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# Existence of Riemannian metrics with positive biorthogonal curvature on simply connected 5-manifolds

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**Abstract.** Using the recent work of Bettiol, we show that a first-order conformal deformation of Wilking's metric of almost-positive sectional curvature on $S^2 \times S^3$ yields a family of metrics with strictly positive average of sectional curvatures of any pair of 2-planes that are separated by a minimal distance in the 2-Grassmanian. A result of Smale allows us to conclude that every closed simply connected 5-manifold with torsion-free homology and trivial second Stiefel–Whitney class admits a Riemannian metric with a strictly positive average of sectional curvatures of any pair of orthogonal 2-planes.

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**Keywords.** Positive biorthogonal curvature, Biorthogonal curvature, 5-manifolds.

**1. Introduction and main results.** Let $(M, g)$ be a compact Riemannian $n$-manifold and let $\sec_g$ be the sectional curvature of the metric. We often abuse notation and denote the Riemannian metric by $(M, g)$ as well. For each 2-plane

$$\sigma \in \mathrm{Gr}_2(T_pM) = \{X \wedge Y \in \Lambda^2 T_pM : ||X \wedge Y||^2 = 1\}, \tag{1.1}$$

let $\sigma^\perp \subset T_pM$ be its orthogonal complement. That is, there is a $g$-orthogonal direct sum decomposition $\sigma \oplus \sigma^\perp = T_pM$ at a point $p \in M$.

**Definition 1.** The biorthogonal curvature of a 2-plane $\sigma \in \mathrm{Gr}_2(T_pM)$ is

$$\sec_g^\perp(\sigma) := \min_{\substack{\sigma' \in \mathrm{Gr}_2(T_pM)\\ \sigma' \subset \sigma^\perp}} \frac{1}{2}(\sec_g(\sigma) + \sec_g(\sigma')) \tag{1.2}$$

(cf. [3, Section 5.4]). We say that $(M, g)$ has positive biorthogonal curvature $\sec_g^\perp > 0$ if (1.2) is positive for every $\sigma \in \mathrm{Gr}_2(T_pM)$ at every point $p \in M$.

A stronger curvature condition is the following. Choose a distance on the Grassmanian bundle $\mathrm{Gr}_2(TM)$ that induces the standard topology.

**Definition 2.** The distance curvature of a 2-plane $\sigma \subset T_pM$ is

$$\sec_g^\theta(\sigma) := \min_{\substack{\sigma' \in \mathrm{Gr}_2(T_pM) \\ \mathrm{dis}(\sigma,\sigma') \geq \theta}} \frac{1}{2}(\sec_g(\sigma) + \sec_g(\sigma')) \tag{1.3}$$

for each $\theta > 0$ (cf. [3, Section 5.2]). We say that $(M, g^\theta)$ has positive distance curvature $\sec_{g^\theta} > 0$ if for every $\theta > 0$, there is a Riemannian metric $(M, g^\theta)$ for which (1.3) is positive for every $\sigma \in \mathrm{Gr}_2(T_pM)$ at every point $p \in M$.

Bettiol [4] classified up to homeomorphism closed simply connected 4-manifolds that admit a Riemannian metric of positive biorthogonal curvature by constructing metrics of positive distance curvature on $S^2 \times S^2$ [2, Theorem, Proposition 5.1], [3, Theorem 6.1], and showing that positive biorthogonal curvature is a property that is closed under connected sums [3, Proposition 7.11], [4, Proposition 3.1].

In this paper, we extend Bettiol's results to dimension five. More precisely, we build upon Bettiol's work and show that an application of a first-order conformal deformation to Wilking's metric $(S^2 \times S^3, g_W)$ of almost-positive sectional curvature [11] yields the main result of this note.

