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On the Ricci curvature of 3-submanifolds in the unit sphere

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Abstract. In this short note, studying 3-dimensional compact and minimal submanifolds of the $(3+p)$-dimensional unit sphere $\mathbb{S}^{3+p}(1)$, we establish two rigidity theorems in terms of the Ricci curvature. The first theorem related to hypersurfaces of $\mathbb{S}^4(1)$ gives a new characterization of the minimal Clifford torus, whereas the second theorem is about the Legendrian submanifolds of $\mathbb{S}^7(1)$ so that a new characterization of the Calabi torus can be presented.

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1. Introduction. Recently, with some new insight, we obtained several optimal pinching results on compact minimal submanifolds of the round sphere in terms of either the scalar curvature or the Ricci curvature (cf. [9,10]). The purpose of this note is to apply this new idea to the Ricci curvature pinching problem for two further interesting situations. Historically, towards that problem, Ejiri [6] for the first time proved an optimal pinching theorem for compact minimal submanifolds in $\mathbb{S}^{n+p}(1)$.

Theorem A. *Let M^n be an n-dimensional simply connected compact orientable minimal submanifold immersed in $\mathbb{S}^{n+p}(1)$ such that the immersion is full. If $n \geq 4$ and the Ricci curvature of M^n satisfies* $\mathrm{Ric}\,(M^n) \geq n-2$, *then M^n is either $\mathbb{S}^n(1)$ (totally geodesic), or $n = 2m$ and $M^n = \mathbb{S}^m(\sqrt{1/2}) \times \mathbb{S}^m(\sqrt{1/2})$, or $n = 4$ and $M^4 = \mathbb{C}P^2(4/3) \to \mathbb{S}^7(1)$. Here, $\mathbb{C}P^2(4/3)$ denotes the two-dimensional complex projective space with holomorphic sectional curvature* $4/3$.

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However, we should see that even though in the above theorem the pinching constant for Ricci curvature is optimal if n is even, it is not if n is odd (cf. [12, Theorem 3.2] for $n \geq 5$). In particular, we observe that Theorem A is valid also for the case $n = 3$. This latter assertion follows actually from the combinations of Shen [18, Theorem 1], Li [12, Theorem 3.3], the result of Chern–do Carmo–Kobayashi [3] for $p \leq 2$, and Li–Li [15, Theorem 3] for $p \geq 3$. Moreover, this observation further motivates us to consider the following interesting question:

Question. *For* 3*-dimensional compact minimal submanifolds of the unit sphere* $\mathbb{S}^{3+p}(1)$*, what is the best possible Ricci curvature condition so that submanifolds next to the totally geodesic one can be characterized?*

Regarding a special case about Lagrangian submanifolds in the nearly Kähler 6-sphere $\mathbb{S}^6(1)$, very recently in [10], we adopted a strategy that has been used successfully in [9]. As the result, we can improve the previous results of Li [14] and Antić–Djorić–Vrancken [1] by giving a satisfactory answer to the above question (cf. [10, Main Theorem]).

In this paper, with continuing the concern about the above Question, we shall study two further interesting situations: the hypersurfaces of $\mathbb{S}^4(1)$ and the Legendrian submanifolds of $\mathbb{S}^7(1)$. Before stating the main results, we look at two examples.

Example 1.1. The Clifford torus $\mathbf{Cl}_{1,2} := \mathbb{S}^1(\sqrt{1/3}) \times \mathbb{S}^2(\sqrt{2/3})$ in $\mathbb{S}^4(1)$ is a compact minimal hypersurface with two distinct principal curvatures $\kappa_1 = \pm\sqrt{2}$ and $\kappa_2 = \kappa_3 = \mp\sqrt{1/2}$. Its Ricci curvature satisfies $0 \leq \mathrm{Ric}(\mathbf{Cl}_{1,2}) \leq 3/2$.

Remark 1.1. Conversely, making use of the integral formula technique due to Peng–Terng [17], Li [13] proved that if a compact minimal hypersurface M in $\mathbb{S}^4(1)$ satisfies the Ricci curvature condition $0 \leq \mathrm{Ric}(M) \leq 3/2$, then it must be the case $M = \mathbf{Cl}_{1,2}$.

Example 1.2. The Calabi torus $\mathbf{Ca}_{1,2}$ in $\mathbb{S}^7(1)$.

