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Quantitative weakly compact sets and Banach-Saks sets in ℓ_1

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Abstract. In this paper, we show a quantitative version of the theorem stating that relatively weakly compact sets in ℓ_1 coincide with those having the Banach-Saks property. Namely, we prove that the measure of the weak noncompactness based on the Eberlein double limit criterion is equal to the measure of the non-Banach-Saks property defined by the arithmetic separation of sequences.

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1. Introduction. In this article, we aim to quantify the relationship of compact sets and the Banach-Saks sets in the Banach space ℓ_1 by using measures of weak noncompactness and the Banach-Saks property. Measures of noncompactness and weak noncompactness have been widely applied in functional analysis, both in applications and Banach space theory. In the area of differential and integral equations, they become indispensable to characterize compact sets and weakly compact sets, and then to get fixed points and further solutions to equations, see [6, 12, 13, 25] for example. On the other hand, they are widely used in Banach space theory to get deeper understanding of the implications through quantitative means. The quantitative methods provide different angles to view the theoretical results. There is a new trend to investigate the quantified properties of Banach spaces, see, e.g., [9, 10, 14, 17] and their references. From the applications in both equations and theories, we may infer that representations of the measures are always crucial. Every representation has its own advantages. To satisfy different goals, there have appeared many different measures of noncompactness and weak noncompactness, and

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the relationship between these measures is of interest, see [1,7,22] for example. In view of [3,15,17], it is specially interesting to study on which spaces the measures are equal. In the sequel, we work on De Blasi's [11] measure of weak noncompactness ω and the measure γ based on the Eberlein double limit criterion for weakly compact sets. Results in [3,4] showed that the two measures are not equivalent in general. We will prove that γ is exactly 2ω in ℓ_1.

A Banach space X is said to have the Banach-Saks property if every bounded sequence (x_n) in X has a subsequence (x'_n) such that the Cesàro means $(x'_1 + \cdots + x'_n)/n$ converge. As a weaker form, the weak Banach-Saks property of Banach spaces has been introduced. It means that every weakly convergent sequence has a subsequence whose Cesàro means converge in norm. For example, the spaces c_0, ℓ_1, and $L_1[0,1]$ have the weak Banach-Saks property. As for localization, a bounded subset A of a Banach space is said to be a Banach-Saks set if every sequence in A has a subsequence whose Cesàro sum converges. An analogue of the weak Banach-Saks set could be defined. The Banach-Saks property connects closely to reflexivity and weak compactness. Any Banach space with the Banach-Saks property was shown to be reflexive by a so-called summability method [23]. Meanwhile there are reflexive spaces without the Banach-Saks property [5]. Via the Rosenthal ℓ_1 theorem, every Banach-Saks set has been proved to be relatively weakly compact [21]. But in general, the reverse is not true because of the counterexample by Schreier [24] (see also, Baernstein [5]). Many mathematicians keep trying to quantify the involving Banach-Saks properties, see, e.g., [9,18,19].

Suppose that a sequence (x_n) is contained in a relatively weakly compact set of a Banach space X, then there is a subsequence (x_{n_k}) weakly converging to some point. If additionally the space X has the weak Banach-Saks property, then we could get a subsequence of (x_{n_k}), whose Cesàro sum converges in norm. Consequently, we may see that Banach-Saks sets coincide with relatively weakly compact sets in Banach spaces having the weak Banach-Saks property. Recently Kryczka [19] proved a quantitative Szlenk theorem which states the equivalence of the relatively weak compactness and the Banach-Saks property in $L_1[0,1]$. Inspired by their works, in Section 2, we prove a similar quantitative equivalence result in ℓ_1 by proving an equality in terms of measures of the Banach-Saks property and weak noncompactness.

2. Quantitative Banach-Saks property. Let X be a real infinite dimensional Banach space normed with $\|\cdot\|$, and let X^* be its dual. B_X is the unit ball of X. For any subset $E \subset X$, $\mathrm{co}(E)$ is the convex hull of E. Denote by $\mathcal{B}(X)$ the collection of all nonempty bounded subsets of X. By $\mathbf{N}$ we understand the set of positive integers, and $|A|$ is the cardinality of a subset $A \subset \mathbf{N}$. Recall the Hausdorff measure of noncompactness $\chi : \mathcal{B}(X) \to \mathbf{R}$ on X.

$$\chi(A) := \inf\{t > 0 \mid A \subset K + tB_X, K \subset X \text{ is compact}\}, \tag{1}$$

where $A \in \mathcal{B}(X)$. The Hausdorff measure is widely applied and the representation of the measure is of interest. In the Banach space ℓ_1, χ may be expressed by the following formula (see [2, p. 5]).

$$\chi(A) = \lim_{n\to\infty} \sup_{x\in A} \sum_{k=n}^{\infty} |x(k)| \tag{2}$$

for any $A \in \mathcal{B}(\ell_1)$.

