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A new stabilization scenario for Timoshenko systems with thermo-diffusion effects in second spectrum perspective

A.J.A. Ramos, M. Aouadi, D.S. Almeida Júnior,
M.M. Freitas, and M.L. Araújo

Abstract. In this work, we analyze a truncated version for the Timoshenko beam model with thermal and mass diffusion effects derived by Aouadi et al. (Z Angew Math Phys 70:117, 2019). In particular, we study some issues related to the second spectrum of frequency according to a procedure due to Elishakoff (in: Advances in mathematical modelling and experimental methods for materials and structures, solid mechanics and its applications, Springer, Berlin, 2010). In Aouadi et al. (2019), the lack of exponential stability for the classical Timoshenko beam with thermo-diffusion effects without assuming the nonphysical condition of equal wave speeds has be proved. By using the classical Faedo–Galerkin method combined with the a priori estimates, we prove the existence and uniqueness of a global solution of the truncated version of this problem. Then we prove that this solution is exponentially stable without assuming the condition of equal wave speeds.

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Keywords. Timoshenko, Thermo-diffusion, Well-posedness, Exponential stability.

1. Introduction. Recently, Aouadi et al. [5] introduced a new Timoshenko beam model with thermal and mass diffusion effects given by

$$\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0 \quad \text{in} \quad]0, L[\times]0, \infty[, \tag{1.1}$$

$$\rho_2\psi_{tt} - \alpha\psi_{xx} + \kappa(\varphi_x + \psi) - \gamma_1\theta_x - \gamma_2 P_x = 0 \quad \text{in} \quad]0, L[\times]0, \infty[, \tag{1.2}$$

$$c\theta_t + dP_t - K\theta_{xx} - \gamma_1\psi_{xt} = 0 \quad \text{in} \quad]0, L[\times]0, \infty[, \tag{1.3}$$

$$d\theta_t + rP_t - \hbar P_{xx} - \gamma_2\psi_{xt} = 0 \quad \text{in} \quad]0, L[\times]0, \infty[, \tag{1.4}$$

where φ is the transverse displacement, ψ is the rotation of the neutral axis due to bending, θ is the temperature, and P is the chemical potential. The constants ρ_1, ρ_2, κ, α, γ_1, γ_2, c, r, d, $\hbar$, and K are physical positive parameters. They showed, without assuming the well-known equal wave speeds condition $\chi := \kappa/\rho_1 - b/\rho_2 = 0$, the lack of exponential stability for the problem. Based on [5] and the recent studies due to Almeida Júnior et al. [1–4], we consider the truncated version given by

$$\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0 \quad \text{in} \quad]0, L[\times]0, \infty[, \tag{1.5}$$

$$-\rho_2\varphi_{xtt} - \alpha\psi_{xx} + \kappa(\varphi_x + \psi) - \gamma_1\theta_x - \gamma_2 P_x = 0 \quad \text{in} \quad]0, L[\times]0, \infty[, \tag{1.6}$$

$$c\theta_t + dP_t - K\theta_{xx} - \gamma_1\psi_{xt} = 0 \quad \text{in} \quad]0, L[\times]0, \infty[, \tag{1.7}$$

$$d\theta_t + rP_t - \hbar P_{xx} - \gamma_2\psi_{xt} = 0 \quad \text{in} \quad]0, L[\times]0, \infty[, \tag{1.8}$$

with the initial conditions

$$\begin{aligned} &\varphi(x, 0) = \varphi_0(x),\ \varphi_t(x, 0) = \varphi_1(x),\ \varphi_{tt}(x, 0) = \varphi_2(x), \quad x \in (0, L), \\ &\psi(x, 0) = \psi_0(x),\ \theta(x, 0) = \theta_0(x),\ P(x, 0) = P_0(x), \quad x \in (0, L), \end{aligned} \tag{1.9}$$

and boundary conditions of Dirichlet-Neumann-type

$$\begin{aligned} &\varphi(0, t) = \varphi(L, t) = \psi_x(0, t) = \psi_x(L, t) = 0, \quad t \geq 0 \\ &\theta(0, t) = \theta(L, t) = P(0, t) = P(L, t) = 0, \quad t \geq 0. \end{aligned} \tag{1.10}$$

The truncated version (1.5)–(1.10) is obtained by following the procedure of Elishakoff [7] which involves replacing the term ψ_{tt} in (1.2) by $-\varphi_{xtt}$ based on d'Alembert's principle for dynamic equilibrium. This eliminates the second spectrum of frequency and its damaging consequences for wave propagation speed (see the first results in [1] and also in [9]). Therefore, the goal of this work is to prove the well-posedness of problem (1.5)–(1.10) and the exponential stability of solutions without assuming the nonphysical condition of equal wave speeds.

In order to derive the dissipative nature of the system (1.5)–(1.10), we define its functional energy of solutions

$$\begin{aligned} E(t) := &\frac{\rho_1}{2}\int_0^L |\varphi_t|^2 dx + \frac{\rho_1\rho_2}{2\kappa}\int_0^L |\varphi_{tt}|^2 dx + \frac{\rho_2}{2}\int_0^L |\varphi_{xt}|^2 dx \\ &+ \frac{\alpha}{2}\int_0^L |\psi_x|^2 dx + \frac{\kappa}{2}\int_0^L |\varphi_x + \psi|^2 dx + \frac{1}{2}(c - d^2/r)\int_0^L |\theta|^2 dx \\ &+ \frac{1}{2}\int_0^L \left|\frac{d}{\sqrt{r}}\theta + \sqrt{r}P\right|^2 dx, \end{aligned} \tag{1.11}$$

which preserves its positivity property for

$$cr - d^2 > 0. \tag{1.12}$$

2. Well-posedness. In this section, the existence and uniqueness of weak and strong solutions to (1.5)–(1.10) will be proved. To this end, we will use the Faedo–Galerkin approximations and pass to the limit by using compactness arguments (see also [6]).

We introduce the phase space

$$\mathcal{H} := H_0^1(0,L) \times H_0^1(0,L) \times L^2(0,L) \times H_*^1(0,L) \times L^2(0,L) \times L^2(0,L),$$

and

$$\mathcal{H}_1 := (H^2(0,L) \cap H_0^1(0,L))^2 \times H_0^1(0,L) \times H_*^2(0,L) \times H_0^1(0,L) \times H_0^1(0,L),$$

where

$$L_*^2(0,L) := \left\{ u \in L^2(0,L) : \int_0^L u(x)dx = 0 \right\},$$

and

$$H_*^1(0,L) := H^1(0,L) \cap L_*^2(0,L), \quad H_*^2(0,L) := H^2(0,L) \cap H_*^1(0,L).$$

In order to state our main result, we begin with a precise definition of a weak solution to (1.5)–(1.10).