Let $\tau : \mathbb{S}^2(1) \hookrightarrow \mathbb{R}^3$ be the inclusion mapping, and $\gamma : \mathbb{S}^1(1) \to \mathbb{S}^3(1)$ be the standard embedding with a parametrization

$$\gamma(t) = \Big(\tfrac{\sqrt{3}}{2}\exp(\tfrac{it}{\sqrt{3}}), \tfrac{1}{2}\exp(-i\sqrt{3}t)\Big) = (\gamma_1(t), \gamma_2(t)) \in \mathbb{C}^2, \quad i = \sqrt{-1}.$$

Putting $F : \mathbb{S}^1(1) \times \mathbb{S}^2(1) \to \mathbb{S}^7(1)$ such that

$$F(t, x) = (\gamma_1(t)\tau(x), \gamma_2(t)) \in \mathbb{C}^4 \simeq \mathbb{R}^8, \tag{1.1}$$

then following [16], we call $\mathbf{Ca}_{1,2} := F(\mathbb{S}^1(1) \times \mathbb{S}^2(1))$ in $\mathbb{S}^7(1)$ the *Calabi torus*.

Considering the unit sphere $\mathbb{S}^7(1)$ as a Sasakian space form with standard contact metric structure (φ, ξ, η, g) (cf. [8]), the Calabi torus $\mathbf{Ca}_{1,2}$ in $\mathbb{S}^7(1)$ then becomes a minimal Legendrian submanifold. Exactly, one can choose a local orthonormal frame field $\{E_1, E_2, E_3\}$ on $\mathbf{Ca}_{1,2}$ such that the second fundamental form h takes the form (cf. [5] and [16]):

$$\begin{cases} h(E_1, E_1) = \frac{2}{\sqrt{3}}\varphi E_1, \quad h(E_2, E_2) = h(E_3, E_3) = -\frac{1}{\sqrt{3}}\varphi E_1, \\ h(E_1, E_2) = -\frac{1}{\sqrt{3}}\varphi E_2, \quad h(E_1, E_3) = -\frac{1}{\sqrt{3}}\varphi E_3, \quad h(E_2, E_3) = 0. \end{cases} \tag{1.2}$$

From the Gauss equation and (1.2), the curvature tensor of $\mathbf{Ca}_{1,2}$ satisfies

$$\begin{aligned} &R(E_1,E_2)E_3 = R(E_2,E_3)E_1 = R(E_3,E_1)E_2 = 0,\\ &R(E_1,E_2)E_2 = R(E_1,E_3)E_3 = 0, \\ &R(E_2,E_3)E_3 = (4/3)E_2. \end{aligned} \tag{1.3}$$

Thus its Ricci curvature satisfies $0 \le \mathrm{Ric}(\mathbf{Ca}_{1,2}) \le 4/3$. Moreover, we can say more about the sectional curvatures. Indeed, for any plane section σ of $T_p(\mathbf{Ca}_{1,2})$, we can find an orthonormal basis $\{X, Y\}$ of σ such that $X = \cos\theta E_2(p) + \sin\theta E_3(p)$ and $Y = \sin\rho E_1(p) - \cos\rho\sin\theta E_2(p) + \cos\rho\cos\theta E_3(p)$ with $\theta, \rho \in \mathbb{R}$. Then direct calculations show that the sectional curvature of σ satisfies

$$K(\sigma) = R(X,Y,Y,X) = (4/3)\cos^2\rho. \tag{1.4}$$

Remark 1.2. The above immersion $F : \mathbb{S}^1(1)\times\mathbb{S}^2(1) \to \mathbb{S}^7(1)$ is called the Calabi torus because it is one of the *generalized Calabi* product Legendrian immersions that have been intensively studied by Castro–Li–Urbano [4] (cf. [4, Theorem 3.1]). Obviously, $\mathbf{Ca}_{1,2}$ with the induced metric is isometric with the Riemannian product $\mathbb{S}^1(\sqrt{3}) \times \mathbb{S}^2(\sqrt{3}/2)$.

