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Approximation of analytic functions by an absolutely convergent Dirichlet series

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Abstract. In the paper, an absolutely convergent Dirichlet series whose shifts approximate a wide class of analytic functions is constructed. This series is close in the mean to the Riemann zeta-function.

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1. Introduction. Let $\zeta(s)$, $s = \sigma + it$, be the Riemann zeta-function, i.e., for $\sigma > 1$,

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}.$$

The function $\zeta(s)$ has an analytic continuation to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. It is well known that the function $\zeta(s)$ has a universality property discovered by Voronin [13] on the approximation of a wide class of analytic functions by shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$. Let $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, and by $H_0(K)$ with $K \in \mathcal{K}$ the class of continuous non-vanishing functions on K that are analytic in the interior of K. Then the last version of the Voronin theorem, see, for example, [5], says that, for every $K \in \mathcal{K}$, $f(s) \in H_0(K)$, and $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s\in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0. \qquad (1.1)$$

The latter inequality shows that there are infinitely many shifts $\zeta(s+i\tau)$ that approximate uniformly on K a given function $f(s) \in H_0(K)$ with accuracy

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$\varepsilon > 0$. On the other hand, no concrete such shift is known. In [7] and [9], it was obtained that "lim inf" in (1.1) can be replaced by "lim" for all but at most countably many $\varepsilon > 0$.

Some other zeta-functions and their classes are universal in the above sense as well, see [1,3,6,11] and the very informative survey paper [8].

The Riemann zeta-function $\zeta(s)$, for $s \in D$, cannot be written as a convergent Dirichlet series, and is defined by analytic continuation. The aim of this note is to present a certain absolutely convergent Dirichlet series in D having an approximation property similar to that of the function $\zeta(s)$. Obviously, an absolutely convergent Dirichlet series cannot be universal. Therefore, such a series must be close in a certain sense to $\zeta(s)$.

For $u > 0$, let $\theta > 1/2$ be a fixed number and, for $m \in \mathbb{N}$,

$$v_u(m) = \exp\left\{ -\left(\frac{m}{u}\right)^{\theta} \right\},$$

where $\exp\{a\} = \mathrm{e}^a$. Define the series

$$\zeta_u(s) = \sum_{m=1}^{\infty} \frac{v_u(m)}{m^s}.$$

It will be proved below that the latter series is absolutely convergent in the half plane $\sigma > 1/2$, thus, for $s \in D$ as well.

For the precise statement of the approximation theorem for the function $\zeta_u(s)$, we need one particular topological group. Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, $\mathbb{P}$ denote the set of all prime numbers, and

$$\Omega = \prod_{p \in \mathbb{P}} \gamma_p,$$

where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$. By the Tikhonov theorem, the infinite-dimensional torus Ω with pointwise multiplication and the product topology is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$ ($\mathcal{B}(\mathbb{X})$ is the Borel σ-field of the space $\mathbb{X}$), the probability Haar measure m_H can be defined. This leads to the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $H(D)$ the space of analytic functions on D endowed with the topology of uniform convergence on compacta, by $\omega(p)$ the pth component of an element $\omega \in \Omega$, $p \in \mathbb{P}$, and on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H(D)$-valued random element

$$\zeta(s, \omega) = \prod_{p \in \mathbb{P}} \left(1 - \frac{\omega(p)}{p^s}\right)^{-1}.$$

Note that the latter infinite product converges uniformly on compact subsets of the strip D for almost all $\omega \in \Omega$, see [1] or [5].

The main result of the paper is the following theorem.

Theorem 1.1. *Suppose that $n_T \to \infty$ and $n_T \ll T^2$ as $T \to \infty$. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then the limit*

$$\lim_{T \to \infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta_{n_T}(s + i\tau) - f(s)| < \varepsilon \right\}$$

$$= m_H \left\{ \omega \in \Omega : \sup_{s \in K} |\zeta(s,\omega) - f(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

Theorem 1.1 implies that there exists $T_0 = T_0(f, K, \varepsilon) > 0$ such that, for every $T \geq T_0$, there are infinitely many shifts $\zeta_{n_T}(s + i\tau)$ approximating a given function $f(s)$.