Definition 2.1. Given initial data $(\varphi_0, \varphi_1, \varphi_2, \psi_0, \theta_0, P_0) \in \mathcal{H}$, a function $U = (\varphi, \varphi_t, \varphi_{tt}, \psi, \theta, P) \in C(0,T;\mathcal{H})$ is said to be a weak solution of (1.5)–(1.10) if for almost every $t \in [0,T]$,

$$\rho_1 \frac{d}{dt}(\varphi_t, u) + \kappa(\varphi_x + \psi, u_x) = 0, \tag{2.1}$$

$$\rho_1 \frac{d}{dt}(\varphi_{tt}, v) + \kappa \frac{d}{dt}(\varphi_x + \psi, v_x) = 0, \tag{2.2}$$

$$\rho_2 \frac{d}{dt}(\varphi_t, w_x) + \alpha(\psi_x, w_x) + \kappa(\varphi_x + \psi, w) + (\gamma_1\theta + \gamma_2 P, w_x) = 0, \tag{2.3}$$

$$\frac{d}{dt}(c\theta + dP, \xi) + K(\theta_x, \xi_x) + \gamma_1 \frac{d}{dt}(\psi, \xi_x) = 0, \tag{2.4}$$

$$\frac{d}{dt}(d\theta + rP, \zeta) + \hbar(P_x, \zeta_x) + \gamma_2 \frac{d}{dt}(\psi, \zeta_x) = 0, \tag{2.5}$$

for all $u, v, \xi, \zeta \in H_0^1(0,L)$, $w \in H_*^1(0,L)$, and

$$\big(\varphi(0), \varphi_t(0), \varphi_{tt}(0), \psi(0), \theta(0), P(0)\big) = \big(\varphi_0, \varphi_1, \varphi_2, \psi_0, \theta_0, P_0\big).$$

Theorem 2.2. *Suppose that condition* (1.12) *holds. Then we have:*
(i) If the initial data $(\varphi_0, \varphi_1, \varphi_2, \psi_0, \theta_0, P_0) \in \mathcal{H}$, *then problem* (1.5)–(1.10) *has a weak solution satisfying*

$$\varphi \in L^\infty\big(0,T; H_0^1(0,L)\big),\ \psi \in L^\infty\big(0,T; H_*^1(0,L)\big),$$
$$\varphi_t \in L^\infty\big(0,T; H_0^1(0,L)\big),\ \varphi_{tt} \in L^\infty\big(0,T; L^2(0,L)\big),$$
$$\theta \in L^\infty\big(0,T; L^2(0,L)\big),\ P \in L^\infty\big(0,T; L^2(0,L)\big).$$

(ii) If the initial data $(\varphi_0, \varphi_1, \varphi_2, \psi_0, \theta_0, P_0) \in \mathcal{H}_1$, *then problem* (1.5)–(1.10) *has a unique stronger weak solution satisfying*

$$\varphi \in L^\infty\big(0,T; H^2(0,L) \cap H_0^1(0,L)\big),\ \psi \in L^\infty\big(0,T; H_*^2(0,L)\big),$$

$$\varphi_t \in L^\infty\big(0,T;H^2(0,L)\cap H_0^1(0,L)\big),$$
$$\varphi_{tt} \in L^\infty\big(0,T;H_0^1(0,1)\big),\ \ \theta \in L^\infty\big(0,T;H_0^1(0,L)\big),$$
$$P \in L^\infty\big(0,T;H_0^1(0,L)\big).$$

(iii) In both cases, the solution $(\varphi,\varphi_t,\varphi_{tt},\psi,\theta,P)$ *depends continuously on the initial data in* $\mathcal{H}$*. In particular, problem* (1.5)–(1.10) *has a unique weak solution.*

Proof. The proof is given by the Faedo–Galerkin method. We only briefly present the main (six) steps.

Step 1 – Approximate problem. Let us consider the initial data $(\varphi_0,\varphi_1,\varphi_2,\psi_0,\theta_0,P_0)\in\mathcal{H}$. Let $\{\omega_j\}_{j=1}^\infty$ and $\{\mu_j\}_{j=1}^\infty$ be orthogonal bases for $H^2(0,L)\cap H_0^1(0,L)$ and $H_*^2(0,L)$, respectively, which are both orthonormal in $L^2(0,L)$. Now we denote the finite-dimensional subspaces, for any integer $n\in\mathbb{N}$, by

$$H_n = span\{\omega_1,\omega_2,...,\omega_n\},\quad V_n = span\{\mu_1,\mu_2,...,\mu_n\}.$$

We will find an approximate solution of the form

$$\varphi^n(x,t)=\sum_{j=1}^n a_{j,n}\omega_j(x),\quad \psi^n(x,t)=\sum_{j=1}^n b_{j,n}\mu_j(x), \tag{2.6}$$

$$\theta^n(x,t)=\sum_{j=1}^n c_{j,n}\omega_j(x),\quad P^n(x,t)=\sum_{j=1}^n d_{j,n}\omega_j(x), \tag{2.7}$$

to the following approximate problem

$$\rho_1(\varphi_{tt}^n,u)+\kappa(\varphi_x^n+\psi^n,u_x)=0, \tag{2.8}$$
$$\rho_1(\varphi_{ttt}^n,v)+\kappa(\varphi_{xt}^n+\psi_t^n,v_x)=0, \tag{2.9}$$
$$\rho_2(\varphi_{tt}^n,w_x)+\alpha(\psi_x^n,w_x)+\kappa(\varphi_x^n+\psi^n,w)+(\gamma_1\theta^n+\gamma_2P^n,w_x)=0, \tag{2.10}$$
$$(c\theta_t^n+dP_t^n,\xi)+K(\theta_x^n,\xi_x)+\gamma_1(\psi_t^n,\xi_x)=0, \tag{2.11}$$
$$(d\theta_t^n+rP_t^n,\zeta)+\hbar(P_x^n,\zeta_x)+\gamma_2(\psi_t^n,\zeta_x)=0, \tag{2.12}$$

for all $u,v,\xi,\zeta\in H_n$, $w\in V_n$ with initial conditions

$$(\varphi^n(0),\varphi_t^n(0),\varphi_{tt}^n(0),\psi^n(0),\theta^n(0),P^n(0))=(\varphi_0^n,\varphi_1^n,\varphi_2^n,\psi_0^n,\theta_0^n,P_0^n) \tag{2.13}$$

satisfying

$$(\varphi_0^n,\varphi_1^n,\varphi_2^n,\psi_0^n,\theta_0^n,P_0^n)\to(\varphi_0,\varphi_1,\varphi_2,\psi_0,\theta_0,P_0)\ \text{ strongly in }\ \mathcal{H}.$$

From the application of the standard ODE theory, we can obtain a local solution $\big(\varphi^n(t),\varphi_t^n(t),\varphi_{tt}^n(t),\psi^n(t),\theta^n(t),\ P^n(t)\big)$ on the maximal interval $[0,t_n)$ with $0<t_n\le T$ for every $n\in\mathbb{N}$.