To introduce our first result, let M^3 be a compact minimal hypersurface in the unit sphere $\mathbb{S}^4(1)$ with h the second fundamental form and N the unit normal vector field. Denote by g the metric on $\mathbb{S}^4(1)$ as well as the induced metric on M^3. Let UM^3 be the unit tangent bundle over M^3, i.e., $U_qM^3 = \{v \in T_qM^3 \,|\, g(v,v) = 1\}$ for $q \in M^3$. Then we can define a function $f_q(v) = g(A_Nv, v)$ on U_qM^3, where A_N denotes the shape operator of M^3 in $\mathbb{S}^4(1)$. Since U_qM^3 is compact, there exists an element $e \in U_qM^3$ such that $f_q(e) = \max_{v\in U_qM^3} f_q(v)$. Put

$$\mathcal{U}_q := \{u \in U_qM^3 \mid f_q(u) = f_q(e)\}.$$

Then, for any fixed $e \in \mathcal{U}_q$, we have a well-defined function $\Phi_e : U_qM^3 \to \mathbb{R}$ defined by

$$\Phi_e(v) = [g(A_Ne, v)]^2, \quad v \in U_qM^3.$$

Now, our first result can be stated as:

Theorem 1.1. *Let M^3 be a compact minimal hypersurface in the unit sphere $\mathbb{S}^4(1)$. If the Ricci curvature $\mathrm{Ric}(v) := \mathrm{trace}\{X \mapsto R(X,v)v\}/\|v\|^2$ of M^3 satisfies, at any $q \in M^3$, $\mathrm{Ric}(v) \ge \frac{3}{2} - \frac{3}{4}\Phi_e(v)$ for all $v \in U_qM^3$ and a fixed $e \in \mathcal{U}_q$, then either*

(a) $M^3 = \mathbb{S}^3(1)$ is totally geodesic with $\mathrm{Ric}(v) \equiv 2$, or
(b) $M^3 = \mathbf{Cl}_{1,2}$, which satisfies $\mathrm{Ric}(v) \equiv \frac{3}{2} - \frac{3}{4}\Phi_e(v)$ and $0 \le \mathrm{Ric}(v) \le \frac{3}{2}$.

To introduce the second result of this paper, let M^3 be a compact minimal Legendrian submanifold in the unit sphere $\mathbb{S}^7(1)$, being equipped with the standard contact metric structure $\{\varphi, \xi, \eta, g\}$. Then, letting h be the second fundamental form, we can follow [2] and consider the function $F_q(v) = g(h(v,v), \varphi v)$, defined on U_qM^3. In view of the compactness of U_qM^3, there is an element $e \in U_qM^3$ such that $F_q(e) = \max_{v\in U_qM^3} F_q(v)$. Similarly, we put

$$\mathcal{V}_q := \{u \in U_qM^3 \mid F_q(u) = F_q(e)\}.$$

Then, for any fixed $e \in \mathcal{V}_q$, we have a well-defined function $\Psi_e : U_qM^3 \to \mathbb{R}$ defined by

$$\Psi_e(v) = [g(h(e,e), \varphi v)]^2, \quad v \in U_qM^3.$$

Our second result can be stated as:

Theorem 1.2. *Let M^3 be a compact minimal Legendrian submanifold in the unit sphere $\mathbb{S}^7(1)$. If the Ricci curvature* $\mathrm{Ric}(v) = \mathrm{trace}\{X \mapsto R(X,v)v\}/\|v\|^2$ *of M^3 satisfies, at any $q \in M^3$, $\mathrm{Ric}(v) \geq \frac{4}{3} - \Psi_e(v)$ for all $v \in U_qM^3$ and a fixed $e \in \mathcal{V}_q$, then either*

(a) $M^3 = \mathbb{S}^3(1)$ is totally geodesic with $\mathrm{Ric}(v) \equiv 2$, *or*
(b) $M^3 = \mathbf{Ca}_{1,2}$, which satisfies $\mathrm{Ric}(v) \equiv \frac{4}{3} - \Psi_e(v)$ *and* $0 \leq \mathrm{Ric}(v) \leq \frac{4}{3}$.

Remark 1.3. The statement of our Ricci curvature condition in Theorems 1.1 and 1.2 is a little different from that of Ejiri's Theorem A. Actually, in Theorem A, the Ricci curvature condition $\mathrm{Ric}\,(M^n) \geq n-2$ just means that $\mathrm{Ric}\,(v) := \mathrm{trace}\,\{X \mapsto R(X,v)v\}/\|v\|^2 \geq n-2$ for all nonzero $v \in T_qM^n$ and any $q \in M^n$.