χ characterizes compact sets in X as $\chi(A) = 0$ if and only if A is relatively compact. Replacing the compact set K in formula (1) by a weakly compact one, it is then the De Blasi measure of weak noncompactness ω. By the well-known Schur theorem, we observe easily that $\omega(A) = \chi(A)$. Recall that the measures ω and χ satisfy all the axiomatic principles for regular measures, see [6].

Another measure of weak noncompactness we are interested in is the measure γ based on the classical double-limit criterion of Eberlein. For any $A \in \mathcal{B}(X)$,

$$\gamma(A) := \sup |\lim_n \lim_m \langle f_n, x_m\rangle - \lim_m \lim_n \langle f_n, x_m\rangle|,$$

where the supremum is taken over all sequences $f_n \in B_{X^*}$ and $x_m \in A$ such that the double limits exist. It has been proved in [4] that the measures ω and γ are not equivalent, and they are shown in [3] to have the relationship $\gamma(A) \le 2\omega(A)$ for any $A \in \mathcal{B}(X)$.

Kryczke et al. [20] found that the measure γ can be expressed exactly in terms of the James convex separation criterion of weak compactness (see [16]). In detail, they proved

$$\gamma(A) = \sup\{\text{csep}(x_n) \mid (x_n) \subset \text{co}(A)\},$$

where

$$\text{csep}(x_n) := \inf_m d\{\text{co}\{x_n\}_{n=1}^{m}, \text{co}\{x_n\}_{n=m+1}^{\infty}\}. \tag{3}$$

Applying this result, we will see in next theorem that the reverse relationship of ω and γ could also be verified particularly in ℓ_1.

Theorem 1. *For any nonempty bounded subset A of ℓ_1, $\gamma(A) = 2\omega(A)$.*

Proof. With the comments above in mind, we only need to prove the inequality $\gamma(A) \ge 2\omega(A)$. Without loss of generality, we may suppose $\omega(A) = \chi(A) = \theta > 0$ since the case is trivial when $\theta = 0$. By fomula (2), for any $\varepsilon > 0$, there exists an integer $K_0 \in \mathbf{N}$ such that for any $x \in A$,

$$\|x\|_{K_0} := \sum_{k=K_0+1}^{\infty} |x(k)| < \theta + \varepsilon.$$

It is easy to see that $\sup_{x\in A} \sum_{k=n}^{\infty} |x(k)|$ is decreasing in n. Then for any $K \in \mathbf{N}$, there is $x \in A$ with $\theta - \varepsilon < \|x\|_K$. Thus there exist $x_1 \in A$ and $K_1 > K_0$ such that

$$\|x_1\|_{K_0} > \theta - \varepsilon, \text{ and } \|x_1\|_{K_0}^{K_1} := \sum_{k=K_0+1}^{K_1} |x_1(k)| > \theta - \varepsilon.$$

Proceeding this process, we inductively produce a sequence $(x_n) \subset A$ and an increasing sequence $(K_n) \subset \mathbf{N}$ such that for any $m \in \mathbf{N}$,

$$\|x_m\|_{K_{m-1}} > \theta - \varepsilon, \text{ and } \|x_m\|_{K_{m-1}}^{K_m} := \sum_{k=K_{m-1}+1}^{K_m} |x_m(k)| > \theta - \varepsilon,$$

We observe that the sequence (x_n) satisfies a nice property. To specify that, let us take just two elements x_1 and x_2 as an example. Clearly,

$$\begin{aligned}\|x_1 + x_2\| &\geq \|x_1 + x_2\|_{K_0} = \|x_1 + x_2\|_{K_0}^{K_1} + \|x_1 + x_2\|_{K_1}^{K_2} + \|x_1 + x_2\|_{K_2} \\ &\geq \sum_{k=K_0+1}^{K_1} |x_1(k) + x_2(k)| + \sum_{k=K_1+1}^{K_2} |x_1(k) + x_2(k)| - \|x_1 + x_2\|_{K_2} \\ &\geq \sum_{k=K_0+1}^{K_1} (|x_1(k)| - |x_2(k)|) + \sum_{k=K_1+1}^{K_2} (|x_2(k)| - |x_1(k)|) - \|x_1\|_{K_2} - \|x_2\|_{K_2} \\ &= (\|x_1\|_{K_0}^{K_1} - \|x_2\|_{K_0}^{K_1}) + (\|x_2\|_{K_1}^{K_2} - \|x_1\|_{K_1}^{K_2}) - \|x_1\|_{K_2} - \|x_2\|_{K_2} \\ &= (\|x_1\|_{K_0}^{K_1} - \|x_1\|_{K_1}) + (\|x_2\|_{K_1}^{K_2} - \|x_2\|_{K_0}^{K_1} - \|x_2\|_{K_2}) \\ &\geq 2(\theta - 3\varepsilon).\end{aligned}$$