**Theorem A.** *For every $\theta > 0$, there is a Riemannian metric $(S^2 \times S^3, g^\theta)$ such that*

- *(a)* $\sec_{g^\theta}^\theta > 0$;
- *(b) there is a limit metric $g^0$ such that $g^\theta \to g^0$ in the $C^k$-topology as $\theta \to 0$ for $k \geq 0$;*
- *(c) $g^\theta$ is arbitrarily close to Wilking's metric $g_W$ of almost-positive curvature in the $C^k$-topology for $k \geq 0$;*
- *(d)* $\mathrm{Ric}_{g^\theta} > 0$;
- *(e) there is a 2-plane $\sigma \in \mathrm{Gr}_2(T_pS^2 \times S^3)$ with $\sec_{g^\theta}(\sigma) < 0$;*

*In particular, there is a Riemannian metric of positive biorthogonal curvature on $S^2 \times S^3$.*

The next corollary is a consequence of coupling Theorem A with a classification result of Smale [8].

**Corollary B.** *Every closed simply connected 5-manifold with torsion-free homology and zero second Stiefel–Whitney class admits a Riemannian metric of positive biorthogonal curvature.*

The hypothesis imposed on the homology and the second Stiefel–Whitney class of the manifolds of Corollary B are technical in nature; cf. Remark 2. Indeed, an examination of the canonical metric on the Wu manifold yields the following proposition.

**Proposition C.** *The symmetric space metric* $(\mathrm{SU}(3)/\mathrm{SO}(3), g)$ *has positive biorthogonal curvature.*

The Wu manifold has second homology group of order two and nontrivial second Stiefel–Whitney class.

## 2. Constructions of Riemannian metrics of positive biorthogonal curvature.

**2.1. Wilking's metric of almost-positive curvature on $S^2 \times S^3$.** We follow the exposition in [11, Section 5] to describe Wilking's construction of a metric of almost-positive curvature on the product of projective spaces $\mathbb{R}P^2 \times \mathbb{R}P^3$ and its pullback to $S^2 \times S^3$ under the covering map; see [12, Section 5] for a discussion relating these two constructions.

The unit tangent sphere bundle of the 3-sphere

$$T_1(S^3) = S^2 \times S^3, \tag{2.1}$$

embeds into $\mathbb{R}^4 \times \mathbb{R}^4 = \mathbb{H} \times \mathbb{H}$ as a pair of orthogonal unit quaternions

$$S^3 \times S^2 = \{(p, v) \in \mathbb{H} \times \mathbb{H} : |p| = |v| = 1, \langle p, v\rangle = 0\} \subset \mathbb{H} \times \mathbb{H}, \tag{2.2}$$

where $\langle x, y\rangle = \mathrm{Re}(\bar{x}y)$, $|x|^2 = \langle x, x\rangle$, and $\bar{x}$ denotes the quarternion conjugation of $x$. The group $G = \mathrm{Sp}(1) \times \mathrm{Sp}(1) \simeq S^3 \times S^3$ acts on $S^2 \times S^3$ by

$$(q_1, q_2) \star (p, v) = (q_1 p \bar{q}_2, q_1 v \bar{q}_2) \tag{2.3}$$

for $q_1, q_2 \in \mathrm{Sp}(1)$ and $(p, v) \in S^2 \times S^3$. This action is effectively free and transitive. The isotropy group of the point $(1, i) \in S^2 \times S^3$ is

$$H = \{(e^{i\phi}, e^{i\phi}) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1)\} \subset G. \tag{2.4}$$

Thus, $S^2 \times S^3 \simeq G/H$ is a homogeneous space.

In order to put a metric on $S^2 \times S^3$, Wilking first defines a left invariant metric $g$ on $G = \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ as follows. Let

$$g_0((X, Y), (X', Y')) = \langle X, Y\rangle + \langle X', Y'\rangle, \tag{2.5}$$

for $(X, Y), (X', Y') \in \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) = \mathrm{Im}(\mathbb{H}) \oplus \mathrm{Im}(\mathbb{H})$, denote a bi-invariant metric. In terms of $g_0$, the metric $g$ is

$$g((X, Y), (X', Y')) = g_0(\Phi(X, Y), (X', Y')), \tag{2.6}$$

where $\Phi$ is a $g_0$-symmetric, positive definite endomorphism of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ given by

$$\Phi = \mathrm{Id} - \frac{1}{2}P, \tag{2.7}$$

and $P$ is the $g_0$-orthogonal projection onto the diagonal subalgebra

$$\Delta\mathfrak{sp}(1) \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(1); \tag{2.8}$$

see [11, p. 125].

