbb{R}^2$ and the equivalence relation $\sim$ is defined as

$$(u, x) \sim (v, y) \Longleftrightarrow \begin{cases} x = y, \\ \omega(u \sharp \overline{v}) = 0, \\ c_1(u \sharp \overline{v}) = 0. \end{cases}$$

Here $\overline{v}$ is the disc with the opposite orientation on the domain and $u \sharp \overline{v}$ is the glued sphere. Each $[u, x] \in \widetilde{P(H)}$ has a Conley-Zehnder index $\mu_{CZ}([u, x]) \in \mathbb{Z}$. We normalize μ_{CZ} so that the Conley-Zehnder index of a local maximum of a C^2-small Morse function is equal to n. The Conley-Zehnder index gives a grading of the Floer chain complex and the Floer homology. We also have the action functional A_H on $\widetilde{P(H)}$ as follows:

$$A_H([u, x]) = -\int_{D^2} u^*\omega + \int_0^1 H(t, x(t))dt.$$

Then the Floer chain complex $CF_*(H)$ is defined as

$$CF_*(H) = \left\{ \sum_{z \in \widetilde{P(H)}} a_z \cdot z \;\middle|\; a_z \in \mathbb{Q}, \forall C \in \mathbb{R}, \sharp\{z \in \widetilde{P}(H) \mid a_z \neq 0, A_H(z) > C\} < \infty \right\}.$$

The boundary operator d_F has the following form:

$$d_F(z) = \sum_{w \in \widetilde{P(H)}} n(z, w)w.$$

The coefficient $n(z, w) \in \mathbb{Q}$ is the number of solutions of the following Floer equation modulo the natural $\mathbb{R}$-action [4, 13]. Let J_t be an almost complex structure on M parametrized by $t \in S^1$,

$$\begin{gathered} z = [v_-, x_-], w = [v_+, x_+], \\ u : \mathbb{R} \times S^1 \longrightarrow M, \\ \partial_s u(s, t) + J_t(u(s, t))(\partial_t u(s, t) - X_{H_t}(u(s, t))) = 0, \\ \lim_{s \to -\infty} u(s, t) = x_-(t), \lim_{s \to +\infty} u(s, t) = x_+(t), (v_- \sharp u, x_+) \sim (v_+, x_+). \end{gathered}$$

The Floer homology $HF_*(H)$ is the homology of the chain complex $(CF_*(H), d_F)$. We introduce the notion of the Novikov ring of (M, ω). We define an abelian group Γ by

$$\Gamma = \frac{\pi_2(M)}{\mathrm{Ker}\omega \cap \mathrm{Ker}c_1}$$

where $\omega : \pi_2(M) \to \mathbb{R}$ is the integration of the symplectic form ω and $c_1 : \pi_2(M) \to \mathbb{Z}$ is the integration of the first Chern class. We define the degree of $u \in \Gamma$ by $-2c_1(u)$. The Novikov ring $\Lambda_{(M,\omega)}$ is defined as the set of possibly infinite sums of Γ with suitable convergence, i.e.,

$$\Lambda_{(M,\omega)} = \left\{ \sum_{u\in\Gamma} a_u \cdot u \;\middle|\; a_u \in \mathbb{Q}, \forall C \in \mathbb{R}, \sharp\{u \in \Gamma \mid a_u \neq 0, \omega(u) < C\} < \infty \right\}.$$

Then the Floer homology is isomorphic to the singular homology group with the Novikov ring coefficient [4,13].

$$HF_*(H) \cong H_{*-n}(M : \mathbb{Q}) \otimes \Lambda_{(M,\omega)}.$$

3. Generic Conley conjecture. We prove Theorem 1.1 in this section. Throughout this section, we assume that (M,ω) is a $2n$-dimensional closed symplectic manifold with the minimum Chern number N and it also satisfies at least one of the following conditions.

(1) n is odd.
(2) $H_{odd}(M : \mathbb{Q}) \neq 0$.
(3) $N > 1$.

The purpose of this section is to construct a subset $X \subset \mathrm{Ham}(M,\omega)$ and a family of subsets $\{Y_k \subset \mathrm{Ham}(M,\omega)\}$ $(1 \leq k < +\infty)$ which satisfy the following conditions.

- $X \subset \mathrm{Ham}(M,\omega)$ is a C^∞-dense subset.
- $Y_k \subset \mathrm{Ham}(M,\omega)$ are C^∞-open dense subsets.
- Any element of X has infinitely many simple contractible periodic orbits.
- $X = \bigcap_{k=1}^{\infty} Y_k$ holds.