Remark 1.4. In [5], Dillen and Vrancken classified all 3-dimensional compact minimal *C-totally real* (this is equivalent to *Legendrian*) submanifolds in $\mathbb{S}^7(1)$ with *nonnegative sectional curvatures*. Their classification consists of three examples amongst which the two cases in Theorem 1.2 appeared.

2. Basic lemmas. In this section, we briefly review several facts and lemmas on 3-dimensional submanifolds M^3 in the unit sphere $\mathbb{S}^{3+p}(1)$. This includes the hypersurface case for $p = 1$ (cf. [17]) and the Legendrian submanifold case for $p = 4$ (cf. [8]).

2.1. Hypersurfaces in $\mathbb{S}^4(1)$. In this subsection, we assume that M^3 is a minimal hypersurface in the unit sphere $\mathbb{S}^4(1)$. For the sake of simplicity, we adopt the notations of Peng and Terng [17]. We begin with the following well-known result.

Lemma 2.1. *Let M^3 be a minimal hypersurface in the unit sphere $\mathbb{S}^4(1)$. Then it holds that*

$$\tfrac{1}{2}\Delta\|h\|^2 = \|\nabla h\|^2 + \|h\|^2(3 - \|h\|^2). \tag{2.1}$$

Next, noting that $g(A_NX, Y) = g(h(X,Y), N)$, for later purposes, we state the following easy fact:

Lemma 2.2. *Let M^3 be a minimal hypersurface in the unit sphere $\mathbb{S}^4(1)$ with unit normal vector field N. Then, for each $q \in M^3$, there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of T_qM^3 and numbers κ_1, κ_2 such that the second fundamental form h of M^3 satisfies*

$$\begin{aligned} &h(e_1,e_1) = (\kappa_1+\kappa_2)N, \quad h(e_2,e_2) = -\kappa_1 N,\\ &h(e_3,e_3) = -\kappa_2 N, \quad h(e_i,e_j) = 0, \ \text{for } i \neq j, \end{aligned} \tag{2.2}$$

where $\kappa_1 + \kappa_2 = \max\{\kappa_1+\kappa_2, -\kappa_1, -\kappa_2\} = \max_{v \in U_qM^3} g(A_Nv, v)$.

2.2. Legendrian submanifolds in $\mathbb{S}^7(1)$. In this subsection, we assume that M^3 is a minimal Legendrian submanifold in the unit sphere $\mathbb{S}^7(1)$ which is regarded as a Sasakian space form with contact metric structure $\{\varphi, \xi, \eta, g\}$. According to Chern–do Carmo–Kobayashi [3], and by using the notations introduced in [8] for *integral* (in the present case, this is also equivalent to the *Legendrian* condition) submanifolds, we have (cf. also [16, Lemma 2.1])

Lemma 2.3. *Let M^3 be a minimal Legendrian submanifold in the unit sphere $\mathbb{S}^7(1)$. Then, in terms of $H_i = (h^{i^*}_{jk})$, we have the Laplacian of $\|h\|^2$ as below:*

$$\tfrac{1}{2}\Delta\|h\|^2 = \|\bar\nabla^\xi h\|^2 + 4\|h\|^2 - \sum_{i,j} N(H_iH_j - H_jH_i) - \sum_{i,j}(S_{ij})^2, \tag{2.3}$$

where $\|\bar\nabla^\xi h\|^2 = \sum_{i,j,k,l}(h^{l^*}_{ij,k})^2$, $S_{ij} = \mathrm{trace}(H_iH_j)$, and $N(A) = \sum_{i,j}(a_{ij})^2$ for $A = (a_{ij})$.

Next, following an idea due to Ejiri [7] and based on [8, Lemma 3.6], we can easily get the following result.