Wilking's doubling trick guarantees the existence of a diffeomorphism

$$G/H \simeq \Delta G \backslash G \times G/\{1_G\} \times H, \tag{2.9}$$

where $\Delta G\backslash$ denotes the quotient by the left diagonal action of $G$ on $G \times G$ and $H$ acts on the second factor from the right. Consider the product $(G \times G, g + g)$ (cf. (2.6)) and the induced metric on $S^2 \times S^3 \simeq \Delta G\backslash G \times G/\{1_G\} \times H$ that we denote by $g_W$. That is, Wilking's metric $(S^2 \times S^3, g_W)$ is the metric that makes the quotient submersion

$$(G \times G, g \oplus g) \to (\Delta G\backslash G \times G/\{1_G\} \times H, g_W) \tag{2.10}$$

into a Riemannian submersion. Wilking has shown that $(S^2 \times S^3, g_W)$ has almost-positive curvature, with flat 2-planes located on two hypersurfaces. These hypersurfaces are both diffeomorphic to $S^2 \times S^2$, and they intersect along an $\mathbb{R}P^3$ [11, Corollary 3, Proposition 6]. However, except for points that lie on four disjoint copies of $S^2$ inside these two hypersurfaces, there is a unique flat 2-plane. At each point in these four 2-spheres, there is a one parameter family of flat 2-planes and neither the distance curvature nor the biorthogonal curvature of the metric $g_W$ are strictly positive at any of these points.

## 3. Proofs.

**3.1. Proof of Theorem A.** We follow Bettiol's construction of metrics of positive distance curvature on $S^2 \times S^2$ [2, Theorem], [3, Theorem 6.1], and apply a first-order conformal deformation to Wilking's metric $(S^2 \times S^3, g_W)$ that was described in Section 2.1. This yields metrics of positive distance curvature as in Definition 2, which converge to a metric $g^0$ as $\theta$ tends to zero in the $C^k$-topology.

**Definition 3.** Let $(M, g)$ be a compact Riemannian manifold, then, for any function $\phi : M \to \mathbb{R}$, and for any small enough $s > 0$, the following is also a Riemannian metric on $M$

$$g_s = (1 + s\phi)g, \tag{3.1}$$

called the first-order conformal deformation of $g$.

The variation of sectional curvature of a metric under the first order conformal deformation is given by the following lemma [9]; cf. [3, Chapter 3, Corollary 3.4].

**Lemma 1.** *Let $(M, g)$ be a Riemannian manifold with sectional curvature $\sec_g \geq 0$, and let $X, Y \in T_pM$ be $g$-orthonormal vectors such that $\sec_g(X \wedge Y) = 0$. Consider a first-order conformal deformation $g_s = (1+s\phi)g$ of $g$. The first variation of $\sec_{g_s}(X \wedge Y)$ is*

$$\frac{\mathrm{d}}{\mathrm{d}s}\sec_{g_s}(X \wedge Y)|_{s=0} = -\frac{1}{2}\mathrm{Hess}\,\phi(X, X) - \frac{1}{2}\,\mathrm{Hess}\,\phi(Y, Y). \tag{3.2}$$

We will also need the following elementary fact [3, Chapter 3, Lemma 3.5].

**Lemma 2.** *Let $f : [0, S] \times K \to \mathbb{R}$ be a smooth function, where $S > 0$ and $K$ is a compact subset of a manifold. Assume that $f(0, x) \geq 0$ for all $x \in K$, and $\frac{\partial f}{\partial s} > 0$ if $f(0, x) = 0$. Then there exists $s_* > 0$ such that $f(s, x) > 0$ for all $x \in K$ and $0 < s < s_*$.*