The above conditions imply that X is a C^∞-residual subset of $\mathrm{Ham}(M,\omega)$ and generically Hamiltonian diffeomorphisms have infinitely many simple contractible periodic orbits.

As in [6], our proof is based on applications of the Birkhoff–Moser fixed point theorem (local theory) and Floer homology theory (global theory). Roughly speaking, the Birkhoff–Moser fixed point theorem guarantees infinitely many periodic orbits of a symplectic map near non-hyperbolic fixed points which satisfies some generic conditions. For the reader's convenience, we briefly recall the statement and properties of the Birkhoff–Moser fixed point theorem.

Theorem 3.1 (Birkhoff–Moser fixed point theorem [14]). *Let ϕ be a symplectic map defined in an open neighborhood of the origin ($= p$) in $(\mathbb{R}^{2n}, \omega_0)$ and the origin is a fixed point of ϕ. Here ω_0 is the standard symplectic form $\sum_{i=1}^n x_i \wedge y_i$ on $\mathbb{R}^{2n}$. Let $\lambda_1, \ldots, \lambda_m, \lambda_1^{-1}, \ldots, \lambda_m^{-1}$ be all the eigenvalues of the differential map*

$$d\phi_p : T_pM \longrightarrow T_pM$$

on the unit circle in $\mathbb{C}$. Assume that ϕ satisfies the following conditions:

(1) $m \geq 1$.
(2) $\prod_{k=1}^m \lambda_k^{j_k} \neq 1$ *for* $1 \leq \sum_{k=1}^m |j_k| \leq 4$.
(3) *The Taylor coefficient of ϕ up to order 3 satisfies a non-degenerate condition.*

Then ϕ possesses infinitely many periodic orbits in any neighborhood of p.

The meaning of "non-degenerate" in (3) is difficult to state briefly because its meaning becomes clear in the proof of the theorem. We just introduce an example of the "non-degenerate" condition.

Example 3.1. (Non-degeneracy condition [14]) Let ϕ be a symplectic map defined in an open neighborhood of the origin in $(\mathbb{R}^{2n}, \omega_0)$ and the origin is a fixed point of ϕ. Assume that ϕ can be written in the following form:

$$\begin{cases} \phi((x_1, \ldots, x_n, y_1, \ldots, y_n)) = (x_1^{(1)}, \ldots, x_n^{(1)}, y_1^{(1)}, \ldots, y_n^{(1)}), \\ x_k^{(1)} = x_k \cos \Phi_k - y_k \sin \Phi_k + f_k, \\ y_k^{(1)} = x_k \sin \Phi_k + y_k \cos \Phi_k + f_{k+n}, \\ \Phi_k = \alpha_k + \sum_{l=1}^{n} \beta_{kl}(x_l^2 + y_l^2). \end{cases}$$

The error terms f_k are assumed to have vanishing derivatives up to order 3 at the origin. Then non-degeneracy means that the matrix (β_{kl}) is non-singular.

Moser first proved the Birkhoff–Moser fixed point theorem for the above special case. Then he proved that general cases can be reduced to this special case. So roughly speaking, "non-degenerate" means that it can be reduced to the above form so that the matrix (β_{kl}) is non-singular.

The Birkhoff–Moser fixed point theorem (and its proof in [14]) implies the following fact. Let $x \in P(H)$ be a non-degenerate contractible periodic orbit of a Hamiltonian function $H \in C^\infty(S^1 \times M)$. We also assume that there is at least one eigenvalue of the differential map

$$d\phi_H : T_{x(0)}M \longrightarrow T_{x(0)}$$

on the unit circle and all eigenvalues on the unit circle are pairwise distinct. Then we can perturb H to $\widetilde{H}$ near x so that it satisfies all required conditions in the statement of the Birkhoff–Moser fixed point theorem. Moreover, these conditions are satisfied in a sufficiently small open neighborhood of $\phi_{\widetilde{H}}$ and hence all of them possess infinitely many simple contractible periodic orbits.

