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Continuous functionals for unbounded convergence in Banach lattices

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Abstract. Recently, the different types of unbounded convergences (uo, un, uaw, uaw^*) in Banach lattices were studied. In this paper, we study the continuous functionals with respect to unbounded convergences. We first characterize the continuity of linear functionals for these convergences. Then we define the corresponding unbounded dual spaces and get their exact form. Based on these results, we discuss order continuity and reflexivity of Banach lattices. Some related results are obtained as well.

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Keywords. Riesz space, Banach lattice, Unbounded order convergence, Unbounded norm convergence, Unbounded absolute weak convergence, Unbounded absolute weak* convergence.

1. Introduction. A net $(x_\alpha)_{\alpha\in A}$ in a Riesz space E is order convergent to $x \in E$ (write $x_\alpha \xrightarrow{o} x$) if there exists a net (y_β), possibly over a different index set, such that $y_\beta \downarrow 0$ and for each $\beta \in B$, there exists $\alpha_0 \in A$ satisfying $|x_\alpha - x| \le y_\beta$ for all $\alpha \ge \alpha_0$. The *unbounded order convergence* was considered firstly by Nakano in [7] and introduced in [2,9]. A net (x_α) in a Banach lattice E is unbounded order (resp. norm, absolute weak) convergent to some x, denoted by $x_\alpha \xrightarrow{uo} x$ (resp. $x_\alpha \xrightarrow{un} x$, $x_\alpha \xrightarrow{uaw} x$), if the net $(|x_\alpha - x| \wedge u)$ converges to zero in order (resp. norm, weak) for all $u \in E_+$. A net (x'_α) in a dual Banach lattice E' is unbounded absolute weak* convergent to some x', denoted by $x'_\alpha \xrightarrow{uaw^*} x'$, if $|x'_\alpha - x'| \wedge u' \xrightarrow{w^*} 0$ for all $u' \in E'_+$. Recently, there are different kind of results involving these convergences (see [3–5,8,10]). In [3,4], some properties of uo-convergence in Riesz spaces and Banach lattices is studied. For the properties of un, uaw, and uaw^*-convergence, we refer to [5,8,10].

It can be easily verified that, in $l_p\,(1 \le p < \infty)$, uo, un, and uaw, and uaw^*-convergence of nets are the same as the coordinate-wise convergence.

In $L_p(\mu)\,(1 \le p < \infty)$ for a finite measure μ, *uo*-convergence for sequences is the same as almost everywhere convergence, *un* and *uaw*-convergence for sequences are the same as convergence in measure. In $L_p(\mu)\,(1 < p < \infty)$ for a finite measure μ, uaw^*-convergence for sequences is also the same as convergence in measure.

In [4], Gao et al. studied the continuity of the linear functionals for *uo*-convergence. The aim of the present paper is the continuity of linear functionals for different types of unbounded convergences $(uo; un; uaw; uaw^*)$ in Banach lattices. A linear functional f on a Banach lattice E is said to be *uo* (resp. *un*, *uaw*, uaw^*)-continuous whenever $f(x_\alpha) \to 0$ for every *uo* (resp. *un*, *uaw*, uaw^*)-null net (x_α) in E. In the first part of the paper, we investigate the continuity of the linear functional f and prove that the carrier of f is finite-dimensional. Then we assume that the net (x_α) is norm bounded. We characterize the continuity of such functionals and obtain the exact form of the corresponding dual spaces. As an application of these results, we conclude the paper with characterizations of the order continuity and reflexivity of Banach lattices.

