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Four-dimensional quadratic forms over $\mathbb{C}((t))(X)$

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Abstract. For quadratic forms in 4 variables defined over the rational function field in one variable over $\mathbb{C}((t))$, the validity of the local-global principle for isotropy with respect to different sets of discrete valuations is examined.

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1. Introduction. Let E be a field of characteristic different from 2 and let $E(X)$ denote the rational function field in one variable over E.

For $E = \mathbb{C}((t))$, the field of Laurent series in one variable over the complex numbers, the quadratic form

$$Y_1^2 + tY_2^2 + tY_3^2 + X(Y_1^2 + Y_2^2 + tY_4^2)$$

in the variables Y_1, Y_2, Y_3, Y_4 over $E(X)$ has no non-trivial zero, but it has a non-trivial zero over the completion of $E(X)$ with respect to any non-trivial valuation on $E(X)$ that is trivial on E. This is in contrast to the situation when E is a finite field, by the Hasse-Minkowski theorem (see [6, Chapter VI, Theorem 66.1]). Note that, in both cases, the field E has a unique extension of each degree in a fixed algebraic closure.

By a $\mathbb{Z}$-*valuation* we mean a valuation with value group $\mathbb{Z}$. A quadratic form is *isotropic* if it has a non-trivial zero, otherwise it is *anisotropic*. Without any restrictions on the base field E other than $\mathsf{char}(E) \neq 2$, any anisotropic quadratic form over $E(X)$ of dimension at most 3 remains anisotropic over the

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completion of $E(X)$ with respect to some $\mathbb{Z}$-valuation on $E(X)$ that is trivial on E; this follows for example from Milnor's exact sequence [4, Theorem IX.3.1]. The case of 4-dimensional quadratic forms is the first case over $E(X)$ where the validity of such a local-global principle for isotropy depends on the base field E.

When E is a non-dyadic local field, using a result of Lichtenbaum [5], one obtains that a 4-dimensional anisotropic quadratic form over $E(X)$ remains anisotropic over the completion of $E(X)$ with respect to some $\mathbb{Z}$-valuation on $E(X)$ that is trivial on E (see [1, Remark 3.8]). This resembles the case where E is a finite field.

In contrast to the situations where E is a finite field or a local field, for $E = \mathbb{C}(\!(t)\!)$, the example of the quadratic form above shows that the local-global principle for isotropy of 4-dimensional quadratic forms over $E(X)$ fails with respect to $\mathbb{Z}$-valuations that are trivial on E. However, anisotropy of this quadratic form can be detected over the larger field $\mathbb{C}(X)(\!(t)\!)$ by using Springer's theorem (see [4, Proposition VI.1.9]).

Consider the more general situation where the field E is complete with respect to a non-dyadic $\mathbb{Z}$-valuation v. In this case, a local-global principle for isotropy was obtained in [1] using a geometric setup. Let $\mathcal{O}_v$ denote the valuation ring of v. By a *model for $E(X)$ over $\mathcal{O}_v$* we mean a two-dimensional integral normal projective flat $\mathcal{O}_v$-scheme $\mathscr{X}$ whose function field is isomorphic to $E(X)$. Codimension-one points on a model of $E(X)$ over $\mathcal{O}_v$ correspond to certain $\mathbb{Z}$-valuations on $E(X)$. For a model $\mathscr{X}$ of $E(X)$ over $\mathcal{O}_v$, let $\Omega_{\mathscr{X}}$ denote the set of $\mathbb{Z}$-valuations given by codimension-one points of $\mathscr{X}$. Consider the set $\Omega = \bigcup_{\mathscr{X}} \Omega_{\mathscr{X}}$ where the union is taken over all models $\mathscr{X}$ of $E(X)$ over $\mathcal{O}_v$. It follows from [1, Theorem 3.1 and Remark 3.2] that an anisotropic quadratic form over $E(X)$ remains anisotropic over the completion of $E(X)$ with respect to some $\mathbb{Z}$-valuation in Ω. One may ask whether this remains true if one replaces Ω by $\Omega_{\mathscr{X}}$ for some well-chosen model $\mathscr{X}$ of $E(X)$ over $\mathcal{O}_v$.