2. The function $\zeta_u(s)$. Denote by $\Gamma(s)$ the Euler gamma-function, and define

$$l_u(s) = \frac{s}{\theta} \Gamma\left(\frac{s}{\theta}\right) u^s,$$

where $\theta > 1/2$ is from the definition of $v_u(m)$. For convenience, recall some properties of $\Gamma(s)$.

Lemma 2.1. *For arbitrary* $\sigma_1 < \sigma_2$*, there exists* $c > 0$ *such that the estimate*

$$\Gamma(\sigma + it) \ll \exp\{-c|t|\}$$

holds uniformly in σ*,* $\sigma_1 \leq \sigma \leq \sigma_2$*.*

Proof of the lemma can be found, for example, in [4].

Lemma 2.2. *For positive* a *and* b*,*

$$\frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \Gamma(s) a^{-s} \, \mathrm{d}s = \mathrm{e}^{-a}.$$

The equality of the lemma is the classical Mellin formula, for the proof, see, for example, [12].

Lemma 2.3. *The series for* $\zeta_u(s)$ *is absolutely convergent for* $\sigma > 1/2$*. Moreover, the equality*

$$\zeta_u(s) = \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} \zeta(s+z) l_u(z) \frac{\mathrm{d}z}{z}$$

is valid.

Proof. In view of Lemma 2.2,

$$\begin{aligned} &\frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} \frac{1}{\theta} \Gamma\left(\frac{s}{\theta}\right) \left(\frac{u}{m}\right)^s \mathrm{d}s \\ &= \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \Gamma(s) \left(\left(\frac{m}{u}\right)^\theta\right)^{-s} \mathrm{d}s \\ &= \exp\left\{-\left(\frac{m}{u}\right)^\theta\right\}. \end{aligned} \tag{2.1}$$

Therefore, by Lemma 2.1,

$$\exp\left\{-\left(\frac{m}{u}\right)^{\theta}\right\} \ll_{\theta} \int_{-\infty}^{\infty} \left|\Gamma\left(1+\frac{it}{\theta}\right)\right| \frac{u^{\theta}}{m^{\theta}}\,\mathrm{d}t \ll_{\theta,u} m^{-\theta}.$$

Since $\theta > 1/2$, this shows the absolute convergence in the half-plane $\sigma > 1/2$ of the series for $\zeta_u(s)$.

Now, using (2.1) and the definition of $l_u(s)$, we find

$$\begin{aligned}\zeta_u(s) &= \frac{1}{2\pi i}\sum_{m=1}^{\infty}\frac{1}{m^s}\int_{\theta-i\infty}^{\theta+i\infty}\frac{z}{\theta}\Gamma\left(\frac{z}{\theta}\right)\left(\frac{u}{m}\right)^{z}\frac{\mathrm{d}z}{z}\\ &= \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\frac{l_u(z)}{z}\sum_{m=1}^{\infty}\frac{1}{m^{s+z}}\,\mathrm{d}z = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\zeta(s+z)\frac{l_u(z)\mathrm{d}z}{z}.\end{aligned}$$

□

3. Mean distance between $\zeta_{n_T}(s)$ and $\zeta(s)$. For n_T sufficiently large, the coefficients of the series for $\zeta_{n_T}(s)$ are close to 1. This suggests that the function $\zeta_{n_T}(s)$ is close to $\zeta(s)$ even in the strip D. Actually, those two functions are close in the mean.

Lemma 3.1. *Suppose that $n_T \to \infty$ and $n_T \ll T^2$ as $T \to \infty$. Then, for every compact set $K \subset D$,*

$$\lim_{T\to\infty}\frac{1}{T}\int_{0}^{T}\sup_{s\in K}|\zeta(s+i\tau)-\zeta_{n_T}(s+i\tau)|\,\mathrm{d}\tau = 0.$$

Proof. By Lemma 2.3, for $s \in D$, we have

$$\zeta_{n_T}(s) = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty}\zeta(s+z)l_{n_T}(z)\frac{\mathrm{d}z}{z}. \tag{3.1}$$

