



The generating graph of a profinite group

ANDREA LUCCHINI 

Abstract. Let G be a 2-generated group. The generating graph $\Gamma(G)$ of G is the graph whose vertices are the elements of G and where two vertices g and h are adjacent if $G = \langle g, h \rangle$. This definition can be extended to a 2-generated profinite group G , considering in this case topological generation. We prove that the set $V(G)$ of non-isolated vertices of $\Gamma(G)$ is closed in G and that, if G is prosoluble, the graph $\Delta(G)$ obtained from $\Gamma(G)$ by removing its isolated vertices is connected with diameter at most 3. However we construct an example of a 2-generated profinite group G with the property that $\Delta(G)$ has 2^{\aleph_0} connected components. This implies that the so called “swap conjecture” does not hold for finitely generated profinite groups. We also prove that if an element of $V(G)$ has finite degree in the graph $\Gamma(G)$, then G is finite.

Mathematics Subject Classification. 20D60.

Keywords. Generating graph, Profinite groups, Swap conjecture.

1. Introduction. Given a group G , the generating graph $\Gamma(G)$ is the graph with vertex set G where two elements x and y are adjacent if and only if $G = \langle x, y \rangle$. There could be many isolated vertices in this graph. All of the elements in the Frattini subgroup will be isolated vertices, but we can also find isolated vertices outside the Frattini subgroup (for example, the elements of the Klein subgroup are isolated vertices in $\Gamma(\text{Sym}(4))$). Several strong structural results about $\Gamma(G)$ are known in the case where G is simple, and this reflects the rich group theoretic structure of these groups. For example, if G is a non-abelian simple group, then the only isolated vertex of $\Gamma(G)$ is the identity and the graph $\Delta(G)$ obtained by removing the isolated vertex is connected with diameter two and, if G is sufficiently large, admits a Hamiltonian cycle. In [5], it is proved that $\Delta(G)$ is connected, with diameter at most 3, if G is a finite soluble group.

Clearly the definitions of $\Gamma(G)$ and $\Delta(G)$ can be extended to the case of a 2-generated profinite group G (in this case, we consider topological generation, i.e. we say that X generates G if the abstract subgroup generated by X is dense in G). We denote by $V(G)$ the set of the vertices of $\Delta(G)$. We will prove in Section 2 (see Proposition 6) that $V(G)$ is a closed subset of G . The profinite group G , being a compact topological group, can be seen as a probability space. If we denote by μ the normalized Haar measure on G , so that $\mu(G) = 1$, we may consider the probability $\mu(V(G))$ that a vertex of the generating graph $\Gamma(G)$ is non-isolated. By [4, Remark 2.7(ii)], if G is a 2-generated pronilpotent group, then $\mu(V(G)) \geq 6/\pi^2$. However it is possible to construct a 2-generated prosoluble group G with $\mu(V(G)) = 0$. Indeed let $H = C_2^2$ and let h_1, h_2, h_3 be the non-trivial elements of H . For each odd prime p , write $N_p = C_p^3$ and define $G = \left(\prod_{p \text{ odd}} N_p\right) \rtimes H$ where for each odd prime p , the subgroup N_p is H -stable and for $(n_{p,1}, n_{p,2}, n_{p,3}) \in N_p$ and $h_j \in H$,

$$(n_{p,1}, n_{p,2}, n_{p,3})^{h_j} = \begin{cases} (n_{p,1}, n_{p,2}^{-1}, n_{p,3}^{-1}) & \text{if } j = 1, \\ (n_{p,1}^{-1}, n_{p,2}, n_{p,3}^{-1}) & \text{if } j = 2, \\ (n_{p,1}^{-1}, n_{p,2}^{-1}, n_{p,3}) & \text{if } j = 3. \end{cases}$$

The vertex $((n_{p,i})_{1 \leq i \leq 3, p \text{ odd}}; h)$ is non-isolated in $\Gamma(G)$ if and only if $h = h_j$ for some $1 \leq j \leq 3$ and $n_{p,j} \neq 0$ for all p (see [4, Example 2.8] for more details). This implies

$$\mu(V(G)) = \frac{3}{4} \prod_{p \text{ odd}} \left(1 - \frac{1}{p}\right) = 0.$$

The neighborhood of a vertex g of the generating graph of G , denoted by $\mathcal{N}_G(g)$, is the set of vertices of $\Gamma(G)$ adjacent to g . The degree of g , denoted by $\delta_G(g)$, is the number of edges of $\Gamma(G)$ incident with g . Since $G = \langle g, x \rangle$ if and only if there is no open maximal subgroup of G containing g and x , it follows that

$$\mathcal{N}_G(g) = G \setminus \bigcup_{M \in \mathcal{M}_g} M,$$

where \mathcal{M}_g is the set of the open maximal subgroups of G containing g . In particular, $\mathcal{N}_G(g)$ is a closed subset of $V(G)$. The surprising result is that if $\delta_G(g)$ is finite for some $g \in V(G)$, then G is finite. Indeed we have:

Theorem 1. *Assume that G is an infinite 2-generated profinite group. Then the degree of any non-isolated vertex in the generating graph $\Gamma(G)$ is infinite.*

This is a consequence of the following result, concerning the generating graph of a finite group G .