Step 2 – A priori estimate. Replacing u by φ_t^n in (2.8), v by φ_{tt}^n in (2.9), w by ψ_t^n in (2.10), ξ by θ^n in (2.11), and ζ by P^n, we obtain

$$\frac{d}{dt}E^n(t)+K\int_0^L|\theta_x^n|^2dx+\hbar\int_0^L|P_x^n|^2dx=0, \tag{2.14}$$

where

$$E^n(t) := \frac{\rho_1}{2}\int_0^L |\varphi_t^n|^2 dx + \frac{\rho_1\rho_2}{2\kappa}\int_0^L |\varphi_{tt}^n|^2 dx + \frac{\rho_2}{2}\int_0^L |\varphi_{xt}^n|^2 dx$$
$$+\frac{\alpha}{2}\int_0^L |\psi_x^n|^2 dx + \frac{\kappa}{2}\int_0^L |\varphi_x^n + \psi^n|^2 dx + \frac{1}{2}(c - d^2/r)\int_0^L |\theta^n|^2 dx$$
$$+\frac{1}{2}\int_0^L \left|\frac{d}{\sqrt{r}}\theta^n + \sqrt{r}P^n\right|^2 dx.$$

Then integrating (2.14) from 0 to $t < t_n$, we obtain from our choice of initial data that for all $t \in [0, T]$ and for every $n \in \mathbb{N}$,

$$E^n(t) + K\int_0^t\int_0^L |\theta_x^n(s)|^2 dxds + \hbar\int_0^t\int_0^L |P_x^n(s)|^2 dxds \le C_1, \tag{2.15}$$

where C_1 is a positive constant depending on the initial data. Thus, approximate solutions are defined on the whole range $[0, T]$.

Step 3 – Passing to the limit. From (2.15) and definition of $E^n(t)$, we deduce that

$$\begin{cases} \{\varphi^n\} \text{ is bounded in } L^\infty\big(0,T;H_0^1(0,L)\big), \\ \{\varphi_t^n\} \text{ is bounded in } L^\infty\big(0,T;H_0^1(0,L)\big), \\ \{\varphi_{tt}^n\} \text{ is bounded in } L^\infty\big(0,T;L^2(0,L)\big), \\ \{\psi^n\} \text{ is bounded in } L^\infty\big(0,T;H_*^1(0,L)\big), \\ \{\theta^n\} \text{ is bounded in } L^\infty\big(0,T;L^2(0,L)\big) \cap L^2\big(0,T;H_0^1(0,L)\big), \\ \{P^n\} \text{ is bounded in } L^\infty\big(0,T;L^2(0,L)\big) \cap L^2\big(0,T;H_0^1(0,L)\big). \end{cases}$$

Then we can extract a subsequence of $\{\varphi^n\}$, $\{\psi^n\}$, $\{\theta^n\}$, and $\{P^n\}$ still denoted by $\{\varphi^n\}$, $\{\psi^n\}$, $\{\theta^n\}$, and $\{P^n\}$, such that

$$\begin{cases} \varphi^n \to \varphi \text{ weakly star in } L^\infty\big(0,T;H_0^1(0,L)\big), \\ \varphi_t^n \to \varphi_t \text{ weakly star in } L^\infty\big(0,T;H_0^1(0,L)\big), \\ \varphi_{tt}^n \to \varphi_{tt} \text{ weakly star in } L^\infty\big(0,T;L^2(0,L)\big), \\ \psi^n \to \psi \text{ weakly star in } L^\infty\big(0,T;H_*^1(0,L)\big), \\ \theta^n \to \theta \text{ weakly star in } L^\infty\big(0,T;L^2(0,L)\big), \\ \theta^n \to \theta \text{ weakly in } L^2\big(0,T;H_0^1(0,L)\big), \\ P^n \to P \text{ weakly star in } L^\infty\big(0,T;L^2(0,L)\big), \\ P^n \to P \text{ weakly in } L^2\big(0,T;H_0^1(0,L)\big). \end{cases}$$

Therefore the above limits allow us to pass to the limit in the approximate problem (2.8)–(2.12) to get a weak solution satisfying

$$\varphi \in L^\infty\big(0,T;H_0^1(0,L)\big),\ \psi \in L^\infty\big(0,T;H_*^1(0,L)\big),$$

$$\varphi_t \in L^\infty\big(0,T;H_0^1(0,L)\big),\ \varphi_{tt} \in L^\infty\big(0,T;L^2(0,L)\big),$$
$$\theta \in L^\infty\big(0,T;L^2(0,L)\big),\ P \in L^\infty\big(0,T;L^2(0,L)\big).$$

Step 4 – Initial data. By using Aubin-Lions lemma, see [8], we arrive at

$$\varphi^n \to \varphi \text{ strongly in } C(0,T;L^2(0,L)), \tag{2.16}$$
$$\varphi^n_t \to \varphi_t \text{ strongly in } C(0,T;L^2(0,L)). \tag{2.17}$$