Wilking's metric $(S^2 \times S^3, g_W)$ has positive sectional curvature away from a hypersurface $Z$; see the discussion at the end of Section 2.1. The biorthogonal and distance curvatures are positive inside $Z$ except for points that lie in four disjoint copies of $S^2$. Every point in these four 2-spheres carries an $S^1$ worth of flat 2-planes. Denote these four 2-spheres by

$$\{S^2_i : i = 1, 2, 3, 4\}. \tag{3.3}$$

We only deform Wilking's metric near these four submanifolds. Let

$$\chi_i : S^2 \times S^3 \to \mathbb{R} \tag{3.4}$$

denote a bump function of $S_i^2$, i.e., a nonnegative function that is identically zero outside a tubular neighborhood of $S_i^2$, and identically one in a smaller tubular neighborhood of $S_i^2$. Finally, we define four functions

$$\{\psi_i : S^2 \times S^3 \to \mathbb{R} : i = 1, 2, 3, 4\} \tag{3.5}$$

as

$$\psi_i(p) = \mathrm{dist}_{g_W}(p, S_i^2)^2 \tag{3.6}$$

for $p \in S^2 \times S^3$, where $\mathrm{dist}_{g_W}$ is the metric distance function on $(S^2 \times S^3, g_W)$. Let $\phi : S^2 \times S^3 \to \mathbb{R}$ be a function defined as

$$\phi = -\chi_1\psi_1 - \chi_2\psi_2 - \chi_3\psi_3 - \chi_4\psi_4, \tag{3.7}$$

and consider the first-order conformal deformation of $g_W$ given by

$$g_s = (1 + s\phi)g_W. \tag{3.8}$$

Note that at a point $p \in S_i^2$, we have

$$\mathrm{Hess}\,\phi(X, X) = -\mathrm{Hess}\,\psi_i(X, X) = -2g_W(X_\perp, X_\perp)^2 = -2\|X_\perp\|^2_{g_W}, \tag{3.9}$$

where $X_\perp$ denotes the component of $X$ perpendicular to $S_i^2$. For each $\theta > 0$, consider the compact subset of $(S^2 \times S^3) \times \mathrm{Gr}_2(T(S^2 \times S^3)) \times \mathrm{Gr}_2(T(S^2 \times S^3))$ given by

$$K_\theta := \{(p, \sigma, \sigma') : \sigma, \sigma' \in \mathrm{Gr}_2(T_p(S^2 \times S^3)), \mathrm{dist}(\sigma, \sigma') \geq \theta\}, \tag{3.10}$$

and define

$$\begin{gathered} f : [0, S] \times K_\theta \to \mathbb{R}, \\ f(s, (p, \sigma, \sigma')) := \frac{1}{2}(\sec_{g_s}(\sigma) + \sec_{g_s}(\sigma')). \end{gathered} \tag{3.11}$$

Notice that $f(0, (p, \sigma, \sigma')) \geq 0$ since $\sec_{g_s} \geq 0$. Furthermore, $f(0, (p, \sigma, \sigma')) = 0$ only for

$$p \in S_1^2 \cup S_2^2 \cup S_3^2 \cup S_4^2 \tag{3.12}$$

since these are the only points of $S^2 \times S^3$ that have vanishing biorthogonal and distance curvatures. Let $(p, \sigma, \sigma')$ be such that $f(0, (p, \sigma, \sigma')) = 0$ and let $\sigma = X \wedge Y$ and $\sigma' = Z \wedge W$, where $X, Y$ are $g_W$-orthonormal, and $Z, W$ are $g_W$-orthonormal. Then, by Lemma 1 and equation (3.9), at these points of $K_\theta$, we have

$$\begin{aligned} \frac{\partial f}{\partial s}|_{s=0} &= \frac{\mathrm{d}}{\mathrm{d}s}(\sec_{g_s}(X \wedge Y) + \sec_{g_s}(Z \wedge W))|_{s=0} \\ &= -\frac{1}{2}\mathrm{Hess}\,\phi(X, X) - \frac{1}{2}\mathrm{Hess}\,\phi(Y, Y) - \frac{1}{2}\mathrm{Hess}\,\phi(Z, Z) - \frac{1}{2}\mathrm{Hess}\,\phi(W, W) \\ &= \|X_\perp\|^2_{g_W} + \|Y_\perp\|^2_{g_W} + \|Z_\perp\|^2_{g_W} + \|W_\perp\|^2_{g_W} > 0. \end{aligned} \tag{3.13}$$