Theorem 2. *Let G be a 2-generated finite group. If $g \in V(G)$, then $\delta_G(g) \geq 2^{t-2}$, where t is the length of a chief series of G .*

The results concerning the connectivity of $\Delta(G)$ when G is a finite soluble group can be extended with standard arguments to the prosoluble case.

Theorem 3. *If G is a 2-generated prosoluble group, then $\Delta(G)$ is connected and $\text{diam}(\Delta(G)) \leq 3$.*

The bound $\text{diam}(\Delta(G)) \leq 3$ given in Theorem 3 is best possible. In [5], a soluble 2-generated group G of order $2^{10} \cdot 3^2$ with $\text{diam}(\Delta(G)) = 3$ is constructed. However, in [5], it is proved that $\text{diam}(\Delta(G)) \leq 2$ in some relevant cases. These results can be extended to the profinite case, with the same arguments used in the proof of Theorem 3. Suppose that a 2-generated prosoluble soluble group G has the property that $|\text{End}_G(V)| > 2$ for every non-trivial irreducible G -module V which is G -isomorphic to a complemented chief factor of G . Then $\text{diam}(\Delta(G)) \leq 2$. In particular, $\text{diam}(\Delta(G)) \leq 2$ if the derived subgroup of G is pronilpotent or has odd order (as a supernatural number).

No example is known of a 2-generated finite group G for which $\Delta(G)$ is disconnected, and it is an open problem whether or not $\Delta(G)$ is connected when G is an arbitrary finite group. The situation is different in the profinite case.

Theorem 4. *There exists a 2-generated profinite group G with the property that $V(G)$ is the disjoint union of 2^{\aleph_0} connected components. Moreover each connected component is dense in $V(G)$.*

The previous theorem implies that the “swap conjecture” is not satisfied by the 2-generated profinite groups. Recall that the swap conjecture concerns the connectivity of the graph $\Sigma_d(G)$ in which the vertices are the ordered generating d -tuples and two vertices (x_1, \dots, x_d) and (y_1, \dots, y_d) are adjacent if and only if they differ only by one entry. Tennant and Turner [8] conjectured that the swap graph is connected for every group. Roman’kov [6] proved that the free metabelian group of rank 3 does not satisfy this conjecture but no counterexample is known in the class of finite groups. However, by Lemma 10 in Section 3, it follows from Theorem 4 that there exists a 2-generated profinite group with the property that the graph $\Sigma_2(G)$ has 2^{\aleph_0} connected components.

2. Some properties of $\Gamma(G)$. We begin this section by proving a criterion to decide when a vertex x of the generating graph $\Gamma(G)$ of a profinite group G is non-isolated.

Lemma 5. *Let G be a 2-generated profinite group. Then $x \in V(G)$ if and only if $xN \in V(G/N)$ for every open normal subgroup N of G .*

Proof. Let \mathcal{N} be the set of the open normal subgroups of G . It is clear that if $x \in V(G)$, then $xN \in V(G/N)$ for every $N \in \mathcal{N}$. Conversely, assume $xN \in V(G/N)$ for every $N \in \mathcal{N}$. Given $N \in \mathcal{N}$, let $\Omega_N = \{y \in G \mid \langle x, y \rangle N = G\}$. Notice that if $y \in \Omega_N$, then $yN \subseteq \Omega_N$ and consequently Ω_N , being a union of cosets of the open subgroup N , is a non-empty closed subset of G . If $N_1, \dots, N_t \in \mathcal{N}$, then $\emptyset \neq \Omega_{N_1 \cap \dots \cap N_t} \subseteq \Omega_{N_1} \cap \dots \cap \Omega_{N_t}$. Since G is compact, $\bigcap_{N \in \mathcal{N}} \Omega_N \neq \emptyset$. Let $y \in \bigcap_{N \in \mathcal{N}} \Omega_N$. Since $\langle x, y \rangle N = G$ for every $N \in \mathcal{N}$, we have $\langle x, y \rangle = G$, and consequently $x \in V(G)$. \square

Proposition 6. *If G is a 2-generated profinite group, then $V(G)$ is a closed subset of G .*

Proof. We prove that $G \setminus V(G)$ is an open subset of G . Let $x \notin V(G)$. By Lemma 5, there exists $N \in \mathcal{N}$ such that $x \notin V(G/N)$. This means that $\langle x, y \rangle N \neq G$ for every $y \in G$, and consequently $\langle xn, y \rangle \neq G$ for every $n \in N$ and $y \in G$. This implies $xN \cap V(G) = \emptyset$, so xN is an open neighbourhood of x contained in $G \setminus V(G)$. \square