The aim of this note is to show that this is not the case: if the residue field of v is separably closed, then, for any model $\mathscr{X}$ of $E(X)$ over $\mathcal{O}_v$, there exists an anisotropic 4-dimensional quadratic form over $E(X)$ which is isotropic over the completion of $E(X)$ with respect to any $w \in \Omega_{\mathscr{X}}$ (Corollary 2). Let $\pi \in \mathcal{O}_v$ be a uniformiser of v. For any model $\mathscr{X}$ of $E(X)$ over $\mathcal{O}_v$, the set $\{w(\pi) \mid w \in \Omega_{\mathscr{X}}\}$ is finite and hence it has an upper bound. However, for any positive integer r, the quadratic form

$$\varphi_r = (X^r - \pi)Y_1^2 + (X^{r+1} + \pi)Y_2^2 + \pi X Y_3^2 + X(X^r + \pi)Y_4^2$$

is anisotropic over $E(X)$, but it is isotropic over the completion of $E(X)$ with respect to any $\mathbb{Z}$-valuation w on $E(X)$ with $w(\pi) < r$ (Theorem). The construction of φ_r is inspired by the example in [1, Remark 3.6] of an anisotropic 6-dimensional quadratic form over $\mathbb{Q}_p(X)$ where p is an odd prime.

2. Results. We assume some familiarity with basic quadratic form theory over fields, for which we refer to [4]. We first fix some notation and recall some results.

By a *quadratic form* or simply a *form* we mean a regular quadratic form. Let E always be a field of characteristic different from 2 and let $E^\times$ denote its multiplicative group. For $a_1, \ldots, a_n \in E^\times$, the diagonal form $a_1X_1^2 + \cdots + a_nX_n^2$ is denoted by $\langle a_1, \ldots, a_n \rangle$.

Let v be a $\mathbb{Z}$-valuation on E. We denote the corresponding valuation ring, its maximal ideal, and its residue field respectively by $\mathcal{O}_v, \mathfrak{m}_v$, and κ_v. For an element $a \in \mathcal{O}_v$, let $\overline{a}$ denote the residue class $a + \mathfrak{m}_v$ in κ_v. The completion of E with respect to v is denoted by E_v. We say that v is *henselian* if it extends uniquely to every finite field extension of E. Complete discretely valued fields are henselian (see [2, Theorem 1.3.1 and Theorem 4.1.3]). We recall a consequence of Hensel's lemma:

Lemma. *Let v be a henselian $\mathbb{Z}$-valuation on E such that $v(2) = 0$. Then:*

(a) *For $u_1, u_2 \in \mathcal{O}_v^\times$, the quadratic form $\langle u_1, u_2 \rangle$ over E is isotropic if and only if $\overline{u_1u_2} \in -\kappa_v^{\times 2}$.*
(b) *If κ_v is separably closed, then every 3-dimensional form over E is isotropic.*

Proof. (a) For $u_1, u_2 \in \mathcal{O}_v^\times$, since v is henselian and $v(2) = 0$, it follows by [2, Theorem 4.1.3(4)] that $u_1u_2 \in -E^{\times 2}$ if and only if $\overline{u_1u_2} \in -\kappa_v^{\times 2}$.