Let $0 < \hat{\theta} < 2\theta$. Then (3.1), properties of the function $\zeta(s)$, and the residue theorem imply

$$\zeta_{n_T}(s) - \zeta(s) = \frac{1}{2\pi i}\int_{-\hat{\theta}-i\infty}^{-\hat{\theta}+i\infty}\zeta(s+z)l_{n_T}(z)\frac{\mathrm{d}z}{z} + \frac{l_{n_T}(1-s)}{1-s}. \tag{3.2}$$

Denote the points of the set K by $s = \sigma + iv$. Then there exists $\varepsilon > 0$ such that $1/2 + 2\varepsilon \le \sigma \le 1 - \varepsilon$ for all $s \in K$. Taking

$$\hat{\theta} = \sigma - \frac{1}{2} - \varepsilon > 0,$$

we obtain from (3.2) that, for all $s \in K$,

$$\zeta_{n_T}(s+i\tau)-\zeta(s+i\tau)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\zeta\left(\sigma+iv+i\tau-\sigma+\frac{1}{2}+it+\varepsilon\right)$$
$$\times\, l_{n_T}\left(\frac{1}{2}+\varepsilon-s+i(v+t)\right)\frac{\mathrm{d}t}{1/2+\varepsilon-s+i(v+t)}+\frac{l_{n_T}(1-s-i\tau)}{1-s-i\tau}.$$

Hence, putting t in place of $v+t$ gives

$$\begin{aligned}\zeta_{n_T}(s+i\tau)-\zeta(s+i\tau)=&\frac{1}{2\pi i}\int_{-\infty}^{\infty}\zeta\left(\frac{1}{2}+\varepsilon+i\tau+it\right)\\&\times\frac{l_{n_T}(1/2+\varepsilon-s+it)}{1/2+\varepsilon-s+it}\,\mathrm{d}t+\frac{l_{n_T}(1-s-i\tau)}{1-s-i\tau}\\ \ll&\int_{-\infty}^{\infty}\left|\zeta\left(\frac{1}{2}+\varepsilon+i\tau+it\right)\right|\\&\times\sup_{s\in K}\frac{|l_{n_T}(1/2+\varepsilon-s+it)|}{|1/2+\varepsilon-s+it|}\,\mathrm{d}t\\&+\sup_{s\in K}\frac{|l_{n_T}(1-s-i\tau)|}{|1-s-i\tau|}.\end{aligned}$$

Therefore, in view of Lemma 2.1,

$$\frac{1}{T}\int_0^T\sup_{s\in K}|\zeta_{n_T}(s+i\tau)-\zeta(s+i\tau)|\,\mathrm{d}\tau\ll I_{1T}+I_{2T},\tag{3.3}$$

where

$$I_{1T}=\int_{-\infty}^{\infty}\left(\frac{1}{T}\int_0^T\left|\zeta\left(\frac{1}{2}+\varepsilon+i\tau+it\right)\right|\,\mathrm{d}\tau\right)\sup_{s\in K}\frac{|l_{n_T}(1/2+\varepsilon-s+it)|}{|1/2+\varepsilon-s+it|}\,\mathrm{d}t$$

and

$$I_{2T}=\frac{1}{T}\int_0^T\sup_{s\in K}\left|\frac{l_{n_T}(1-s-i\tau)}{1-s-i\tau}\right|\,\mathrm{d}\tau.$$

It is well known that, for fixed σ, $1/2<\sigma<1$,

$$\int_{-T}^{T}|\zeta(\sigma+it)|^2\,\mathrm{d}t\ll_\sigma T.$$

Therefore, for all $t\in\mathbb{R}$ and $T\geq 1$,

$$\frac{1}{T}\int_0^T \left|\zeta\left(\frac{1}{2}+\varepsilon+i\tau+it\right)\right| \mathrm{d}\tau \le \left(\frac{1}{T}\int_0^T \left|\zeta\left(\frac{1}{2}+\varepsilon+i\tau+it\right)\right|^2 \mathrm{d}\tau\right)^{1/2}$$
$$\ll \left(\frac{1}{T}\int_{-|t|}^{T+|t|} \left|\zeta\left(\frac{1}{2}+\varepsilon+i\tau\right)\right|^2 \mathrm{d}\tau\right)^{1/2} \tag{3.4}$$
$$\ll_\varepsilon \left(\frac{T+|t|}{T}\right)^{1/2} \ll_\varepsilon 1+|t|.$$