Consequently

$$\big(\varphi(0),\varphi_t(0)\big) = \big(\varphi_0,\varphi_1\big).$$

Now, we multiply (2.9) by a test function

$$\eta \in H^1(0,T), \quad \eta(0)=1, \quad \eta(T)=0,$$

and integrate the result over $[0,T]$ to obtain

$$-\rho_1(\varphi_2^n,v) - \rho_1\int_0^T(\varphi_{tt}^n,v)\eta_t dt + \kappa\int_0^T\frac{d}{dt}(\varphi_x^n+\psi^n,v_x)\eta dt = 0$$

for all $v \in H_0^1(0,L)$. Taking the limit $n\to\infty$, we obtain

$$-\rho_1(\varphi_2,v) - \rho_1\int_0^T(\varphi_{tt},v)\eta_t dt + \kappa\int_0^T\frac{d}{dt}(\varphi_x+\psi,v_x)\eta dt = 0 \tag{2.18}$$

for all $v \in H_0^1(0,L)$. On the other hand, multiplying (2.2) by η and integrating the result over $[0,T]$, we obtain

$$-\rho_1(\varphi_{tt}(0),v) - \rho_1\int_0^T(\varphi_{tt},v)\eta_t dt + \kappa\int_0^T\frac{d}{dt}(\varphi_x+\psi,v_x)\eta dt = 0 \tag{2.19}$$

for all $v \in H_0^1(0,L)$. Combining (2.18) and (2.19), we conclude that $\varphi_{tt}(0) = \varphi_2$. Analogously, we obtain

$$\big(\psi(0),\theta(0),P(0)\big) = \big(\psi_0,\theta_0,P_0\big).$$

Step 5—Stronger solutions. Suppose that the initial data in the approximate problem (2.8)–(2.12) satisfies $(\varphi_0,\varphi_1,\varphi_2,\psi_0,\theta_0,P_0)\in\mathcal{H}_1$ and

$$(\varphi_0^n,\varphi_1^n,\varphi_2^n,\psi_0^n,\theta_0^n,P_0^n) \to (\varphi_0,\varphi_1,\varphi_2,\psi_0,\theta_0,P_0) \ \text{ strongly in } \ \mathcal{H}_1. \tag{2.20}$$

Replacing u by $-\varphi_{xxt}^n$ in (2.8), v by $-\varphi_{xxtt}^n$ in (2.9), w by $-\psi_{xxt}^n$ in (2.10), ξ by $-\theta_{xx}^n$ in (2.11), and ζ by $-P_{xx}$ in (2.12), we see that

$$\frac{d}{dt}F^n(t) + K\int_0^L|\theta_{xx}^n|^2dx + \hbar\int_0^L|P_{xx}^n|^2dx = 0, \tag{2.21}$$

where

$$F^n(t) := \frac{\rho_1}{2}\int_0^L|\varphi_{xt}^n|^2dx + \frac{\rho_1\rho_2}{2\kappa}\int_0^L|\varphi_{xtt}^n|^2dx + \frac{\rho_2}{2}\int_0^L|\varphi_{xxt}^n|^2dx$$

$$+\frac{\alpha}{2}\int_0^L |\psi_{xx}^n|^2dx + \frac{\kappa}{2}\int_0^L |\varphi_{xx}^n + \psi_x^n|^2dx + \frac{1}{2}(c - d^2/r)\int_0^L |\theta_x^n|^2dx$$
$$+\frac{1}{2}\int_0^L \left|\frac{d}{\sqrt{r}}\theta_x^n + \sqrt{r}P_x^n\right|^2 dx.$$

Then from (2.21), we obtain that for all $t \in [0,T]$, $n \in \mathbb{N}$,

$$F^n(t) + K\int_0^t\int_0^L |\theta_{xx}^n(s)|^2dxds + \hbar\int_0^t\int_0^L |P_{xx}^n(s)|^2dxds \le C_2, \qquad (2.22)$$

where C_2 is a positive constant independent of t and n but depending on the initial data. From (2.22), we deduce that

$$\begin{cases} \{\varphi^n\} \text{ is bounded in } L^\infty\big(0,T;H^2(0,L)\cap H_0^1(0,L)\big),\\ \{\varphi_t^n\} \text{ is bounded in } L^\infty\big(0,T;H^2(0,L)\cap H_0^1(0,L)\big),\\ \{\varphi_{tt}^n\} \text{ is bounded in } L^\infty\big(0,T;H_0^1(0,L)\big),\\ \{\psi^n\} \text{ is bounded in } L^\infty\big(0,T;H_*^2(0,L)\big),\\ \{\theta^n\} \text{ is bounded in } L^\infty\big(0,T;H_0^1(0,L)\big)\cap L^2\big(0,T;H^2(0,L)\cap H_0^1(0,L)\big),\\ \{P^n\} \text{ is bounded in } L^\infty\big(0,T;H_0^1(0,L)\big)\cap L^2\big(0,T;H^2(0,L)\cap H_0^1(0,L)\big). \end{cases}$$

This implies that

$$\begin{cases} \varphi^n \to \varphi \text{ weakly star in } L^\infty\big(0,T;H^2(0,L)\cap H_0^1(0,L)\big),\\ \varphi_t^n \to \varphi_t \text{ weakly star in } L^\infty\big(0,T;H^2(0,L)\cap H_0^1(0,L)\big),\\ \varphi_{tt}^n \to \varphi_{tt} \text{ weakly star in } L^\infty\big(0,T;H_0^1(0,L)\big),\\ \psi^n \to \psi \text{ weakly star in } L^\infty\big(0,T;H_*^2(0,L)\big),\\ \theta^n \to \theta \text{ weakly star in } L^\infty\big(0,T;H_0^1(0,L)\big),\\ \theta^n \to \theta \text{ weakly in } L^2\big(0,T;H^2(0,L)\cap H_0^1(0,L)\big),\\ P^n \to P \text{ weakly star in } L^\infty\big(0,T;H_0^1(0,L)\big),\\ P^n \to P \text{ weakly in } L^2\big(0,T;H^2(0,L)\cap H_0^1(0,L)\big). \end{cases}$$

From the above limits, we conclude that $(\varphi,\varphi_t,\varphi_{tt},\psi,\theta,P)$ is a stronger weak solution satisfying

$$\begin{aligned} &\varphi \in L^\infty\big(0,T;H^2(0,L)\cap H_0^1(0,L)\big),\ \psi \in L^\infty\big(0,T;H_*^2(0,L)\big),\\ &\varphi_t \in L^\infty\big(0,T;H^2(0,L)\cap H_0^1(0,L)\big),\\ &\varphi_{tt} \in L^\infty\big(0,T;H_0^1(0,1)\big),\ \theta \in L^\infty\big(0,T;H_0^1(0,L)\big),\\ &P \in L^\infty\big(0,T;H_0^1(0,L)\big). \end{aligned}$$

Step 6 – Continuous dependence. Firstly, we consider the case of stronger solutions. Let $U(t) = (\varphi,\varphi_t,\varphi_{tt},\psi,\theta,P)$ and $V(t) = (\tilde{\varphi},\tilde{\varphi}_t,\tilde{\varphi}_{tt},\tilde{\psi},\tilde{\theta},\tilde{P})$ be the stronger weak solutions of the problem (1.5)–(1.8) corresponding to the initial data $U(0) = (\varphi_0,\varphi_1,\varphi_2,\psi_0,\ \theta_0,P_0)$, $V(0) = (\tilde{\varphi_0},\tilde{\varphi_1},\tilde{\varphi_2},\tilde{\psi_0},\tilde{\theta_0},\tilde{P_0}) \in$