Recall that a Riesz space E is an ordered vector space in which $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$ exist for every $x, y \in E$. The positive cone of E is denoted by E_+, i.e., $E_+ = \{x \in E : x \ge 0\}$. For any vector x in E, define $x^+ := x \vee 0, x^- := (-x) \vee 0, |x| := x \vee (-x)$. An operator $T : E \to F$ between two Riesz spaces is said to be *positive* if $Tx \ge 0$ for all $x \ge 0$. A net (x_α) in a Riesz space is called disjoint whenever $\alpha \ne \beta$ implies $|x_\alpha| \wedge |x_\beta| = 0$ (denoted by $x_\alpha \perp x_\beta$). A set A in E is said to be *order bounded* if there exsits some $u \in E_+$ such that $|x| \le u$ for all $x \in A$. The solid hull $Sol(A)$ of A is the smallest solid set including A and it equals the set $Sol(A) := \{x \in E : \exists y \in A, |x| \le |y|\}$. An operator $T : E \to F$ is called order bounded if it maps order bounded subsets of E to order bounded subsets of F. A Banach lattice E is a Banach space $(E, \|\cdot\|)$ such that E is a Riesz space and its norm satisfies the following property: for each $x, y \in E$ with $|x| \le |y|$, we have $\|x\| \le \|y\|$. Recall that a vector $e > 0$ in Banach lattice lattice E is an atom if for any $u, v \in [0, e]$ with $u \wedge v = 0$, either $u = 0$ or $v = 0$. In this case, the band generated by e is $span\{e\}$. Moreover, the band projection $P_e : E \to span\{e\}$ defined by

$$P_e x = \sup_n (x^+ \wedge ne) - \sup_n (x^- \wedge ne)$$

exsits, and there is a unique positive linear functional f_e on E such that $P_e(x) = f_e(x)e$ for all $x \in E$. We call f_e the coordinate functional with the atom e. Clearly, the span of any finite set of atoms is also a projection band.

For undefined terminology, notation, and basic theory of Riesz spaces, Banach lattices, and linear operators, we refer to [1,6].

2. Results. Let us determine continuous functionals with respect to unbounded convergences on ℓ_1.

Example 2.1. Let (x_α) be a *uo*-null, *un*-null, *uaw*-null, uaw^*-null, and disjoint net in ℓ_1. Clearly, (x_α) is coordinate-wise convergent. For a vector $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n, ...)$ satisfying $\lambda(x_\alpha) \to 0$, it can be easily verified that $\lambda \in c_{00}$.

According to the above example, we can find that the carriers of the *uo*-continuous, *un*-continuous, *uaw*-continuous, *uaw**-continuous, and disjoint continuous functionals λ on l_1 are finite-dimensional. It is natural to ask whether the carriers are finite-dimensional in more general situations. The following results confirm the hypothesis.

For an operator $T : E \to F$ between two Riesz spaces, we shall say that its modulus $|T|$ exists (or that T possesses a modulus) whenever $|T| := T \vee (-T)$ exists. The carrier of T is denoted by C_T with $C_T := \{x \in E : |T|(|x|) = 0\}^d$.

Theorem 2.2. *Let E be an atomic Banach lattice and F a Banach lattice. For a nonzero linear operator $T : E \to F$, assume that the modulus $|T|$ exsits, then C_T is generated by finitely many atoms if one of the following conditions is satisfied.*

(1) $Tx_\alpha \to 0$ for every disjoint net $(x_\alpha) \subset E$.
(2) $Tx_\alpha \to 0$ for every uo-null net $(x_\alpha) \subset E$.
(3) $Tx_\alpha \to 0$ for every un-null net $(x_\alpha) \subset E$ and E has order continuous norm.
(4) $Tx_\alpha \to 0$ for every uaw-null net $(x_\alpha) \subset E$.
(5) $Tx_\alpha \to 0$ for every uaw*-null net $(x_\alpha) \subset E$ whenver E is a dual Banach lattice.

Proof. (1). We claim that C_T can not contain an infinite disjoint set of nonzero vectors. Suppose that there exsits an infinite positive disjoint sequence of nonzero vectors $(x_n)_{n\in\mathbb{N}}$ in C_T. Clearly, $|T|(x_n) > 0$ for all $n \in \mathbb{N}$. Hence there exsits $y_n \in [-x_n, x_n]$ such that $T(y_n) \neq 0$. Since (y_n) is also a disjoint sequence, $\left(\frac{y_n}{\|T(y_n)\|}\right)$ is disjoint, but for any $n \in \mathbb{N}$, one has $T\left(\frac{y_n}{\|T(y_n)\|}\right) = 1$ and so $\to 0$ is absurd.