(b) Every 3-dimensional form over E contains a 2-dimensional form isometric to $\lambda\langle 1, u \rangle$ for some $u \in \mathcal{O}_v^\times$ and $\lambda \in E^\times$. If κ_v is separably closed, then $\overline{u} \in -\kappa_v^{\times 2}$ and hence $\langle 1, u \rangle$ is isotropic by (a). □

The set of all $\mathbb{Z}$-valuations on $E(X)$ is denoted by $\Omega_{E(X)}$. For $r \in \mathbb{N}$, we define

$$\Omega_r = \{w \in \Omega_{E(X)} \mid w(E^\times) = i\mathbb{Z} \text{ for some } 0 \le i \le r\}.$$

With this notation, Ω_0 is the set of all E-trivial $\mathbb{Z}$-valuations on $E(X)$. We recall that any monic irreducible polynomial $p \in E[X]$ determines a unique $\mathbb{Z}$-valuation v_p on $E(X)$ which is trivial on E and such that $v_p(p) = 1$. There is further a unique $\mathbb{Z}$-valuation v_∞ on $E(X)$ such that $v_\infty(f) = -\deg(f)$ for any $f \in E[X] \setminus \{0\}$. Moreover, every $\mathbb{Z}$-valuation w on $E(X)$ trivial on E is either equal to v_∞ or to v_p for some monic irreducible polynomial $p \in E[X]$ (see [2, Theorem 2.1.4]), and in either of the two cases the residue field is a finite field extension of E.

Theorem. *Let v be a henselian $\mathbb{Z}$-valuation on E such that $v(2) = 0$. Assume that κ_v is separably closed. Let $\pi \in E^\times$ be such that $v(\pi) = 1$ and let $r \in \mathbb{N}$. Then the quadratic form*

$$\varphi_r = \langle X^r - \pi, X^{r+1} + \pi, \pi X, X(X^r + \pi) \rangle$$

is isotropic over $E(X)_w$ for every $\mathbb{Z}$-valuation $w \in \Omega_{r-1}$, but anisotropic over $E(X)_w$ for some $w \in \Omega_r$.

Proof. Set $F = E(X)$. We first show that φ_r is isotropic over F_w for all $w \in \Omega_{r-1}$. Consider $w \in \Omega_{r-1}$.

Case 1: $w(\pi) = 0 = w(X)$. Then κ_w is a finite extension of E. Since v is henselian, there is a unique extension v' of v to κ_w, and v' is again henselian.

Furthermore, it follows by [2, Theorem 3.3.4] that $v'(\kappa_w^\times)$ is isomorphic to $\mathbb{Z}$ and $\kappa_{v'}$ is separably closed. It follows by part (b) of the Lemma that every 3-dimensional quadratic form over κ_w is isotropic. We have that $w = v_p$ for some monic irreducible polynomial $p \in E[X]$ such that $p \neq X$. Note that, in this case, at least three diagonal coefficients of φ_r are units in $\mathcal{O}_w$. It follows by Springer's theorem [4, Proposition VI.1.9] that φ_r is isotropic over F_w.

Case 2: $0 \leq w(\pi) < r$ and $1 \leq w(X)$. Let $u = (X^r\pi^{-1} - 1)(X^{(r+1)}\pi^{-1} + 1)$. Then $w(u) = 0$ and $\overline{u} = -1 \in -\kappa_w^{\times 2}$. It follows by part (a) of the Lemma that the form $\pi^{-1}\langle X^r - \pi, X^{r+1} + \pi\rangle$ is isotropic over F_w. Thus φ_r is isotropic over F_w.

Case 3: $w(X) < 0 \leq w(\pi) < r$. Note that, if $w(\pi) = 0$, then $w = v_\infty$ and $\kappa_w = E$, and otherwise $w|_E$ is equivalent to v and $\kappa_v \subseteq \kappa_w$; since $-1 \in \kappa_v^{\times 2}$, we get in either case that $-1 \in \kappa_w^{\times 2}$. Consider $u = (1 + \pi X^{-(r+1)})(1 + \pi X^{-r})$. We have that $w(u) = 0$ and $\overline{u} = 1 \in \kappa_w^{\times 2} = -\kappa_w^{\times 2}$. It follows by part (a) of the Lemma that the form $X^{-(r+1)}\langle X^{r+1} + \pi, X(X^r + \pi)\rangle$ is isotropic over F_w. Thus φ_r is isotropic over F_w.