$\mathcal{H}_1$, respectively. Then $(\Phi, \Phi_t, \Phi_{tt}, \Psi, \Theta, \Upsilon) = U(t)-V(t)$ satisfies the following equations:

$$\rho_1\Phi_{tt} - \kappa(\Phi_x + \Psi)_x = 0, \tag{2.23}$$

$$-\rho_2\Phi_{xtt} - b\Psi_{xx} + \kappa(\Phi_x + \Psi) - \gamma_1\Theta_x - \gamma_2\Upsilon_x = 0, \tag{2.24}$$

$$c\Theta_t + d\Upsilon_t - K\Theta_{xx} - \gamma_1\Psi_{xt} = 0, \tag{2.25}$$

$$d\Theta_t + r\Upsilon_t - \hbar\Upsilon_{xx} - \gamma_2\Psi_{xt} = 0, \tag{2.26}$$

with initial data $(\Phi(0), \Phi_t(0), \Phi_{tt}(0), \Psi(0), \Theta(0), \Upsilon(0)) = U(0) - V(0)$.

We multiply (2.23) by Φ_t, (2.24) by Ψ_t, (2.25) by Θ, and (2.26) by Υ and integrate the result over $(0, L)$ to derive

$$\frac{d}{dt}\widehat{E}(t) = -K\int_0^L |\Theta_x^n|^2 dx - \hbar\int_0^L |\Upsilon_x^n|^2 dx, \tag{2.27}$$

where $\widehat{E}(t)$ is the energy corresponding to $U(t) - V(t)$ defined by

$$\widehat{E}(t) = \frac{\rho_1\rho_2}{2\kappa}\int_0^L \Phi_{tt}^2 dx + \frac{\rho_2}{2}\int_0^L \Phi_{xt}^2 dx + \frac{\rho_1}{2}\int_0^L \Phi_t^2 dx$$
$$+\frac{\kappa}{2}\int_0^L (\Phi_x + \Psi)^2 dx + \frac{\alpha}{2}\int_0^L \Psi_x^2 dx + \frac{1}{2}(c - d^2/r)\int_0^L |\Theta^n|^2 dx$$
$$+\frac{1}{2}\int_0^L \left|\frac{d}{\sqrt{r}}\Theta^n + \sqrt{r}\Upsilon^n\right|^2 dx.$$

Integrating (2.27) over $(0, t)$, we get that there exists a constant $C_T > 0$ such that for any $t \in [0, T]$,

$$\widehat{E}(t) \le C_T\widehat{E}(0),$$

which implies the continuous dependence of stronger weak solutions on the initial data. Then we know that the stronger weak solution of problem (1.5)–(1.10) is unique. The continuous dependence and uniqueness for weak solutions can be proved by using density arguments (weak solutions are limits of stronger weak solutions). Combining the above analysis, we complete the proof of Theorem 2.2. □

3. Exponential decay. In this section, we use the energy method to prove that $E(t)$, the energy of system (1.5)–(1.10) given by (1.11), decays exponentially. For this, we assume that $(\varphi, \varphi_t, \varphi_{tt}, \psi, \theta, P)$ is a solution of the system (1.5)–(1.10) with the regularity stated in Theorem 2.2 and we suppose that condition (1.12) holds true.

Theorem 3.1. *Suppose that the hypotheses of Theorem 2.2 hold. Then there exist two positive constants M and η such that*

$$E(t) \le ME(0)e^{-\eta t}, \quad \forall t \ge 0. \tag{3.1}$$

The proof of Theorem 3.1 will be established through several lemmas. We have the first lemma regarding the dissipative nature of the energy.

Lemma 3.2. *The energy $E(t)$ of the system (1.5)–(1.10) satisfies the energy dissipation law given by*

$$\frac{d}{dt}E(t) = -K\int_0^L |\theta_x|^2 dx - \hbar\int_0^L |P_x|^2 dx, \quad t \geq 0. \tag{3.2}$$

Proof. Multiplying Eq. (1.5) by φ_t, (1.6) by ψ_t, (1.7) by θ, (1.8) by P, and integrating the result over $[0, L]$, we obtain the desired result. □

We set

$$\mathcal{F}_1(t) := -\rho_1\int_0^L \varphi_t\varphi dx. \tag{3.3}$$

Lemma 3.3. *Suppose that the hypotheses of Theorem 2.2 hold. Then we have*

$$\frac{d}{dt}\mathcal{F}_1(t) \leq -\rho_1\int_0^L |\varphi_t|^2 dx + \kappa c_p\int_0^L |\psi_x|^2\, dx + \frac{3\kappa}{2}\int_0^L |\varphi_x+\psi|^2 dx, \tag{3.4}$$

where $c_p > 0$ is the Poincaré constant.

Proof. Multiplying Eq. (1.5) by φ, integrating over $[0, L]$ using integration by parts, and taking into account the boundary conditions (1.10), we have

$$\rho_1\int_0^L \varphi_{tt}\varphi\, dx + \kappa\int_0^L (\varphi_x+\psi)\varphi_x\, dx = 0. \tag{3.5}$$

Taking into account the identity $\varphi_{tt}\varphi = \frac{\partial}{\partial t}(\varphi_t\varphi) - |\varphi_t|^2$ and Young's inequality, we arrive at

$$-\frac{d}{dt}\left(\rho_1\int_0^L \varphi_t\varphi\, dx\right) \leq -\rho_1\int_0^L |\varphi_t|^2 dx + \frac{\kappa}{2}\int_0^L |\varphi_x+\psi|^2 dx + \frac{\kappa}{2}\int_0^L |\varphi_x|^2\, dx. \tag{3.6}$$

Moreover, we consider the inequality given by $\int_0^L |\varphi_x|^2 dx \leq 2\int_0^L |\varphi_x+\psi|^2 dx + 2c_p\int_0^L |\psi_x|^2 dx$, we complete the proof. □