Then we prove that C_T is generated by finitely many atoms. Let X be a maximal disjoint family of atoms of E and $A = X \cap C_T$. The linear span B of A is a projection band in C_T since A is a finite set (of atoms). If $B \neq C_T$, hence $C_T = B \oplus B^d$, so there exist $0 < x \in C_T$ such that $x \perp B$. Since x is not an atom, there exist u_1, y such that $0 < u_1, y \leq x$ and $u_1 \perp y$. Clearly, $u_1, y \in C_T$. Since $y \perp B$, y is not an atom, and thus there exist u_2, z such that $0 < u_2, z \leq y$ and $u_2 \perp z$. Clearly, $u_2, z \in C_T$. Repeating this process, we obtain an infinite disjoint sequence $(u_n)_{n\in\mathbb{N}}$ in C_T, but we have proved that the carrier of T can not contain an infinite disjoint set of nonzero vectors. Hence $B = C_T$.

(2)–(5). It follows from [3, Corollary 3.6], [10, Lemma 2], and [8, Lemma 2.3] that every disjoint sequence in a Banach lattice E is *uo*-null, *uaw*-null, and *uaw**-null. According to [5, Proposition 3.5], if E has order continuous norm, every disjoint sequence in E is *un*-null, so we can find that $\left(\frac{y_n}{\|T(y_n)\|}\right)$ is *uo*, *un*, *uaw*, and *uaw**-null. The rest of the proof is an application of (1). □

Let $F = \mathbb{R}$, we have the following result.

Corollary 2.3. *Let E be an atomic Banach lattice, f a nonzero linear functional on E, and (x_α) a net in E such that $f(x_\alpha) \to 0$. Then f is the linear* *combination of the coordinate functionals of finitely many atoms if one of the following conditions is satisfied.*

(1) (x_α) *is disjoint.*
(2) ([4, Proposition 2.2]) $x_\alpha \xrightarrow{uo} 0$.
(3) ([5, Corollary 5.4]) $x_\alpha \xrightarrow{un} 0$ *and* E *has order continuous norm.*
(4) $x_\alpha \xrightarrow{uaw} 0$.
(5) Whenever E *is a dual Banach lattices and* $x_\alpha \xrightarrow{uaw^*} 0$.

According to the above results, we can find that the *uo*-continuous, *un*-continuous, *uaw*-continuous, *uaw**-continuous, and disjoint continuous functionals only work on finite-dimensional spaces, hence we study the "bounded" continuous functionals for unbounded convergences in Banach lattices.

Let E be a Banach lattice. A linear functional f on E is said to be (σ)-*order continuous* if $f(x_\alpha) \to 0$ $\big(f(x_n) \to 0\big)$ for any net (sequence) (x_α) $\big((x_n)\big)$ in E that order converges to zero. The set $E_n^\sim$ of all order continuous functionals is called the *order continuous dual* of E. In [4], a linear functional f on E is said to be *bounded uo-continuous* if $f(x_\alpha) \to 0$ for any norm bounded *uo*-null net (x_α) in E. The set of all bounded *uo*-continuous linear functionals on E will be called the unbounded order dual (*uo*-dual for short) of E, and will be denoted by $E_{uo}^\sim$. It is natural to consider the other duals for unbounded convergence like *un*-continuous, *uaw*-continuous, and *uaw**-continuous functionals.

Definition 2.4. Let E be a Banach lattice. A bounded linear functional f on E is said to be *bounded d (un, uaw)-continuous* if $f(x_\alpha) \to 0$ for any norm bounded disjoint (*un*-null, *uaw*-null) net (x_α) in E. The set of all bounded $d\,(un, uaw)$-continuous linear functionals on E will be called the disjoint (unbounded norm, unbounded absolute weak) dual (d-dual, *un*-dual, and *uaw*-dual for short) of E, and will be denoted by $E_d^\sim$ $(E_{un}^\sim, E_{uaw}^\sim)$.

A bounded linear functional f on E' is said to be *bounded uaw*-continuous* if $f(x'_\alpha) \to 0$ for any norm bounded *uaw**-null net (x'_α) in E'. The set of all bounded *uaw**-continuous linear functionals on E' will be called the unbounded absolute weak* dual (*uaw**-dual for short) of E, and will be denoted by $(E')_{uaw^*}^\sim$.

The basic properties of these duals are as follows.

Proposition 2.5. *For a Banach lattice* E*, the following holds.*

(1) $(E')_{uaw^*}^\sim$ *is a closed ideal of* E*;*
(2) $E_{uo}^\sim$ *is a closed ideal of* $E_n^\sim$*;*
(3) $E_{uaw}^\sim$, $E_d^\sim$*, and* $E_{un}^\sim$ *are closed ideals of* E'.