The previous expression is strictly greater than zero. Indeed, since $X \wedge Y$ and $Z \wedge W$ are different 2-planes, $\mathrm{span}\{X, Y, Z, W\}$ is at least three-dimensional while the submanifolds (3.3) are two-dimensional. Hence, at least one of the perpendicular components $X_\perp, Y_\perp, Z_\perp, W_\perp$ is nonzero and (3.13) is greater than zero. Since the assumptions of Lemma 2 for the function (3.11) are satisfied, we conclude that there is an $s_*$ such that $f(s,(p,\sigma,\sigma')) > 0$ for all $(p,\sigma,\sigma') \in K_\theta$ and $0 < s < s_*$. This is precisely the condition $\sec^\theta_{g_s} > 0$ of item (a) of Theorem A. The claims of item (b) and item (c) follow from our construction; cf. [2]. The claim of item (d) follows from [2, Proposition 4.1]. As Bettiol observed in his construction of metrics of positive distance curvature on $S^2 \times S^2$ [2, Section 4.4], for every $\theta > 0$, there are 2-planes in $(S^2 \times S^3, g^\theta)$ with negative sectional curvature. This completes the proof of Theorem A. □

**Remark 1.** The metrics $(S^2 \times S^3, g^\theta)$ of positive distance curvature can be made invariant under the action of certain Deck transformations including the product $\mathbb{Z}/2 \oplus \mathbb{Z}/2$-action. Indeed, it is possible to perform a local conformal deformation on the orbit space $(\mathbb{R}P^2 \times \mathbb{R}P^3, g_W)$ equipped with Wilking's metric of almost positive curvature, and a similar statement to Theorem A holds for $(\mathbb{R}P^2 \times \mathbb{R}P^3, g^\theta)$; cf. [2, Section 4.6].

**3.2. Proof of Corollary B.** We will use a case of the classification up to diffeomorphism of simply connected 5-manifolds with vanishing second Stiefel–Whitney class due to Smale [8, Theorem A].

**Theorem 1.** *A closed simply connected 5-manifold $M$ with torsion-free homology $H_2(M;\mathbb{Z}) = \mathbb{Z}^k$ and zero second Stiefel–Whitney class $w_2(M) = 0$ is determined up to diffeomorphism by its second Betti number $b_2(M)$. In particular, $M$ is diffeomorphic to a connected sum*

$$\{S^5 \# k(S^2 \times S^3) : k = b_2(M)\}. \tag{3.14}$$

Theorem A and Bettiol's result regarding the positivity of biorthogonal curvature under connected sums [3, Proposition 7.11] imply that every 5-manifold in the set (3.14) admits a Riemannian metric of positive biorthogonal curvature. □

**Remark 2.** It is natural to ask if the hypothesis $w_2(M) = 0$ of Corollary B can be removed. Barden has shown that a closed simply connected 5-manifold with torsion-free second homology group is diffeomorphic to a connected sum of copies of $S^2 \times S^3$ and the total space $S^3 \widetilde{\times} S^2$ of the nontrivial 3-sphere bundle over the 2-sphere [1]. It is currently unknown if there is a metric of almost-positive sectional curvature on $S^3 \widetilde{\times} S^2$. Unlike $S^2 \times S^3$, the nontrivial bundle does not arise as a biquotient that satisfies the symmetry hypothesis needed to apply Wilking's doubling trick; see DeVito's classification of free circle actions on $S^3 \times S^3$ in [5].