To explain the last inequality, we may tell that $\|x_2\|_{K_1}^{K_2} - \|x_2\|_{K_0}^{K_1} - \|x_2\|_{K_2} \geq \theta - 3\varepsilon$ since

$$\|x_2\|_{K_1}^{K_2} > \theta - \varepsilon \text{ and } \|x_2\|_{K_0}^{K_1} + \|x_2\|_{K_2} < 2\varepsilon.$$

The latter is true because additionally

$$\|x_2\|_{K_0} = \|x_2\|_{K_0}^{K_1} + \|x_2\|_{K_1}^{K_2} + \|x_2\|_{K_2} < \theta + \varepsilon.$$

By the same process, we have

$$\|x_1\|_{K_0}^{K_1} - \|x_1\|_{K_1} \geq \theta - 3\varepsilon.$$

In fact, this property could be valid not only for x_1 and x_2, but also for any element in the set $\{x_n\}$ and its multiplication with a real number. Namely, for any $x_i \in \{x_n\}$ and $\lambda \in \mathbf{R}$,

$$\begin{aligned}\|\lambda x_i\|_{K_0} &\geq |\lambda| \|x_i\|_{K_{i-1}}^{K_i} - \sum_{j \neq i} |\lambda| \|x_i\|_{K_{j-1}}^{K_j} \\ &\geq |\lambda|(\theta - \varepsilon - 2\varepsilon) \\ &= |\lambda|(\theta - 3\varepsilon).\end{aligned}$$

Applying the similar calculation for x_1 and x_2 to arbitrary finite elements, we have that for any fixed $n \in \mathrm{N}$ and any $\lambda_i \in \mathbf{R}$ with $i = 1, \ldots, n$,

$$\begin{aligned}&\|\sum_{i=1}^n \lambda_i x_i\| \geq \|\sum_{i=1}^n \lambda_i x_i\|_{K_0} \\ &\geq \sum_{j=1}^n \|\sum_{i=1}^n \lambda_i x_i\|_{K_{j-1}}^{K_j} - \|\sum_{i=1}^n \lambda_i x_i\|_{K_n}\end{aligned}$$

$$
\begin{aligned}
&\geq \sum_{j=1}^{n} \Big(|\lambda_j| \|x_j\|_{K_{j-1}}^{K_j} - \sum_{m\neq j}^{n} |\lambda_m| \|x_m\|_{K_{j-1}}^{K_j} \Big) - \sum_{i=1}^{n} |\lambda_i| \|x_i\|_{K_n} \\
&= \Big(|\lambda_1| \|x_1\|_{K_0}^{K_1} - |\lambda_2| \|x_2\|_{K_0}^{K_1} - |\lambda_3| \|x_3\|_{K_0}^{K_1} - \cdots - |\lambda_n| \|x_n\|_{K_0}^{K_1} \\
&\quad - |\lambda_1| \|x_1\|_{K_1}^{K_2} + |\lambda_2| \|x_2\|_{K_1}^{K_2} - |\lambda_3| \|x_3\|_{K_1}^{K_2} - \cdots - |\lambda_n| \|x_n\|_{K_1}^{K_2} \\
&\quad \cdots\cdots\cdots\cdots\cdots \\
&\quad - |\lambda_1| \|x_1\|_{K_{n-1}}^{K_n} - |\lambda_2| \|x_2\|_{K_{n-1}}^{K_n} - |\lambda_3| \|x_3\|_{K_{n-1}}^{K_n} - \cdots + |\lambda_n| \|x_n\|_{K_{n-1}}^{K_n} \Big) \\
&\quad - \sum_{i=1}^{n} |\lambda_i| \|x_i\|_{K_n} \\
&= \sum_{i=1}^{n} \Big(|\lambda_i| \|x_i\|_{Ki-1}^{K_i} - \sum_{j=1}^{i-1} |\lambda_i| \|x_i\|_{K_{j-1}}^{K_j} - \sum_{j=i+1}^{n} |\lambda_i| \|x_i\|_{K_{j-1}}^{K_j} - |\lambda_i| \|x_i\|_{K_n} \Big) \\
&= \sum_{i=1}^{n} \Big(|\lambda_i| \|x_i\|_{K_{i-1}}^{K_i} - \sum_{j\neq i} |\lambda_i| \|x_i\|_{K_{j-1}}^{K_j} \Big) \\
&\geq \sum_{i=1}^{n} (|\lambda_i|(\theta - \varepsilon) - |\lambda_i| \cdot 2\varepsilon) \\
&= (\theta - 3\varepsilon) \sum_{i=1}^{n} |\lambda_i|. \qquad (4)
\end{aligned}
$$