Lemma 3.4. *Suppose that the hypotheses of Theorem 2.2 hold. Then we have*

$$\frac{d}{dt}\mathcal{F}_2(t) \leq -\frac{\rho_1\rho_2}{\kappa}\int_0^L |\varphi_{tt}|^2 dx - \frac{\alpha}{2}\int_0^L |\psi_x|^2 dx + \rho_2\int_0^L |\varphi_{xt}|^2\, dx$$

$$+\frac{3\kappa^2 c_p}{2\alpha}\int_0^L |\varphi_x+\psi|^2\,dx+\frac{3\gamma_1^2 c_p}{2\alpha}\int_0^L |\theta_x|^2 dx+\frac{3\gamma_2^2 c_p}{2\alpha}\int_0^L |P_x|^2 dx, \quad (3.7)$$

where

$$\mathcal{F}_2(t):=\rho_2\int_0^L \varphi_{xt}\varphi_x\,dx. \quad (3.8)$$

Proof. Multiplying Eq. (1.6) by ψ and integrating by parts, we obtain

$$\rho_2\int_0^L \varphi_{tt}\psi_x dx+\alpha\int_0^L |\psi_x|^2 dx+\kappa\int_0^L(\varphi_x+\psi)\psi\,dx-\gamma_1\int_0^L \theta_x\psi\,dx$$
$$-\gamma_2\int_0^L P_x\psi\,dx=0. \quad (3.9)$$

It follows from Eq. (1.5) that $\psi_x=\frac{\rho_1}{\kappa}\varphi_{tt}-\varphi_{xx}$. Then, substituting ψ_x into (3.9), we obtain

$$\frac{d}{dt}\left(\rho_2\int_0^L \varphi_{xt}\varphi_x\,dx\right)-\rho_2\int_0^L |\varphi_{xt}|^2\,dx+\frac{\rho_1\rho_2}{\kappa}\int_0^L |\varphi_{tt}|^2 dx$$
$$+\alpha\int_0^L |\psi_x|^2\,dx+\kappa\int_0^L(\varphi_x+\psi)\psi\,dx-\gamma_1\int_0^L \theta_x\psi\,dx-\gamma_2\int_0^L P_x\psi dx=0,$$

and using Young's and Poincare's inequalities, we arrive at the desired result. □

Let us introduce one more functional which is given by

$$\mathcal{F}_3(t):=\frac{\alpha\rho_1}{\kappa}\int_0^L \psi_x\varphi_t dx-\frac{C_1}{\gamma_1}c\int_0^L \theta\varphi_t dx-\frac{C_1}{\gamma_1}d\int_0^L P\varphi_t dx$$
$$-\rho_2\int_0^L \varphi_{xt}(\varphi_x+\psi)dx. \quad (3.10)$$

Lemma 3.5. *Suppose that the hypotheses of Theorem 2.2 the hold. Then for all $\varepsilon>0$, there are constants $C_i>0$ $(i=1,2,3)$ such that*

$$\frac{d}{dt}\mathcal{F}_3(t)\le-\frac{\rho_2}{2}\int_0^L |\varphi_{xt}|^2 dx-\frac{\kappa}{2}\int_0^L |\varphi_x+\psi|^2 dx+C_2\int_0^L |\theta_x|^2 dx$$
$$+C_3\int_0^L |P_x|^2 dx+\frac{C_1}{\gamma_1}(c+d)\varepsilon\int_0^L |\varphi_{tt}|^2 dx. \quad (3.11)$$

Proof. Multiplying Eq. (1.6) by $(\varphi_x + \psi)$, integrating over $[0, L]$, and using integration by parts, we have

$$-\rho_2 \int_0^L \varphi_{xtt}(\varphi_x + \psi)\, dx + \alpha \int_0^L \psi_x(\varphi_x + \psi)_x\, dx + \kappa \int_0^L |\varphi_x + \psi|^2 dx$$
$$-\gamma_1 \int_0^L \theta_x(\varphi_x + \psi)\, dx - \gamma_2 \int_0^L P_x(\varphi_x + \psi)\, dx = 0. \tag{3.12}$$

Now, using Young's inequality, it follows that

$$-\rho_2 \int_0^L \varphi_{xtt}(\varphi_x + \psi)\, dx + \alpha \int_0^L \psi_x(\varphi_x + \psi)_x\, dx \le -\frac{\kappa}{2} \int_0^L |\varphi_x + \psi|^2 dx$$
$$+\frac{\gamma_1^2}{\kappa} \int_0^L |\theta_x|^2 dx + \frac{\gamma_2^2}{\kappa} \int_0^L |P_x|^2 dx. \tag{3.13}$$

On the other hand, it follows from Eq. (1.5) that $(\varphi_x + \psi)_x = \dfrac{\rho_1}{\kappa}\varphi_{tt}$ and then we can rewrite the above inequality as

$$-\rho_2 \int_0^L \varphi_{xtt}(\varphi_x + \psi)\, dx + \frac{\alpha\rho_1}{\kappa} \int_0^L \varphi_{tt}\psi_x\, dx \le -\frac{\kappa}{2} \int_0^L |\varphi_x + \psi|^2 dx$$
$$+\frac{\gamma_1^2}{\kappa} \int_0^L |\theta_x|^2 dx + \frac{\gamma_2^2}{\kappa} \int_0^L |P_x|^2 dx.$$

Moreover, we consider the identity given by $\varphi_{xtt}(\varphi_x + \psi) = \dfrac{\partial}{\partial t}\big[\varphi_{xt}(\varphi_x + \psi)\big] - \varphi_{xt}(\varphi_x + \psi)_t$ from where we obtain

$$-\frac{d}{dt}\left(\rho_2 \int_0^L \varphi_{xt}(\varphi_x + \psi)\, dx + \rho_2 \int_0^L \varphi_t\psi_x\, dx\right) \le -\rho_2 \int_0^L |\varphi_{xt}|^2 dx$$
$$-C_1 \int_0^L \varphi_{tt}\psi_x\, dx - \frac{\kappa}{2} \int_0^L |\varphi_x + \psi|^2 dx + \frac{\gamma_1^2}{\kappa} \int_0^L |\theta_x|^2 dx$$
$$+\frac{\gamma_2^2}{\kappa} \int_0^L |P_x|^2 dx, \tag{3.14}$$

where $C_1 := \alpha\big(\rho_1/\kappa + \rho_2/\alpha\big)$. On the other hand, multiplying Eq. (1.7) by $C_1\gamma_1^{-1}\varphi_t$, we have

$$\frac{C_1}{\gamma_1}c\int_0^L \theta_t\varphi_t dx + \frac{C_1}{\gamma_1}d\int_0^L P_t\varphi_t dx + \frac{C_1}{\gamma_1}K\int_0^L \theta_x\varphi_{xt}dx - C_1\int_0^L \psi_{xt}\varphi_t dx = 0.$$