Proof. (1). Since $x'_\alpha \xrightarrow{uaw^*} 0 \Leftrightarrow |x'_\alpha| \xrightarrow{uaw^*} 0$, we can assume that (x'_α) is positive. Let f be a bounded *uaw**-continuous functional on E'. For a net (x'_α) satisfying $|x'_\alpha| \xrightarrow{w^*} 0$ in E', clearly, we have $x'_\alpha \xrightarrow{uaw^*} 0$ and $f(x'_\alpha) \to 0$. Since $\big(E', |\sigma|(E', E)\big)' = E$, we have $f \in E$. So $(E')_{uaw^*}^\sim \subset E$.

It is clear that $(E')_{uaw^*}^\sim$ is a linear subspace of E. We claim that $(E')_{uaw^*}^\sim$ is a closed ideal of E. Since $|f|(x) = \sup\{f(y) : |y| \le x\}$, for any $\epsilon > 0$, there exist some α_0 and a net $(y'_\alpha) \subset E'$ such that $|f|(x'_\alpha) \leq f(y'_\alpha) + 2\epsilon$ whenever $\alpha \geq \alpha_0$. It is clear that (y'_α) is also uaw^*-null, hence we have $|f|(x'_\alpha) \to 0$. So $(E')^{\sim}_{uaw^*}$ is a sublattice of E. For the functionals $0 \leq g \leq f \in (E')^{\sim}_{uaw^*}$, clearly, $g(x'_\alpha) \leq f(x'_\alpha) \to 0$, hence $g \in (E')^{\sim}_{uaw^*}$. So $(E')^{\sim}_{uaw^*}$ is an ideal of E. Choose some $g \in (E')^{\sim}_{uaw^*}$ satisfying $\|f - g\| < \epsilon$. Since $f(x'_\alpha) = g(x'_\alpha) + (f - g)(x'_\alpha)$, we have $|f(x'_\alpha)| \leq |g(x'_\alpha)| + |(f - g)(x'_\alpha)|$. Hence $f \in (E')^{\sim}_{uaw^*}$. So $(E')^{\sim}_{uaw^*}$ is a closed ideal of E.

(2) and (3). It is clear that order convergence implies uo-convergence, and norm convergence implies un and uaw-convergence. So we can get that $E^{\sim}_{uo}$ is a subspace of $E^{\sim}_n$ and $E^{\sim}_{uaw}$, and $E^{\sim}_d$ and $E^{\sim}_{un}$ are subspaces of E'. The rest of the proof is similar to (1). □

Recall that the order continuous part E^a of a Banach lattice E is given by

$$E^a = \{x \in E : \text{every monotone increasing sequence in } [0, |x|] \text{ is norm convergent}\}.$$

According to [6, Corollary 2.3.6], it is equivalent to

$$E^a = \{x \in E : \text{every disjoint sequence in } [0, |x|] \text{ is norm convergent}\}.$$

A Banach lattice E is said to be *order continuous* whenever $\|x_\alpha\| \to 0$ for every net $x_\alpha \downarrow 0$ in E. By [6, Proposition 2.4.10], E^a is the largest closed ideal with order continuous norm of E.

The following results show some characterizations of the continuity of bounded uo, un, uaw, uaw^*, and d-continuous functionals.

Theorem 2.6. *Let E be a Banach lattice and $\mathscr{F}_x$ a functional on E' for any $x \in E$. The following conditions are equivalent.*

(1) $\mathscr{F}_x \in (E')^{\sim}_{uaw^*}$.
(2) $\mathscr{F}_x(x'_n) \to 0$ *for any bounded uaw^*-null sequence (x'_n) in E'.*
(3) $\mathscr{F}_x \in (E')^{\sim}_{uo}$.
(4) $\mathscr{F}_x(x'_n) \to 0$ *for any bounded uo-null sequence (x'_n) in E'.*
(5) $\mathscr{F}_x \in (E')^{\sim}_{uaw}$.
(6) $\mathscr{F}_x(x'_n) \to 0$ *for any bounded uaw-null sequence (x'_n) in E'.*
(7) $\mathscr{F}_x \in (E')^{\sim}_d$.
(8) $\mathscr{F}_x(x'_n) \to 0$ *for any bounded disjoint sequence (x'_n) in E'.*
(9) Every disjoint sequence in $[0, |x|]$ is norm convergent to zero.
In addition, if E' has order continuous norm, these conditions are equivalent to
(10) $\mathscr{F}_x \in (E')^{\sim}_{un}$.
(11) $\mathscr{F}_x(x'_n) \to 0$ *for any bounded un-null sequence (x'_n) in E'.*

Proof. (1) ⇒ (2), (3) ⇒ (4), (5) ⇒ (6), and (7) ⇒ (8) are obvious. (1) ⇒ (5) and (2) ⇒ (6) hold since uaw-convergence implies uaw^*-convergence.