Moreover, taking into account Lemma 2.1, we find that, for all $s \in K$,

$$\frac{l_{n_T}(1/2+\varepsilon-s+it)}{1/2+\varepsilon-s+it} \ll_\theta n_T^{1/2+\varepsilon-\sigma}\left|\Gamma\left(\frac{1}{\theta}\left(\frac{1}{2}+\varepsilon-\sigma-iv+it\right)\right)\right|$$
$$\ll_\theta n_T^{-\varepsilon}\exp\left\{-\frac{c}{\theta}|t-v|\right\}$$
$$\ll_{\theta,K} n_T^{-\varepsilon}\exp\{-c_1|t|\}, \quad c_1>0.$$

This and (3.4) show that

$$I_{1T} \ll_{\varepsilon,\theta,K} n_T^{-\varepsilon}\int_{-\infty}^{\infty}(1+|t|)\exp\{-c_1|t|\}\,\mathrm{d}t \ll_{\varepsilon,\theta,K} n_T^{-\varepsilon}. \tag{3.5}$$

Similarly, we obtain that, for $s \in K$,

$$\frac{l_{n_T}(1-s-i\tau)}{1-s-i\tau} \ll_\theta n_T^{1-\sigma}\left|\Gamma\left(\frac{1}{\theta}(1-\sigma-iv-i\tau)\right)\right|$$
$$\ll_\theta n_T^{1/2-2\varepsilon}\exp\left\{-\frac{c}{\theta}|\tau+v|\right\} \ll_{\theta,K} n_T^{1/2-2\varepsilon}\exp\{-c_1|\tau|\}.$$

Therefore,

$$I_{2T} \ll_{\theta,K} n_T^{1/2-2\varepsilon}\frac{1}{T}\int_0^T \exp\{-c_1|\tau|\}\,\mathrm{d}\tau \ll_{\theta,K} \frac{n_T^{1/2-2\varepsilon}}{T}\int_0^\infty \exp\{-c_1\tau\}\,\mathrm{d}\tau$$
$$\ll_{\theta,K} \frac{n_T^{1/2-2\varepsilon}}{T}.$$

Since $n_T \ll T^2$, this, (3.3), and (3.5) prove the lemma. □

4. Limit theorem. The inequality (1.1) and its modification with "lim" are derived in [1,5,7] from a probabilistic limit theorem for measures in the space of analytic functions $H(D)$. For $A \in \mathcal{B}(H(D))$, define

$$P_T(A) = \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \zeta(s+i\tau) \in A\right\},$$

and denote by P_ζ the distribution of the $H(D)$-valued random element $\zeta(s,\omega)$, i.e.,

$$P_\zeta(A) = m_H\{\omega \in \Omega : \zeta(s,\omega) \in A\}.$$

Then the following statement is true [1,5].

Lemma 4.1. *P_T converges weakly to P_ζ as $T \to \infty$. Moreover, the support of the measure P_ζ is the set $S = \{g \in H(D) : g(s) \neq 0$ or $g(s) \equiv 0\}$.*

Several equivalents of weak convergence of probability measures are known, see, for example, [2]. For us, the equivalent in terms of continuity sets is useful. Recall that a set $A \in \mathcal{B}(\mathbb{X})$ is called a continuity set of a measure P on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ if $P(\partial A) = 0$, where ∂A denotes the boundary of the set A.

Lemma 4.2. *Suppose that P_n, $n \in \mathbb{N}$, and P are probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Then P_n converges weakly to P as $n \to \infty$ if and only if, for every continuity set A of P,*

$$\lim_{n\to\infty} P_n(A) = P(A).$$

Proof of the lemma can be found, for example, in [2].