**3.3. Proof of Proposition C.** The symmetric space metric on SU(3)/SO(3) is the metric that makes the canonical surjection

$$\begin{aligned} \pi : \mathrm{SU}(3) &\to \mathrm{SU}(3)/\mathrm{SO}(3), \\ u &\mapsto u\mathrm{SO}(3), \end{aligned} \tag{3.15}$$

into a Riemannian submersion, where SU(3) is equipped with a bi-invariant metric. The left action of SU(3) on SU(3)/SO(3) induced from the left multiplication on SU(3) by (3.15) is transitive and isometric for the symmetric space metric. This means that we can study curvature at one point of SU(3)/SO(3) and isometrically translate the results to any other point. The Cartan decomposition that corresponds to SU(3)/SO(3)

$$T_e\mathrm{SU}(3) \simeq \mathfrak{su}(3) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)^{\perp} \tag{3.16}$$

is orthogonal with respect to the bi-invariant metric and it is precisely the decomposition of $T_e\mathrm{SU}(3)$ into vertical and horizontal subspaces of the Riemannian submersion (3.15). Hence, we have

$$T_{\mathrm{SO}(3)}(\mathrm{SU}(3)/\mathrm{SO}(3)) \simeq \mathfrak{so}(3)^{\perp}. \tag{3.17}$$

To conclude that SU(3)/SO(3) has positive biorthogonal curvature, we need to show that no two flat 2-planes are orthogonal to each other. A result of Tapp [10, Theorem 1.1] implies that a 2-plane on SU(3)/SO(3) is flat if and only if its horizontal lift is flat. Thus, it is enough to consider horizontal flat 2-planes at the identity of SU(3).

A horizontal 2-plane $X \wedge Y \subset \mathfrak{so}(3)^{\perp}$ at the identity of SU(3) is flat if and only if $[X, Y] = 0$. Since the maximal number of linearly independent commuting matrices in $\mathfrak{su}(3)$ is two, every horizontal flat 2-plane corresponds to a maximal abelian subalgebra of $\mathfrak{so}(3)^{\perp}$

$$\mathrm{span}_{\mathbb{R}}\{X, Y\} = \mathfrak{a}_0 \subset \mathfrak{so}(3)^{\perp}\,. \tag{3.18}$$

By a fundamental fact about the Cartan decomposition, see [7, Proposition 7.29] for the precise statement, any two maximal abelian subalgebras of $\mathfrak{so}(3)^{\perp}$ are conjugate by an element of SO(3). This means that by fixing one maximal abelian subalgebra, or one horizontal flat 2-plane, we can parametrize all horizontal flat 2-planes by SO(3). In what follows, we will obtain an explicit parametrization of horizontal flat 2-planes at the identity of SU(3), and so a parametrization of flat 2-planes at a point of SU(3)/SO(3) by choosing a basis for $\mathfrak{su}(3)$, fixing a horizontal flat 2-plane and parametrizing SO(3) by Euler angles. We use this explicit parametrization to show that no two flat 2-planes can be orthogonal. For the basis of $\mathfrak{su}(3)$, we choose $\{-i\lambda_i\}_{i=1,\ldots,8}$, where the $\lambda_i$'s are traceless, self-adjoint 3 by 3 matrices known as the Gell-Mann matrices [6]. The scalar product on $\mathfrak{su}(3)$ that corresponds to the bi-invariant metric is

$$\langle X, Y\rangle = -\frac{1}{2}\mathrm{Tr}(XY) \tag{3.19}$$

for $X, Y \in \mathfrak{su}(3)$ and the basis $\{-i\lambda_i\}_{i=1,\ldots,8}$ is orthonormal with respect to (3.19). The Cartan decomposition (3.16) in this basis is

$$\mathfrak{so}(3) = \mathrm{span}_{\mathbb{R}}\{-i\lambda_2, -i\lambda_5, -i\lambda_7\} \tag{3.20}$$

and

$$\mathfrak{so}(3)^{\perp} = \mathrm{span}_{\mathbb{R}}\{-i\lambda_1, -i\lambda_3, -i\lambda_4, -i\lambda_6, -i\lambda_8\}. \tag{3.21}$$