(1) ⇒ (3). Let (x'_α) be an uo-null net in E' and $\mathscr{F}_x \in (E')^{\sim}_{uaw^*}$. Hence $|x'_\alpha| \wedge u' \leq y'_\beta \downarrow 0$ in E' for all $u' \in E'_+$. Clearly, for a positive element $x \in E$, we have $(|x'_\alpha| \wedge u')(x) \leq (y'_\beta)(x) \to 0$. Since $\mathscr{F}_x \in (E')^{\sim}_{uaw^*}$, $\mathscr{F}_x \in (E')^{\sim}_{uo}$. (2) ⇒ (4) is similar.

(1) ⇒ (7), (2) ⇒ (8), (5) ⇒ (7), and (6) ⇒ (8). It follows from [10, Lemma 2] and [8, Lemma 2.3] that every disjoint net is *uaw*-null and *uaw**-null. According to [3, Corollary 3.6], we have (4) ⇒ (8).

(8) ⇒ (1). Since (x'_n) is a disjoint sequence, a sequence (y'_n) statisfying $\{y'_n \in [-|x'_n|, |x'_n|]\}$ is also disjoint. Therefore $\sup_{y'_n \in [-|x'_n|,|x'_n|]} |\mathscr{F}_x(y'_n)| = |\mathscr{F}_x|(|x'_n|) \to 0$ for any disjoint sequence (x'_n) in $B_{E'}$. Applying [1, Theorem 4.36] to the seminorm $|\mathscr{F}_x|(|\cdot|)$, the identity operator T, and the solid set $B_{E'}$, we have that, for any $\epsilon > 0$, there exists $u' \in E'_+$ such that

$$\sup_{x' \in B_{E'}} |\mathscr{F}_x|(|x'| - |x'| \wedge u') = \sup_{x \in B_{E'}} |\mathscr{F}_x|\big((|x'| - u)^+\big) < \epsilon.$$

For a *uaw**-null net $(x'_\alpha) \subset B_{E'}$, we have $|x'_\alpha| \wedge u' \xrightarrow{w^*} 0$. Hence $|\mathscr{F}_x|(|x'_\alpha| \wedge u') \to 0$. Therefore $|\mathscr{F}_x(x'_\alpha)| \leq |\mathscr{F}_x|(|x'_\alpha|) \to 0$.

(8) ⇔ (9). According to [6, Corollary 2.3.3], let $A = [-|x|, |x|]$ and $B = B_{E'}$. Every disjoint sequence in $[0, |x|]$ is norm convergent to zero if and only if every disjoint sequence in $[-|x|, |x|]$ is uniform convergencent to zero on B. Since (x'_n) is disjoint if and only if $(|x'_n|)$ is disjoint and $\mathscr{F}_x(|x'_n|) = \sup_{g \in [-|x|,|x|]} |g(x'_n)|$, $\mathscr{F}_x(x'_n) \to 0$ for any bounded disjoint sequence (x'_n) in E' if and only if (x'_n) is uniform convergencent to zero on A. We have the result.

(5) ⇔ (11). Suppose now that E' is order continuous. According to [8, Theorem 2.4], the *uaw* and *uaw**-topologies coincide with the *un*-topology. The result can be easily verified. □

Theorem 2.7 (Extension of [4, Theorem 2.3]). *Let E be a Banach lattice. For any $f \in E^\sim_n$, the following conditions are equivalent.*

(1) $f \in E^\sim_{uo}$.
(2) $f(x_n) \to 0$ for any bounded uo-null sequence (x_n) in E.
(3) $f \in E^\sim_{uaw}$.
(4) $f(x_n) \to 0$ for any bounded uaw-null sequence (x_n) in E.
(5) $f \in E^\sim_d$.
(6) $f(x_n) \to 0$ for any bounded disjoint sequence (x_n) in E.
(7) Every disjoint sequence in $[0, |f|]$ is norm convergent to zero.
In addition, if E has order continuous norm, these conditions are equivalent to
(8) $f \in E^\sim_{un}$.
(9) $f(x_n) \to 0$ for any bounded un-null sequence (x_n) in E.