5. Proof of Theorem 1.1. We will apply a method of characteristic functions which is used in the theory of weak convergence of probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We recall that every left continuous non-decreasing function $F(x)$ on $\mathbb{R}$ such that $F(+\infty) = 1$ and $F(-\infty) = 0$ coincides with a certain distribution function. Note that the left continuity of distribution functions can be replaced by right continuity, however, for our aims the left continuity is more convenient. We say that the distribution function $F_n(x)$, $n \in \mathbb{N}$, converges weakly to a distribution function F as $n \to \infty$ if $\lim_{n\to\infty} F_n(x) = F(x)$ for every continuity point x of $F(x)$.

Every distribution function $F(x)$ is uniquely defined by its characteristic function

$$\int_{-\infty}^{\infty} \mathrm{e}^{iux}\, \mathrm{d}F(x), \quad u \in \mathbb{R}.$$

Moreover, the following classical continuity theorem is valid.

Lemma 5.1. *Suppose that $F_n(x)$, $n \in \mathbb{N}$, and $F(x)$ are distribution functions, and $g_n(u)$ and $g(u)$ are the corresponding characteristic functions. If $F_n(x)$, as $n \to \infty$, converges weakly to $F(x)$, then $\lim_{n\to\infty} g_n(u) = g(u)$, $u \in \mathbb{R}$. This convergence is uniform in every finite interval. If $\lim_{n\to\infty} g_n(u) = g(u)$, $u \in \mathbb{R}$, where $g(u)$ is a continuous function at $u = 0$, then there exists a distribution function $F(x)$ such that $F_n(x)$ converges weakly to $F(x)$ as $n \to \infty$. In this case, $g(u)$ is the characteristic function of $F(x)$.*

For the proof of Theorem 1.1, the Mergelyan theorem on the approximation of analytic functions by polynomials [10] is also needed, see the next lemma.

Lemma 5.2. *Suppose that $K \subset \mathbb{C}$ is a compact set with connected complement, and $g(s)$ is a continuous function in K and analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s\in K} |g(s) - p(s)| < \varepsilon.$$

Proof of Theorem 1.1. By Lemma 4.1, P_T converges weakly to P_ζ as $T \to \infty$. Define the set

$$A_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \right\}.$$

The boundaries $\partial A_{\varepsilon_1}$ and $\partial A_{\varepsilon_2}$ do not intersect for different positive ε_1 and ε_2. Therefore, the set A_ε is a continuity set of the measure P_ζ for all but at most countably many $\varepsilon > 0$. This, the weak convergence of P_T, and Lemma 4.2 imply the relation

$$\lim_{T \to \infty} P_T(A_\varepsilon) = P_\zeta(A_\varepsilon) \tag{5.1}$$

for all but at most countably many $\varepsilon > 0$. Moreover, since by Lemma 4.1, the support of P_ζ is the set S, the inequality

$$P_\zeta(\hat{A}_\varepsilon) > 0, \tag{5.2}$$

where $p(s)$ is a polynomial and

$$\hat{A}_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} \left| g(s) - \mathrm{e}^{p(s)} \right| < \frac{\varepsilon}{2} \right\},$$

is true. Lemma 5.2 ensures the choice of the polynomial $p(s)$ satisfying

$$\sup_{s \in K} \left| f(s) - \mathrm{e}^{p(s)} \right| < \frac{\varepsilon}{2}.$$

The latter inequality together with the definitions of the sets A_ε and $\hat{A}_\varepsilon$ implies the inclusion $\hat{A}_\varepsilon \subset A_\varepsilon$. Therefore, in view of (5.2),

$$P_\zeta(A_\varepsilon) > 0.$$

By the definitions of P_T, P_ζ, and A_ε,

$$P_T(A_\varepsilon) = \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\}$$

and

$$P_\zeta(A_\varepsilon) = m_H \left\{ \omega \in \Omega : \sup_{s \in K} |\zeta(s, \omega) - f(s)| < \varepsilon \right\}.$$