Lemma 2.4. *Let M^3 be a minimal Legendrian submanifold in $\mathbb{S}^7(1)$. Then, for each $q \in M^3$, there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of T_qM^3 and numbers $\{\lambda_1, \lambda_2, \mu_1, \mu_2\}$ such that the second fundamental form h of M^3 has the following form:*

$$\begin{cases} h(e_1,e_1) = (\lambda_1+\lambda_2)\varphi e_1, \quad h(e_1,e_2) = -\lambda_1\varphi e_2, \quad h(e_1,e_3) = -\lambda_2\varphi e_3,\\ h(e_2,e_2) = -\lambda_1\varphi e_1 + \mu_1\varphi e_2 + \mu_2\varphi e_3, \quad h(e_2,e_3) = \mu_2\varphi e_2 - \mu_1\varphi e_3,\\ h(e_3,e_3) = -\lambda_2\varphi e_1 - \mu_1\varphi e_2 - \mu_2\varphi e_3, \end{cases} \tag{2.4}$$

where for $F_q(v) = g(h(v,v),\varphi v)$ defined on U_qM^3, it holds

$$\begin{cases} \lambda_1+\lambda_2 = \max_{v\in U_qM^3} F_q(v) \ge 0,\\ \lambda_1+\lambda_2 \ge -2\lambda_1, \quad \lambda_2+\lambda_1 \ge -2\lambda_2,\\ -(\lambda_1+\lambda_2) \le \mu_i \le \lambda_1+\lambda_2, \quad i=1,2. \end{cases} \tag{2.5}$$

Then, with the Legendrian condition, straightforward calculations give

Lemma 2.5. *If (2.4) holds, then, by using the notations of Lemma 2.3, we have*

$$\|h\|^2 = \sum_{i,j,\alpha}(h^\alpha_{ij})^2 = 4\lambda_1^2 + 4\lambda_2^2 + 2\lambda_1\lambda_2 + 4\mu_1^2 + 4\mu_2^2, \tag{2.6}$$

$$\begin{aligned} &\sum_{i,j} N(H_iH_j - H_jH_i) + \sum_{i,j}(S_{ij})^2\\ &\quad = 24(\lambda_1^4 + \lambda_1^3\lambda_2 + \lambda_1^2\lambda_2^2 + \lambda_1\lambda_2^3 + \lambda_2^4) - 36\lambda_1\lambda_2(\mu_1^2+\mu_2^2)\\ &\qquad + 18(\lambda_1^2+\lambda_2^2)(\mu_1^2+\mu_2^2) + 24(\mu_1^2+\mu_2^2)^2. \end{aligned} \tag{2.7}$$

3. Proof of Theorem 1.1. Let M^3 be a compact minimal hypersurface in $\mathbb{S}^4(1)$. For any fixed $q \in M^3$, by choosing the orthonormal basis $\{e_1, e_2, e_3\}$ of T_qM^3

as stated in Lemma 2.2, the Gauss equation implies that the Ricci curvature $R_{ij} = \sum_k g(R(e_i, e_k)e_k, e_j)$ is given by

$$(R_{ij}) = \begin{pmatrix} 2-(\kappa_1+\kappa_2)^2 & 0 & 0 \\ 0 & 2-\kappa_1^2 & 0 \\ 0 & 0 & 2-\kappa_2^2 \end{pmatrix}.$$

According to the assumption of Theorem 1.1 and the statement of Lemma 2.2, the Ricci curvature of M^3 satisfies $\mathrm{Ric}(v) \geq \frac{3}{2} - \frac{3}{4}\Phi_{e_1}(v)$ for all $v \in U_qM^3$, where $\Phi_{e_1}(e_1) = (\kappa_1+\kappa_2)^2$ and $\Phi_{e_1}(e_i) = [g(A_N e_1, e_i)]^2 = 0$ for $i = 2, 3$. It follows that

$$\begin{cases} 2-(\kappa_1+\kappa_2)^2 \geq \frac{3}{2} - \frac{3}{4}(\kappa_1+\kappa_2)^2, \\ 2-\kappa_1^2 \geq \frac{3}{2}, \quad 2-\kappa_2^2 \geq \frac{3}{2}. \end{cases} \tag{3.1}$$

From (3.1), interchanging e_2 and e_3 if necessary, we can assume that

$$0 \leq \kappa_1^2 \leq \tfrac{1}{2}, \quad 0 \leq \kappa_2^2 \leq \tfrac{1}{2}, \quad 0 \leq (\kappa_1+\kappa_2)^2 \leq 2, \quad \kappa_2 \leq \kappa_1. \tag{3.2}$$