Taking into account the identities $\theta_t\varphi_t = \frac{\partial}{\partial t}(\theta\varphi_t) - \theta\varphi_{tt}$, $P_t\varphi_t = \frac{\partial}{\partial t}(P\varphi_t) - P\varphi_{tt}$ and $\psi_{xt}\varphi_t = \frac{\partial}{\partial t}(\psi_x\varphi_t) - \psi_x\varphi_{tt}$, we have

$$\frac{d}{dt}\left(C_1\int_0^L \psi_x\varphi_t dx - \frac{C_1}{\gamma_1}c\int_0^L \theta\varphi_t dx - \frac{C_1}{\gamma_1}d\int_0^L P\varphi_t dx\right)$$
$$= -\frac{C_1}{\gamma_1}c\int_0^L \theta\varphi_{tt}dx - \frac{C_1}{\gamma_1}d\int_0^L P\varphi_{tt}dx + \frac{C_1}{\gamma_1}K\int_0^L \theta_x\varphi_{xt}dx$$
$$+ C_1\int_0^L \psi_x\varphi_{tt}dx. \tag{3.15}$$

Adding (3.14) and (3.15), we obtain

$$\frac{d}{dt}\mathcal{F}_3(t) \le -\rho_2\int_0^L |\varphi_{xt}|^2dx - \frac{\kappa}{2}\int_0^L |\varphi_x+\psi|^2dx + \frac{\gamma_1^2}{\kappa}\int_0^L |\theta_x|^2dx$$
$$+ \frac{\gamma_2^2}{\kappa}\int_0^L |P_x|^2dx - \frac{C_1}{\gamma_1}c\int_0^L \theta\varphi_{tt}dx - \frac{C_1}{\gamma_1}d\int_0^L P\varphi_{tt}dx + \frac{C_1}{\gamma_1}K\int_0^L \theta_x\varphi_{xt}dx$$

and using again Young's inequality, we arrive at the desired result. □

Lemma 3.6. *Suppose that the hypotheses of Theorem 2.2 hold. Then there are constants λ, N_0, $\delta > 0$ such that*

$$\frac{d}{dt}\Big(\lambda E(t) + N_0E(t)\Big) \le -\frac{1}{2}\Big(c - \frac{d^2}{r}\Big)\delta\int_0^L |\theta|^2dx$$
$$-\frac{1}{2}\delta\int_0^L \left|\frac{d}{\sqrt{r}}\theta + \sqrt{r}P\right|^2dx - N_0K\int_0^L |\theta_x|^2dx - N_0\hbar\int_0^L |P_x|^2dx.$$

Proof. First, we define the constant $\lambda_0 := \max\{d^2c_p/Kr,\ rc_p/\hbar\}$. Then, choosing $\lambda > \lambda_0$ and using (3.2), we have

$$\frac{d}{dt}\Big(\lambda E(t) + N_0E(t)\Big) = -\lambda K\int_0^L |\theta_x|^2dx - \lambda\hbar\int_0^L |P_x|^2dx$$
$$-N_0K\int_0^L |\theta_x|^2dx - N_0\hbar\int_0^L |P_x|^2dx. \tag{3.16}$$

Then, using the Poincaré inequality, we get

$$\frac{d}{dt}\Big(\lambda E(t)+N_0E(t)\Big)\leq -\frac{\lambda K}{2c_p}\int_0^L|\theta|^2dx-\frac{\lambda\hbar}{2c_p}\int_0^L|P|^2dx-\frac{\lambda K}{2c_p}\int_0^L|\theta|^2dx$$
$$-\frac{\lambda\hbar}{2c_p}\int_0^L|P|^2dx-N_0K\int_0^L|\theta_x|^2dx-N_0\hbar\int_0^L|P_x|^2dx.$$

Since $\frac{d^2\lambda K}{2rcc_p}\int_0^L|\theta|^2dx>0$ and $-\frac{\lambda\hbar}{2c_p}\int_0^L|P|^2dx<0$, we get the following estimate

$$\frac{d}{dt}\Big(\lambda E(t)+N_0E(t)\Big)\leq -(c-\frac{d^2}{r})\frac{\lambda K}{2cc_p}\int_0^L|\theta|^2dx-\frac{\lambda K}{2c_p}\int_0^L|\theta|^2dx$$
$$-\frac{\lambda\hbar}{2c_p}\int_0^L|P|^2dx-N_0K\int_0^L|\theta_x|^2dx-N_0\hbar\int_0^L|P_x|^2dx. \tag{3.17}$$

Next, we add the term $\frac{d^2}{2r}\int_0^L|\theta|^2dx>0$

$$\frac{d}{dt}\Big(\lambda E(t)+N_0E(t)\Big)\leq -N_0K\int_0^L|\theta_x|^2dx-N_0\hbar\int_0^L|P_x|^2dx$$
$$-\frac{1}{2}\left(\frac{\lambda\hbar}{rc_p}-1\right)r\int_0^L|P|^2dx-\frac{1}{2}\left(\frac{d^2}{r}\int_0^L|\theta|^2dx+r\int_0^L|P|^2dx\right)$$
$$-(c-\frac{d^2}{r})\frac{\lambda K}{2cc_p}\int_0^L|\theta|^2dx-\frac{1}{2}\left(\frac{\lambda Kr}{d^2c_p}-1\right)\frac{d^2}{r}\int_0^L|\theta|^2dx. \tag{3.18}$$

Since $\lambda>\lambda_0=\max\{d^2c_p/Kr,\ rc_p/\hbar\}$, we have $\zeta_1:=\frac{\lambda Kr}{d^2c_p}-1>0$ and $\zeta_2:=\frac{\lambda\hbar}{rc_p}-1>0$. Using Young's inequality, we have

$$2d\int_0^L\theta P dx\leq\frac{d^2}{r}\int_0^L|\theta|^2dx+r\int_0^L|P|^2dx. \tag{3.19}$$

Replacing (3.19) in (3.18), we have

$$\frac{d}{dt}\Big(\lambda E(t)+N_0E(t)\Big)\leq -\big(c-d^2/r\big)\frac{\lambda K}{2cc_p}\int_0^L|\theta|^2dx-\frac{\zeta_1}{2}\frac{d^2}{r}\int_0^L|\theta|^2dx$$
$$-\frac{\zeta_2}{2}r\int_0^L|P|^2dx-d\int_0^L\theta P dx-N_0K\int_0^L|\theta_x|^2dx-N_0\hbar\int_0^L|P_x|^2dx.$$

Therefore,

$$\frac{d}{dt}\Big(\lambda E(t) + N_0 E(t)\Big) \leq -\frac{1}{2}(c - \frac{d^2}{r})\delta \int_0^L |\theta|^2 dx$$
$$-\frac{1}{2}\delta \int_0^L \left|\frac{d}{\sqrt{r}}\theta + \sqrt{r}P\right|^2 dx - N_0 K \int_0^L |\theta_x|^2 dx - N_0 \hbar \int_0^L |P_x|^2 dx, \quad \forall\, t \geq 0,$$

where $\delta := \min\{1,\ \lambda K/cc_p,\ \zeta_1,\ \zeta_2\}$. □

Now we are ready to prove the main result of this paper.