We apply this observation to our proof of Theorem 1.1. Let $\mathcal{H}_{sn} \subset \mathrm{Ham}(M, \omega)$ be the set of strongly non-degenerate Hamiltonian diffeomorphisms (strongly non-degenerate means any iteration of it is non-degenerate). We divide $\mathcal{H}_{sn}$ into the following three pairwise disjoint subsets:

$$\mathcal{H}_{sn}^{(1)} = \left\{ \phi \in \mathcal{H}_{sn} \;\middle|\; \begin{array}{l} \text{the number of simple contractible periodic orbits is finite} \\ \text{and all contractible periodic orbits are hyperbolic} \end{array} \right\},$$

$$\mathcal{H}_{sn}^{(2)} = \left\{ \phi \in \mathcal{H}_{sn} \;\middle|\; \begin{array}{l} \text{the number of simple contractible periodic orbits is finite} \\ \text{and at least one contractible periodic orbit is non-hyperbolic} \end{array} \right\},$$

$$\mathcal{H}_{sn}^{(3)} = \left\{ \phi \in \mathcal{H}_{sn} \;\middle|\; \phi \text{ possesses infinitely many simple contractible periodic orbits} \right\}.$$

First, we prove that $\mathcal{H}_{sn}^{(1)}$ is empty. We fix $\phi \in \mathcal{H}_{sn}^{(1)}$ and let $\{x_l, \ldots, x_l\}$ be the set of all simple contractible periodic orbits of ϕ and let $p_1, \ldots, p_l \in \mathbb{N}$ be their periods. We also choose a common multiple k of $p_1, \ldots, p_l$. Then all periodic orbits of $\psi' = \phi^{(k)}$ are 1-periodic orbits and all of them are hyperbolic. For any capped periodic orbit $\bar{z}$, we have the equation

$$\mu_{CZ}(\bar{z}) = \Delta_{\psi'}(\bar{z})$$

where $\Delta_{\psi'}(\bar{z})$ is the mean index [16]. This implies that

$$\mu_{CZ}(\bar{z}^m) = m\mu_{CZ}(\bar{z})$$

holds for any $m \in \mathbb{N}$. For the iteration $\psi = \psi'^{(2N)}$ and any capped periodic orbit of ψ, the same equation $\mu_{CZ}(\bar{z}) = \Delta_{\psi}(\bar{z})$ holds. Let $\{y_1, \ldots, y_l\}$ be all contractible periodic orbits of ψ. Note that they are $2N$-times iterations of $\{x_1^{(\frac{k}{p_1})}, \ldots, x_l^{(\frac{k}{p_l})}\}$. This means that any capped periodic orbit $\bar{z}$ of $\{y_1, \ldots, y_l\}$ has mean index $\Delta_{\psi}(\bar{z}) = 2Nm$ ($m \in \mathbb{Z}$). This implies that the Conley-Zehnder index of any capped periodic orbit is divided by $2N$ and $HF_{odd}(\psi) = 0$ holds. If n is an odd integer, this is a contradiction because $HF_n(\psi) \neq 0$ holds. So, $\mathcal{H}_{sn}^{(1)}$ is empty if n is odd.

Next, assume that $H_{odd}(M : \mathbb{Q}) \neq 0$ holds. Without loss of generality, we assume that n is an even integer. Then the isomorphism

$$HF_*(\psi) \cong H_{*-n}(M : \mathbb{Q}) \otimes \Lambda_{(M,\omega)}$$

implies that there is at least one capped periodic orbit of ψ whose Conley-Zehnder index is odd. This is a contradiction. So $\mathcal{H}_{sn}^{(1)}$ is empty if $H_{odd}(M : \mathbb{Q})$ $\neq 0$ holds.

Assume that $N > 1$ holds. Without loss of generality, we assume that n is even. Note that $H_{n+2}(M : \mathbb{Q}) \neq 0$ holds in this case. This implies $HF_2(\psi) \neq 0$, but this is impossible because the Conley-Zehnder index of any capped periodic orbit can be divided by $2N$. So we have proved that $\mathcal{H}_{sn}^{(1)}$ is empty in all cases.

Next we fix $\phi \in \mathcal{H}_{sn}^{(2)}$. Let $\{x_1, \ldots, x_l\}$ be the set of all simple periodic orbits and let $p_1, \ldots, p_l$ be their periods. We divide $\{x_1, \ldots, x_l\}$ into hyperbolic periodic orbits and non-hyperbolic periodic orbits. Let $\{x_1, \ldots, x_{l'}\}$ be the set of all non-hyperbolic periodic orbits. We perturb ϕ to $\widetilde{\phi}$ so that all differential maps