Proof. (1) ⇔ (2) ⇔ (6) ⇔ (7). By [4, Theorem 2.3].

(3) ⇔ (9) is similar to (5) ⇔ (11) of Theorem 2.6.

(3) ⇒ (4) and (5) ⇒ (6) are obvious. (3) ⇒ (5) and (4) ⇒ (6) hold since every disjoint net is *uaw*-null.

(6) ⇒ (3). Using [1, Theorem 4.36], the proof is similar to (3) ⇒ (1) of [4, Theorem 2.3] and (8) ⇒ (1) of Theorem 2.6. □

Similarly, we have the following.

Theorem 2.8. *Let E be a Banach lattice. For any $f \in E'$, the following conditions are equivalent.*

(1) $f \in E^{\sim}_{uaw}$.
(2) $f(x_n) \to 0$ *for any bounded uaw-null sequence* (x_n) *in* E.
(3) $f \in E^{\sim}_{d}$.
(4) $f(x_n) \to 0$ *for any bounded disjoint sequence* (x_n) *in* E.
(5) *Every disjoint sequence in* $[0, |f|]$ *is norm convergent to zero.*
In addition, if E *has order continuous norm, these conditions are equivalent to*
(6) $f \in E^{\sim}_{un}$.
(7) $f(x_n) \to 0$ *for any bounded un-null sequence* (x_n) *in* E.

Using the above results, we obtain the exact form of these duals.

Theorem 2.9. *Let* E *be a Banach lattice. The following relations hold.*
(1) $E^{\sim}_{uo} = (E^{\sim}_{n})^a \subset E^{\sim}_{uaw} = E^{\sim}_{d} = (E')^a \subset E^{\sim}_{un} \subset E'$.
(2) $(E')^{\sim}_{uaw^*} = E^a \subset (E')^{\sim}_{uo} = \big((E')^{\sim}_{n}\big)^a \subset (E')^{\sim}_{uaw} = (E')^{\sim}_{d} = (E'')^a \subset (E')^{\sim}_{un} \subset E''$.

Proof. According to Proposition 2.5, we have $(E')^{\sim}_{uaw^*} \subset E$, $E^{\sim}_{uo} \subset E^{\sim}_{n}$, $E^{\sim}_{uaw} \subset E'$, $E^{\sim}_{d} \subset E'$, and $E^{\sim}_{un} \subset E'$. It follows from Theorems 2.6, 2.7, and 2.8 that these duals are the order continuous part, therefore we have the result. □

The following example shows the bounded duals for unbounded convergence in classical Banach lattices.

Example 2.10.

$$(c_0)^{\sim}_{uo} = (c_0)^{\sim}_{uaw} = (c_0)^{\sim}_{d} = (c_0)^{\sim}_{un} = (l_1)^a = l_1,$$
$$(l_1)^{\sim}_{uaw^*} = (l_1)^{\sim}_{uo} = (l_1)^{\sim}_{uaw} = (l_1)^{\sim}_{d} = (l_\infty)^a = (c_0)^a = c_0,$$
$$(l_\infty)^{\sim}_{uaw^*} = (l_\infty)^{\sim}_{uo} = (l_1)^a = l_1,$$
$$(l_\infty)^{\sim}_{uaw} = (l_\infty)^{\sim}_{d} = (l_\infty)^{\sim}_{un} = ba(2^{\mathbb{N}}),$$
$$(L_1[0,1])^{\sim}_{uo} = (L_1[0,1])^{\sim}_{uaw} = (L_1[0,1])^{\sim}_{d} = (L_\infty[0,1])^a = \{0\},$$
$$(L_\infty[0,1])^{\sim}_{uaw^*} = (L_\infty[0,1])^{\sim}_{uo} = (L_1[0,1])^a = L_1[0,1],$$
$$(L_\infty[0,1])^{\sim}_{uaw} = (L_\infty[0,1])^{\sim}_{d} = (L_\infty[0,1])^{\sim}_{un} = (ba[0,1])^a = ba[0,1],$$
$$(C[0,1])^{\sim}_{uo} = (\{0\})^a = \{0\},$$
$$(C[0,1])^{\sim}_{uaw} = (C[0,1])^{\sim}_{d} = (C[0,1])^{\sim}_{un} = rca[0,1].$$

As an application of these results, we conclude the paper with characterizations of the order continuity and reflexivity of Banach lattices.