We have thus shown that φ_r is isotropic over F_w for every $w \in \Omega_{r-1}$. Now we show that φ_r is anisotropic over F_w for some $w \in \Omega_F$.

Let $E' = E(s)$, where $s = \sqrt[r]{\pi}$. Then v extends uniquely to a valuation on E' which we again denote by v. Note that $s^r = \pi$ in E' and hence $v(\pi) = rv(s)$. Then $v' = rv$ is a $\mathbb{Z}$-valuation on E'.

Let $L = E'(X)$ and let $Y = \frac{X}{s}$. Note that $L = E'(Y)$. By [2, Corollary 2.2.2], there exists a unique extension of v' to L such that $v(Y) = 0$ and $\overline{Y}$ is transcendental of $\kappa_{v'}$; we further have that $\kappa_w = \kappa_{v'}(\overline{Y})$ and $w(L^\times) = v'(E'^\times) = \mathbb{Z}$. Since $w(Y) = 0$, we have that $w(X) = w(s) = 1$. We get that

$$\varphi_r = \langle s^r(Y^r - 1), s^r(sY + 1), s^{r+1}Y, s^{r+1}Y(Y^r + 1)\rangle.$$

Consider the forms $\varphi_1 = \langle Y^r - 1,\ sY + 1\rangle$ and $\varphi_2 = \langle Y,\ Y(Y^r + 1)\rangle$.

Since $\overline{Y}^r - 1, \overline{Y}^r + 1 \notin -\kappa_w^{\times 2}$, it follows by Springer's theorem [4, Proposition VI.1.9] that the quadratic form $s^{-r}\varphi_r$ is anisotropic over L_w. Hence φ_r is anisotropic over L_w. We obtain that φ_r is anisotropic over $F_{w|_F}$. Note that $w(\pi) = w(s^r) = rw(s) = r$, thus $w \in \Omega_r$. □

We now provide a different perspective to the above theorem. For a subset $\Omega \subseteq \Omega_{E(X)}$, we say that Ω has the *finite support property* if for every $f \in E(X)^\times$, the set $\{w \in \Omega \mid w(f) \neq 0\}$ is finite. It is well-known that Ω_0 has the finite support property. When E carries a discrete valuation, the set $\Omega_{E(X)}$ does not have the finite support property. However, for any model $\mathscr{X}$ of $E(X)$ over $\mathcal{O}_v$, the set $\Omega_{\mathscr{X}}$ contains Ω_0 and has the finite support property. We show the following:

Corollary 1. *Let v be a henselian $\mathbb{Z}$-valuation on E with $v(2) = 0$. Assume that κ_v is separably closed. Let $\Omega \subseteq \Omega_{E(X)}$ be a subset with the finite support property. Then there exists an anisotropic 4-dimensional quadratic form over $E(X)$ which is isotropic over $E(X)_w$ for every $w \in \Omega$.*

Proof. Let $\pi \in E^\times$ be such that $v(\pi) = 1$. Since Ω has the finite support property, the set $\{w \in \Omega \mid w(\pi) \neq 0\}$ is finite. Set $r = 1 + \max\{w(\pi) \mid w \in \Omega\}$. Clearly $\Omega \subseteq \Omega_{r-1}$. Then the form φ_r in the Theorem is isotropic over $E(X)_w$ for every $w \in \Omega$, but anisotropic over $E(X)$. □

Corollary 2. *Let v be a henselian $\mathbb{Z}$-valuation on E with $v(2) = 0$. Assume that κ_v is separably closed. Let $\mathscr{X}$ be a regular model of $E(X)$ over $\mathcal{O}_v$. Then there exists an anisotropic 4-dimensional quadratic form over $E(X)$ which is isotropic over $E(X)_w$ for every $w \in \Omega_{\mathscr{X}}$.*

Proof. By [3, Chapter II, Lemma 6.1], for every element $f \in E(X)^\times$, the set $\{w \in \Omega_{\mathscr{X}} \mid w(f) \neq 0\}$ is finite, hence the statement follows by Corollary 1. □

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