Thus, $F_T(\varepsilon) \stackrel{def}{=} P_T(A_\varepsilon)$ and $F_\zeta(\varepsilon) \stackrel{def}{=} P_\zeta(A_\varepsilon)$ with respect to ε are distribution functions. Moreover,

$$P_\zeta(B_\varepsilon) - P_\zeta(A_\varepsilon) = P_\zeta(\partial A_\varepsilon),$$

where

$$B_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| \leq \varepsilon \right\}.$$

Thus,

$$P_\zeta(\partial A_\varepsilon) = F_\zeta(\varepsilon + 0) - F_\zeta(\varepsilon).$$

This shows that $F_\zeta(\varepsilon)$ is right continuous, thus continuous, only if the set A_ε is a continuity set of the measure P_ζ. Therefore, in view of (5.1), the distribution function F_T converges weakly to F_ζ as $T \to \infty$. Hence, by the first part of Lemma 5.1,

$$\lim_{T\to\infty} g_T(u) = g_\zeta(u) \tag{5.3}$$

uniformly in u in every finite interval, where $g_T(u)$ and $g_\zeta(u)$ are the characteristic functions of the distribution functions F_T and F_ζ, respectively.

Denote by $\hat{g}_T(u)$ the characteristic function of the distribution function

$$\hat{F}_T(\varepsilon) \stackrel{def}{=} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta_{n_T}(s+i\tau) - f(s)| < \varepsilon\right\}.$$

Using the inequality $|\mathrm{e}^{iu} - 1| \leq |u|$, $u \in \mathbb{R}$, and the triangle inequality

$$\left|\sup_{s\in K} |\zeta_{n_T}(s+i\tau) - f(s)| - \sup_{s\in K} |\zeta(s+i\tau) - f(s)|\right| \\ \leq \sup_{s\in K} |\zeta(s+i\tau) - \zeta_{n_T}(s+i\tau)|,$$

we obtain by (5.3) that

$$\begin{aligned}\hat{g}_T(u) = \int_{-\infty}^{\infty} \mathrm{e}^{iu\varepsilon}\, \mathrm{d}\hat{F}_T(\varepsilon) = \frac{1}{T}\int_0^T \exp\left\{iu \sup_{s\in K} |\zeta_{n_T}(s+i\tau) - f(s)|\right\} \mathrm{d}\tau \\ = \frac{1}{T}\int_0^T \exp\left\{iu\left(\sup_{s\in K} |\zeta(s+i\tau) - f(s)| + \sup_{s\in K} |\zeta_{n_T}(s+i\tau) - f(s)|\right.\right. \\ \left.\left. - \sup_{s\in K} |\zeta(s+i\tau) - f(s)|\right)\right\} \mathrm{d}\tau \\ = \frac{1}{T}\int_0^T \exp\left\{iu \sup_{s\in K} |\zeta(s+i\tau) - f(s)|\right\} \mathrm{d}\tau \\ + O\left(\frac{|u|}{T}\int_0^T \left|\sup_{s\in K} |\zeta_{n_T}(s+i\tau) - f(s)|\right.\right. \\ \left.\left. - \sup_{s\in K} |\zeta(s+i\tau) - f(s)|\right| \mathrm{d}\tau\right) \\ = g_\zeta(u) + o(1) + O\left(\frac{|u|}{T}\int_0^T \sup_{s\in K} |\zeta(s+i\tau) - \zeta_{n_T}(s+i\tau)|\, \mathrm{d}\tau\right)\end{aligned}$$

as $T \to \infty$. Therefore, in view of Lemma 3.1, we have

$$\hat{g}_T(u) = g_\zeta(u) + o(1)$$

as $T \to \infty$, uniformly in u in every finite interval. The function $g_\zeta(u)$, $|u| \leq C$ with every $C > 0$, as a characteristic function is continuous at the point $u = 0$.

This and the second part of Lemma 5.1 show that $\hat{F}_T$, as F_T, converges weakly to F_ζ as $T \to \infty$, and we have

$$\lim_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta_{n_T}(s+i\tau) - f(s)| < \varepsilon\right\}$$
$$= m_H\left\{\omega \in \Omega : \sup_{s\in K} |\zeta(s,\omega) - f(s)| < \varepsilon\right\} > 0$$

for all but at most countably many $\varepsilon > 0$. □

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