Matrices $\lambda_3$ and $\lambda_8$ are diagonal, so we use $-\lambda_3 \wedge \lambda_8$ for the reference horizontal flat 2-plane. Every horizontal flat 2-plane, $X \wedge Y$, with $X, Y \in \mathfrak{so}(3)^{\perp}$ such that $[X, Y] = 0$, can now be written as

$$X \wedge Y = -\mathrm{Ad}_r(\lambda_3 \wedge \lambda_8) \tag{3.22}$$

for some $r \in \mathrm{SO}(3)$. Suppose that $X \wedge Y$ and $X' \wedge Y'$ are two such 2-planes with $X \wedge Y$ given by (3.22) and $X' \wedge Y'$ by

$$X' \wedge Y' = -\mathrm{Ad}_{r'}(\lambda_3 \wedge \lambda_8) \tag{3.23}$$

for some $r' \in \mathrm{SO}(3)$. For the 2-planes (3.22) and (3.23) to be orthogonal, it is necessary and sufficient that the equations

$$\langle \mathrm{Ad}_r\lambda_3, \mathrm{Ad}_{r'}\lambda_3\rangle = 0, \tag{3.24}$$

$$\langle \mathrm{Ad}_r\lambda_3, \mathrm{Ad}_{r'}\lambda_8\rangle = 0, \tag{3.25}$$

$$\langle \mathrm{Ad}_r\lambda_8, \mathrm{Ad}_{r'}\lambda_3\rangle = 0, \tag{3.26}$$

and

$$\langle \mathrm{Ad}_r\lambda_8, \mathrm{Ad}_{r'}\lambda_8\rangle = 0 \tag{3.27}$$

hold. Using the Ad-invariance of the bi-invariant metric, equations (3.24), (3.25), (3.26), and (3.27) can be rewritten as

$$\langle \lambda_3, \mathrm{Ad}_{r^{-1}r'}\lambda_3\rangle = 0, \tag{3.28}$$

$$\langle \lambda_3, \mathrm{Ad}_{r^{-1}r'}\lambda_8\rangle = 0, \tag{3.29}$$

$$\langle \lambda_8, \mathrm{Ad}_{r^{-1}r'}\lambda_3\rangle = 0, \tag{3.30}$$

and

$$\langle \lambda_8, \mathrm{Ad}_{r^{-1}r'}\lambda_8\rangle = 0. \tag{3.31}$$

We now use the Euler angle parametrization of SO(3) to write $r^{-1}r' \in \mathrm{SO}(3)$ as

$$r^{-1}r' = \exp(-i\lambda_2 x)\exp(-i\lambda_5 y)\exp(-i\lambda_2 z), \tag{3.32}$$

where $x, y, z \in \mathbb{R}$. Plugging (3.32) into equations (3.28), (3.29), (3.30), and (3.31) and calculating the traces explicitly, we find

$$\begin{aligned} 0 &= \langle \lambda_3, \mathrm{Ad}_{r^{-1}r'}\lambda_3\rangle \\ &= \frac{1}{4}\cos(2x)\,(3+\cos(2y))\cos(2z) - \sin(2x)\cos(y)\sin(2z), \end{aligned} \tag{3.33}$$

$$0 = \langle \lambda_3, \mathrm{Ad}_{r^{-1}r'}\lambda_8\rangle = -\frac{\sqrt{3}}{2}\cos(2x)\sin^2(y), \tag{3.34}$$

$$0 = \langle \lambda_8, \mathrm{Ad}_{r^{-1}r'}\lambda_3\rangle = -\frac{\sqrt{3}}{2}\cos(2z)\sin^2(y), \tag{3.35}$$

and

$$0 = \langle \lambda_8, \mathrm{Ad}_{r^{-1}r'}\lambda_8\rangle = \frac{1}{4}(1+3\cos(2y)). \tag{3.36}$$

Equations (3.34), (3.35), and (3.36) imply $\cos^2(y) = 1/3$ and $\cos(2x) = \cos(2z) = 0$. Plugging this into equation (3.33), we obtain

$$\langle \lambda_3, \mathrm{Ad}_{r^{-1}r'}\lambda_3\rangle \neq 0, \tag{3.37}$$

and conclude that there is no solution to the system given by equations (3.33), (3.34), (3.35), and (3.36). This shows that no two 2-flat planes are orthogonal. □

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