Proof of Theorem 3.1. We consider the following Lyapunov functional defined by

$$\mathcal{L}(t) := (\lambda + N_0)E(t) + \mathcal{F}_1(t) + N_2\mathcal{F}_2(t) + N_3\mathcal{F}_3(t), \tag{3.20}$$

where $\lambda > \lambda_0 := \max\big\{d^2 c_p/Kr,\ rc_p/\hbar\big\}$ and $N_i,\ i = 0, 2, 3,$ are positive constants to be fixed later. Moreover, the coefficients N_0 and λ will be chosen large enough such that $\mathcal{L}(t)$ and $E(t)$ are equivalent. Indeed, from Young's and Poincaré's inequalities, we infer that there exists a constant $0 < c < N_0 + \lambda$ such that

$$|\mathcal{L}(t) - (N_0 + \lambda)E(t)| \leq |\mathcal{F}_1(t)| + N_2|\mathcal{F}_2(t)| + N_3|\mathcal{F}_3(t)| \leq cE(t),\ \forall t \geq 0.$$

Consequently,

$$(N_0 + \lambda - c)E(t) \leq \mathcal{L}(t) \leq (N_0 + \lambda + c)E(t),\ \forall t \geq 0. \tag{3.21}$$

Substituting the results of Lemmas 3.3, 3.4, and 3.5 in the time derivative of $\mathcal{L}(t)$, we obtain after selecting $\varepsilon := \frac{\gamma_1\rho_1\rho_2}{2\kappa(c+d)C_1N_3}$,

$$\frac{d}{dt}\mathcal{L}(t) \leq -\rho_1 \int_0^L |\varphi_t|^2 dx - \Big(2N_2 - 1\Big)\frac{\rho_1\rho_2}{2\kappa} \int_0^L |\varphi_{tt}|^2 dx$$
$$-\Big(N_3 - 2N_2\Big)\frac{\rho_2}{2} \int_0^L |\varphi_{xt}|^2 dx - \left(N_2 - \frac{2c_p\kappa}{\alpha}\right)\frac{\alpha}{2} \int_0^L |\psi_x|^2 dx$$
$$-\left(N_3 - 3 - \frac{3\kappa c_p}{\alpha}N_2\right)\frac{\kappa}{2} \int_0^L |\varphi_x + \psi|^2 dx - \frac{1}{2}\Big(c - d^2/r\Big)\delta \int_0^L |\theta|^2 dx$$

$$-\frac{1}{2}\delta\int_0^L\left|\frac{d}{\sqrt{r}}\theta+\sqrt{r}P\right|^2dx-\left(KN_0-\frac{3\gamma_1^2c_p}{2\alpha}N_2-C_2N_3\right)\int_0^L|\theta_x|^2dx$$
$$-\left(\hbar N_0-\frac{3\gamma_2^2c_p}{2\alpha}N_2-C_3N_3\right)\int_0^L|P_x|^2dx.$$

By choosing $N_2>\max\left\{1/2,\, 2c_p\kappa/\alpha\right\}$, $N_3>\max\left\{2N_2,\, 3+3\kappa c_pN_2/\alpha\right\}$ and N_0 large enough such that $N_0>\max\left\{\frac{3\gamma_1^2c_p}{2K\alpha}N_2+\frac{C_2}{K}N_3,\, \frac{3\gamma_2^2c_p}{2\hbar\alpha}N_2+\frac{C_3}{\hbar}N_3\right\}$, one can obtain that $\xi_1:=2N_2-1>0$, $\xi_2:=N_3-2N_2>0$, $\xi_3:=N_2-\frac{2c_p\kappa}{\alpha}>0$, $\xi_4:=N_3-3-\frac{3\kappa c_p}{\alpha}N_2>0$, $\xi_5:=KN_1-\frac{3\gamma_1^2c_p}{2\alpha}N_2-C_2N_3>0$, $\xi_6:=\hbar N_1-\frac{3\gamma_2^2c_p}{2\alpha}N_2-C_3N_3>0$.

Now, we can conclude that there exists a positive constant $\omega:=\min\Big\{2,\, \xi_1,\, \xi_2,\, \xi_3,\, \xi_4,\, \delta\Big\}>0$ such that

$$\frac{d}{dt}\mathcal{L}(t)\le-\omega E(t),\quad \forall t\ge 0. \tag{3.22}$$

Combining (3.21) with (3.22), we obtain

$$\frac{d}{dt}\mathcal{L}(t)\le-\frac{\omega}{N_0+\lambda+c}\mathcal{L}(t),\qquad \forall t\ge 0. \tag{3.23}$$

From this and using (3.21) again, we have after integrating over $(0,t)$,

$$E(t)\le\frac{N_0+\lambda+c}{N_0+\lambda-c}E(0)e^{-\frac{\beta t}{N_0+\lambda+c}},\qquad \forall t\ge 0 \tag{3.24}$$

which gives (3.1) with $M:=\frac{N_0+\lambda+c}{N_0+\lambda-c}$ and $\eta:=\frac{\beta}{N_0+\lambda+c}$. This completes the proof. □

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A.J.A. Ramos and M.M. Freitas
Faculty of Mathematics
Federal University of Pará
Raimundo Santana Street,
s/n, Salinópolis PA 68721-000
Brazil
e-mail: ramos@ufpa.br

M.M. Freitas
e-mail: mirelson@ufpa.br

M. Aouadi
Ecole Nationale D'Ingénieurs de Bizerte
Universitá de Carthage
BP 667035 Bizerte
Tunisia
e-mail: moncefaouadi00@gmail.com

D.S. Almeida Júnior
PhD Program in Mathematics
Federal University of Pará
Augusto Corrêa Street, 01
Belém PA 66075-110
Brazil
e-mail: dilberto@ufpa.br

M.L. Araújo
Faculty of Physics
Federal University of Pará
Raimundo Santana Street, s/n
Salinopolis PA 68721-000
Brazil

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