$$(d\phi)^{p_i} : T_{x_i(0)}M \longrightarrow T_{x_i(0)}M$$

have $2n$ pairwise distinct eigenvalues (see the arguments in the proof of [2, Lemma 7.1.5]). The perturbed $\widetilde{\phi}$ may not be strongly non-degenerate and the perturbed periodic orbits $\{\widetilde{x_1}, \ldots, \widetilde{x_{l'}}\}$ may not be non-hyperbolic. We prove that at least one $\widetilde{x}_i$ is non-hyperbolic. Note that $\widetilde{\phi}$ may have more than l simple contractible orbits, but we can assume that the period of a "new" periodic orbit is much greater than $2Nk$. So the existence of "new" periodic orbits does not influence our arguments. Assume that all $\widetilde{x}_i$ are hyperbolic. As in the proof of $\mathcal{H}_{sn}^{(1)} = \emptyset$, we fix $\psi = \widetilde{\phi}^{(2N\times k)}$ where k is a common multiple of $p_1, \ldots, p_l$. Then each periodic orbit $\widetilde{x}_i^{(2N\times \frac{k}{p_i})}$ of ψ has a capping $\bar{z}_i$ so that $\mu_{CZ}(\bar{z}_i) = 0$.

This is a contradiction as in the proof of $\mathcal{H}_{sn}^{(1)} = \emptyset$. So at least one of $\widetilde{x_i}$ is a non-hyperbolic periodic orbit.

Let $\widetilde{x_1}$ be a non-hyperbolic periodic orbit. We can perturb $\widetilde{\phi}$ so that $\widetilde{x_1}$ satisfies all required conditions in the statement of the Birkhoff–Moser fixed point theorem. These arguments imply that we can choose a sequence $\{\phi_k\}_{k\in\mathbb{N}}$ and open neighborhoods W_k of ϕ_k which satisfy the following conditions:

- $\phi_k \longrightarrow \phi$ in the C^∞-topology.
- Any element of W_k satisfies all required conditions in the statement of the Birkhoff–Moser fixed point theorem and hence possesses infinitely many simple contractible periodic orbits.

We define $V(\phi)$ and $V^{(k)}(\phi)$ $(k \in \mathbb{N})$ for $\phi \in \mathcal{H}_{sn}^{(2)}$ as

$$V(\phi) = \bigcup_{i=1}^{\infty} W_i,$$
$$V^{(k)}(\phi) = V(\phi).$$

Next we fix $\phi \in \mathcal{H}_{sn}^{(3)}$. There are the following two possibilities:

(1) There is an open neighborhood U of ϕ (in $\mathrm{Ham}(M,\omega)$) such that

$$U \cap \mathcal{H}_{sn} \subset \mathcal{H}_{sn}^{(3)}$$

holds.

(2) There is no open neighborhood U as above. In other words, we can choose a sequence $\{\phi_k\}_{k\in\mathbb{N}} \subset \mathcal{H}_{sn}^{(2)}$ such that $\phi_k \to \phi$ holds.

In the case of (2), we define $V(\phi)$ and $V^{(k)}(\phi)$ $(k \in \mathbb{N})$ as

$$V(\phi) = \bigcup_{k=1}^{\infty} V(\phi_k),$$
$$V^{(k)}(\phi) = V(\phi).$$

In the case of (1), we define $V(\phi)$ and $V^{(k)}(\phi)$ $(k \in \mathbb{N})$ as

$$V(\phi) = U \cap \mathcal{H}_{sn},$$
$$V^{(k)}(\phi) = \{\psi \in U \mid \psi, \dots, \psi^k \text{ are non-degenerate}\}.$$

Then $V^{(k)}(\phi) \subset U$ is open dense and $V(\phi) = \bigcap_{k=1}^{\infty} V^{(k)}(\phi)$ holds. We can define X and Y_k as

$$X = \bigcup_{\phi\in\mathcal{H}_{sn}} V(\phi),$$
$$Y_k = \bigcup_{\phi\in\mathcal{H}_{sn}} V^{(k)}(\phi).$$

X and $\{Y_k\}$ satisfy the required conditions and we proved Theorem 1.1. □

Acknowledgements. This work was carried out during my stay as a research fellow at the National Center for Theoretical Sciences. The author thanks NCTS for a great research atmosphere and a lot of support. He also gratefully acknowledges his teacher Kaoru Ono for continuous supports and Victor L. Ginzburg for checking the draft and giving me comments and advice. He is supported by the Research Fellowships of Japan Society for the Promotion of Science for Young Scientists.

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Received: 9 February 2021

Revised: 31 May 2021

Accepted: 11 June 2021.