Theorem 2.11. *For a Banach lattice* E*, the following holds.*
(1) E *has order continuous norm if and only if* $(E')^{\sim}_{uaw^*} = E$;
(2) E' *has order continuous norm if and only if* $E^{\sim}_{uaw} = E^{\sim}_{d} = E^{\sim}_{un} = E'$;
(3) *(Extension of* [9, Theorem 5]*)* E *and* E' *are order continuous if and only if* $E^{\sim}_{uo} = E^{\sim}_{uaw} = E^{\sim}_{d} = E^{\sim}_{un} = E'$;
(4) E *is reflexive if and only if* $(E')^{\sim}_{uaw^*} = (E')^{\sim}_{uo} = (E')^{\sim}_{uaw} = (E')^{\sim}_{d} = (E')^{\sim}_{un} = E''$.

Proof. (1). E is order continuous if and only if $E^a = E$. It follows from Theorem 2.9(2) that E is order continuous if and only if $(E')^{\sim}_{uaw^*} = E^a = E$.

(2). E' is order continuous if and only if $(E')^a = E'$. According to Theorem 2.9(1), we have $E^{\sim}_{uaw} = E^{\sim}_{d} = (E')^a$, therefore E' is order continuous if and only if $E^{\sim}_{uaw} = E^{\sim}_{d} = E^{\sim}_{un} = (E')^a = E'$.

(3). E and E' are order continuous if and only if $(E^{\sim}_{n})^a = E'$. Since $E^{\sim}_{uo} = (E^{\sim}_{n})^a$, E and E' are order continuous if and only if $E^{\sim}_{uo} = E^{\sim}_{uaw} = E^{\sim}_{d} = E^{\sim}_{un} = (E^{\sim}_{n})^a = (E')^a = E'$.

(4). E is reflexive if and only if $E^a = E = E''$. Hence E is reflexive if and only if $(E')^{\sim}_{uaw^*} = (E')^{\sim}_{uo} = (E')^{\sim}_{uaw} = (E')^{\sim}_{d} = (E')^{\sim}_{un} = E^a = E = E''$ by $(E')^{\sim}_{uaw^*} = E^a$. □

So far, we still do not know what the exact form of $E^{\sim}_{un}$ is. So, we state the problem here.

Problem 2.12. What is the exact form of $E^{\sim}_{un}$?

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References

[1] Aliprantis, C.D., Burkinshaw, O.: Positive Operators. Springer, Amsterdam (2006)

[2] DeMarr, R.: Partially ordered linear spaces and locally convex linear topological spaces. Illinois J. Math **8**, 601–606 (1964)

[3] Gao, N., Troitsky, V., Xanthos, F.: Uo-convergence and its applications to Cesàro means in Banach lattices. Israel J. Math. **220**(2), 649–689 (2017)

[4] Gao, N., Leung, D.H., Xanthos, F.: Duality for unbounded order convergence and applications. Positivity **22**(3), 711–725 (2018)

[5] Kandić, M., Marabeh, M.A.A., Troitsky, V.G.: Unbounded norm topology in Banach lattices. J. Math. Anal. Appl. **451**(1), 259–279 (2017)

[6] Meyer-Nieberg, P.: Banach Lattices. Universitext, Berlin (1991)

[7] Nakano, H.: Ergodic theorems in semi-ordered linear spaces. Ann.of Math. **49**(2), 538–556 (1948)

[8] Wang, Z., Chen, Z., Chen, J.: On the unbounded convergence in Banach lattices and related operators. arXiv:1903.02168v11 (2019)

[9] Wickstead, A.: Weak and unbounded order convergence in Banach lattices. J. Aust. Math. Soc. Ser. A **24**(3), 312–319 (1977)

[10] Zabeti, O.: Unbounded absolute weak convergence in Banach lattices. Positivity **22**(2), 501–505 (2018)

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