For any $y \in \text{co}\{x_i\}_{i=1}^{m}$ and $z \in \text{co}\{x_i\}_{i=m+1}^{\infty}$ with $m \in \mathbf{N}$, there exist $n > m$, $a_i \geq 0$ with $i = 1, \ldots, m$ and $b_j \geq 0$ with $j = m+1, \ldots, n$, such that

$$
\sum_{i=1}^{m} a_i = 1, \quad \sum_{j=m+1}^{n} b_j = 1,
$$
$$
y = \sum_{i=1}^{m} a_i x_i, \quad z = \sum_{j=m+1}^{n} b_j x_j.
$$

By formula (4), it is easy to see that

$$
\|y - z\| \geq (\theta - 3\varepsilon) \left(\sum_{i=1}^{m} |a_i| + \sum_{j=m+1}^{n} |-b_j| \right) = 2(\theta - 3\varepsilon).
$$

Thus by (3), we have $\text{csep}(x_n) \geq 2(\theta - 3\varepsilon)$.

Now we have proved that for any $\varepsilon > 0$, there is a sequence (x_n) in A such that $\text{csep}(x_n) \geq 2(\theta - 3\varepsilon)$. It further means that $\gamma(A) \geq 2\omega(A)$, and the proof is completed. □

Beauzamy in [8] characterized spaces having the Banach-Saks property by spreading models, i.e., a Banach space X does not have the Banach-Saks property if and only if there exist $\theta > 0$ and a bounded sequence $(x_n) \subset X$ such that for any subsequence (x'_n),

$$\left\| \frac{1}{m} \left(\sum_{n=1}^{k} x'_n - \sum_{n=k+1}^{m} x'_n \right) \right\| \geq \theta.$$

for any positive integers $k \leq m$. Kryczka [18] modified Beauzamy's condition and introduced a deviation φ as the following to denote whether a set of a Banach space is a Banach-Saks set. We may call it the deviation measure of non-Banach-Saksness. For any $A \in \mathcal{B}(X)$,

$$\varphi(A) = \sup\{\text{asep}(x_n) \mid (x_n) \in A\},$$

where

$$\text{asep}(x_n) := \inf \left\| \frac{1}{m} \left(\sum_{n \in C} x_n - \sum_{n \in D} x_n \right) \right\|$$

with the infimum taken over all $m \in \mathbf{N}$ and finite $C, D \subset \mathbf{N}$ having $|C| = |D| = m$ and $\max C < \inf D$. The measure φ satisfies (see [18]) that for any $A, B \in \mathcal{B}(X)$,

(i) $\varphi(A) = 0$ if and only if A is a Banach-Saks set;
(ii) $\varphi(A) \leq \varphi(B)$ whenever $A \subset B$;
(iii) $\varphi(tA) = |t|\varphi(A)$ for $t \in \mathbf{R}$;
(iv) $\varphi(A + B) \leq \varphi(A) + \varphi(B)$ if A and B are convex.

From the definitions (3) and (5), we may get $\gamma(A) \leq \varphi(A)$, which quantifies the result that every Banach-Saks set is weakly compact. A glimpse on the unit bases (e_n) of ℓ_1 gives $\varphi(B_{\ell_1}) = 2$. We will use a quantitative method to state that the compact sets, weakly compact sets, and Banach-Saks sets coincide with each other in ℓ_1, and moreover the measures of these properties are equal.

Theorem 2. *For any nonempty bounded subset A of ℓ_1, $\varphi(A) = \gamma(A) = 2\omega(A) = 2\chi(A)$.*

Proof. It is sufficient to prove $\gamma(A) \geq \varphi(A)$. Suppose that $t > \omega(A)$, then there exists a weakly compact set $K \subset \ell_1$ such that $A \subset K + tB_{\ell_1}$. By Krein's theorem, it is reasonable to assume that K is convex. Noticing the properties of φ and $\varphi(K) = 0$ since ℓ_1 has the weak Banach-Saks property, we get

$$\varphi(A) \leq \varphi(K) + tB_{\ell_1} = 2t.$$

It means $\gamma(A) = 2\omega(A) \geq \varphi(A)$, and the proof is completed. □

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