Now, by using (2.2), we can rewrite

$$\begin{aligned} &\|h\|^2(3-\|h\|^2) \\ &= -2(\kappa_1^2+\kappa_2^2+\kappa_1\kappa_2)(2\kappa_1^2+2\kappa_2^2+2\kappa_1\kappa_2-3) \\ &= -(\kappa_1^2+\kappa_2^2+\kappa_1\kappa_2)\big[3(2\kappa_1^2-1)+3(2\kappa_2^2-1)-2(\kappa_1-\kappa_2)^2\big]. \end{aligned} \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\|h\|^2(3-\|h\|^2) \geq 0. \tag{3.4}$$

Moreover, from the fact $\kappa_1+\kappa_2 = \max\{\kappa_1+\kappa_2, -\kappa_1, -\kappa_2\}$, we see that the equality sign holds in (3.4) if and only if either $\kappa_1 = \kappa_2 = 0$, or $\kappa_1 = \kappa_2 = \sqrt{2}/2$ so that $\|h\|^2 = 3$.

On the other hand, since M^3 is compact, Lemma 2.1 implies that

$$\int_{M^3} \Big\{\|\nabla h\|^2 + \|h\|^2(3-\|h\|^2)\Big\} dV_M = 0. \tag{3.5}$$

Therefore, from (3.4) and the arbitrariness of $q \in M^3$, we have

$$\|\nabla h\|^2 = \|h\|^2(3-\|h\|^2) \equiv 0 \quad \text{on} \quad M^3.$$

This immediately shows that M^3 in $\mathbb{S}^4(1)$ is either totally geodesic, or the Clifford torus (see Lawson [11] or Chern–do Carmo–Kobayashi [3]). □

4. Proof of Theorem 1.2. Let M^3 be a compact minimal Legendrian submanifold in the unit sphere $\mathbb{S}^7(1)$. For any fixed point $q \in M^3$, let $\{e_1, e_2, e_3\}$ be the orthonormal basis of T_qM^3 as stated in Lemma 2.4 such that $e_1 = e$. Then, by using the Gauss equation and the fact that $R_{ij} = \sum_k g(R(e_i, e_k)e_k, e_j)$, we can express (R_{ij}) by

$$\begin{pmatrix} 2-2\lambda_1^2-2\lambda_2^2-2\lambda_1\lambda_2 & (\lambda_1-\lambda_2)\mu_1 & (\lambda_1-\lambda_2)\mu_2 \\ (\lambda_1-\lambda_2)\mu_1 & 2-2\lambda_1^2-2\mu_1^2-2\mu_2^2 & 0 \\ (\lambda_1-\lambda_2)\mu_2 & 0 & 2-2\lambda_2^2-2\mu_1^2-2\mu_2^2 \end{pmatrix}.$$

According to the assumption of Theorem 1.2 and the statement of Lemma 2.4, the Ricci curvature of M^3 satisfies $\mathrm{Ric}(v) \geq \frac{4}{3} - \Psi_{e_1}(v)$ for all $v \in U_qM^3$, where $\Psi_{e_1}(e_1) = (\lambda_1 + \lambda_2)^2$ and $\Psi_{e_1}(e_i) = [g(h(e_1, e_1), \varphi e_i)]^2 = 0$ for $i = 2, 3$. It follows, by taking $v = e_1, e_2, e_3$ with $\mathrm{Ric}(e_i) = R_{ii}$, that

$$\begin{cases} 2 - 2\lambda_1^2 - 2\lambda_2^2 - 2\lambda_1\lambda_2 \geq \frac{4}{3} - (\lambda_1 + \lambda_2)^2, \\ 2 - 2\lambda_1^2 - 2\mu_1^2 - 2\mu_2^2 \geq \frac{4}{3}, \\ 2 - 2\lambda_2^2 - 2\mu_1^2 - 2\mu_2^2 \geq \frac{4}{3}. \end{cases} \tag{4.1}$$

From (4.1), interchanging e_2 and e_3 if necessary, we can assume that $\lambda_2^2 \leq \lambda_1^2$. Then the inequalities in (4.1) together with (2.5) imply the following relations:

$$0 \leq \lambda_1 \leq \tfrac{\sqrt{3}}{3}, \quad \lambda_2 \leq \lambda_1, \quad -\tfrac{1}{3}\lambda_1 \leq \lambda_2, \quad \mu_1^2 + \mu_2^2 \leq \tfrac{1}{3} - \lambda_1^2. \tag{4.2}$$

On the other hand, as M^3 is compact, by using Lemmas 2.3 and 2.5, we have the integral identity

$$\begin{aligned} 0 = \int_{M^3} \Big\{ &\|\bar{\nabla}^{\xi} h\|^2 - \big\{24[\lambda_1^4 + \lambda_2^4 + \lambda_1^2\lambda_2^2 + \lambda_1\lambda_2(\lambda_1^2 + \lambda_2^2)] \\ &+ 18(\mu_1^2 + \mu_2^2)(\lambda_1 - \lambda_2)^2 + 24(\mu_1^2 + \mu_2^2)^2 \\ &- 16(\mu_1^2 + \mu_2^2) - 16(\lambda_1^2 + \lambda_2^2 + \tfrac{1}{2}\lambda_1\lambda_2)\big\}\Big\}\, dV_M. \end{aligned} \tag{4.3}$$

Now we put

$$\begin{aligned} \Omega := {}& 24[\lambda_1^4 + \lambda_2^4 + \lambda_1^2\lambda_2^2 + \lambda_1\lambda_2(\lambda_1^2 + \lambda_2^2)] + 18(\mu_1^2 + \mu_2^2)(\lambda_1 - \lambda_2)^2 \\ &+ 24(\mu_1^2 + \mu_2^2)^2 - 16(\mu_1^2 + \mu_2^2) - 16(\lambda_1^2 + \lambda_2^2 + \tfrac{1}{2}\lambda_1\lambda_2). \end{aligned} \tag{4.4}$$

To achieve the desired goal, we first make the following claim.

Claim. *Under the restrictions of (4.2), it holds that $\Omega \leq 0$, and $\Omega = 0$ if and only if either $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 0$, or $\lambda_1 = \lambda_2 = \sqrt{3}/3$ and $\mu_1 = \mu_2 = 0$.*

To verify the Claim, it suffices to consider the point $q \in M^3$ for which $h \neq 0$. In that case, we have $\lambda_1 > 0$, and by (4.2), we further get

$$3\lambda_1 + 5\lambda_2 = \tfrac{4}{3}\lambda_1 + 5(\tfrac{1}{3}\lambda_1 + \lambda_2) > 0.$$

Thus, the Claim follows immediately from (4.2) and the following expression

$$\begin{aligned} \Omega = {}& 24\big[2(\mu_1^2 + \mu_2^2) + 2\lambda_1^2 + \lambda_1\lambda_2 + 2\lambda_2^2\big]\big(\mu_1^2 + \mu_2^2 + \lambda_1^2 - \tfrac{1}{3}\big) \\ &- 3\big[2(\mu_1^2 + \mu_2^2) + 2\lambda_1^2 + \lambda_1\lambda_2 + 2\lambda_2^2\big](3\lambda_1 + 5\lambda_2)(\lambda_1 - \lambda_2) \\ &- 24(\mu_1^2 + \mu_2^2)^2 - 3(\lambda_1 - \lambda_2)^2(2\lambda_1^2 + 2\lambda_2^2 - 3\lambda_1\lambda_2) \\ &- 12(5\lambda_1^2 + 5\lambda_2^2 + 4\lambda_1\lambda_2)(\mu_1^2 + \mu_2^2). \end{aligned} \tag{4.5}$$

Finally, from the Claim and the arbitrariness of $q \in M^3$, by using (4.3), we conclude that M^3 is a Legendrian submanifold with C-parallel second fundamental form (i.e. $\bar{\nabla}^{\xi} h = 0$). Moreover, it is either totally geodesic and $M^3 = \mathbb{S}^3(1)$, or by continuity, it is such that $\lambda_1 = \lambda_2 = \sqrt{3}/3$, $\mu_1 = \mu_2 = 0$ and $\|h\|^2 = 10/3$ hold identically on M^3.

In the latter case, it is easily seen from (2.4) and the Gauss equation that the compact Legendrian submanifold M^3 in $\mathbb{S}^7(1)$ is of nonnegative sectional curvatures, and the sectional curvatures have the same expression as we have described for the Calabi torus $\mathbf{Ca}_{1,2}$ in the introduction. Then, according to [5, Main theorem] and its proof ([5, Example 5.3] in particular), we conclude that M^3 is congruent to $\mathbf{Ca}_{1,2}$ with the immersion given by (1.1).

This completes the proof of Theorem 1.2. □

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