Proof of Theorem 3. By [5, Theorem 1], for every N in the set \mathcal{N} of the open normal subgroups of G , the graph $\Delta(G/N)$ is connected and $\text{diam}(G/N) \leq 3$. Let a and b be two distinct elements of $V(G)$. For every $N \in \mathcal{N}$, let $\text{dist}_N(a, b)$ be the distance in the graph $\Delta(G/N)$ of the two vertices aN and bN . Let

$$t := \max_{N \in \mathcal{N}} \text{dist}_N(a, b).$$

Clearly $t \leq 3$. Set $\mathcal{M} = \{N \in \mathcal{N} \mid \text{dist}_N(a, b) = t\}$. If $N \in \mathcal{N}$ and $M \in \mathcal{M}$, then $N \cap M \in \mathcal{M}$, so $\bigcap_{M \in \mathcal{M}} M = 1$. For every $M \in \mathcal{M}$, let

$$\Omega_M = \{(x_1, \dots, x_t) \in G^t \mid \langle x_1, x_2 \rangle M = \dots = \langle x_{t-1}, x_t \rangle M = G, x_1 M = aM, x_t M = bM\}.$$

If $(x_1, \dots, x_t) \in \Omega_M$, then $(x_1, \dots, x_t)M^t \subseteq \Omega_M$, so Ω_M is a closed subset of G^t . If $M_1, \dots, M_u \in \mathcal{M}$, then $\emptyset \neq \Omega_{M_1 \cap \dots \cap M_u} \subseteq \Omega_{M_1} \cap \dots \cap \Omega_{M_u}$. Since G is compact, $\bigcap_{M \in \mathcal{M}} \Omega_M \neq \emptyset$. Let $(x_1, \dots, x_t) \in \bigcap_{M \in \mathcal{M}} \Omega_M$. Since $\langle x_1, x_2 \rangle M = \dots = \langle x_{t-1}, x_t \rangle M = G$ for every $M \in \mathcal{M}$, we have $\langle x_1, x_2 \rangle = \dots = \langle x_{t-1}, x_t \rangle = G$. Moreover $x_1 \in \bigcap_{M \in \mathcal{M}} aM = \{a\}$ and $x_t \in \bigcap_{M \in \mathcal{M}} bM = \{b\}$. We conclude that (x_1, \dots, x_t) is a path in $\Delta(G)$ joining the vertices $a = x_1$ and $b = x_t$. \square

3. An example. Let $G_p = (\text{SL}(2, 2^p))^{\delta_p}$, where p is a prime and δ_p is the largest positive integer with the property that the direct power $(\text{SL}(2, 2^p))^{\delta_p}$ can be generated by 2-elements. The graph $\Delta(G_p)$ is connected for every prime p , and, by [2, Theorem 1.3], there exists an increasing sequence $(p_n)_{n \in \mathbb{N}}$ of odd primes such that $\text{diam}(\Delta(G_{p_n})) \geq 2^n$ for every $n \in \mathbb{N}$. Consider the cartesian product

$$G = \prod_{n \in \mathbb{N}} G_{p_n}$$

with the product topology. Notice that G is a 2-generated profinite group. Moreover $\Delta(G)$ is the infinite tensor product of the finite graphs $\Delta(G_{p_n})$, $n \in \mathbb{N}$, and $V(G) = \prod_{n \in \mathbb{N}} V(G_{p_n})$. Indeed $\langle (x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \rangle = G$ if and only if $\langle x_n, y_n \rangle = G_{p_n}$ for every $n \in \mathbb{N}$. First we describe the connected components of the graph $\Delta(G)$.

Lemma 7. *Let $x = (x_n)_{n \in \mathbb{N}} \in V(G)$ and let Ω_x be the connected component of $\Delta(G)$ containing x . Then $y = (y_n)_{n \in \mathbb{N}}$ belongs to Ω_x if and only if*

$$\sup_{n \in \mathbb{N}} \text{dist}_{\Delta(G_{p_n})}(x_n, y_n) < \infty.$$

Proof. Assume that $y = (y_n)_{n \in \mathbb{N}} \in \Omega_x$ and let $m = \text{dist}_{\Delta(G)}(x, y)$. It follows that $\text{dist}_{\Delta(G_{p_n})}(x_n, y_n) \leq m$ for every $n \in \mathbb{N}$. Conversely assume $y = (y_n)_{n \in \mathbb{N}} \in V(G)$ with $\text{dist}_{\Delta(G_{p_n})}(x_n, y_n) \leq m$ for every $n \in \mathbb{N}$. There is a path

$$x_n = x_{n,0}, x_{n,1}, \dots, x_{n,\mu_n} = y_n,$$

with $\mu_n \leq m$, joining x_n and y_n in the graph $\Delta(G_{p_n})$. For $0 \leq i \leq m$, set

$$\tilde{x}_{n,i} = \begin{cases} x_{n,i} & \text{if } i < \mu_n, \\ x_{n,\mu_n} & \text{if } i \geq \mu_n \text{ and } m - \mu_n \text{ is even,} \\ x_{n,\mu_n-1}x_{n,\mu_n} & \text{if } i \geq \mu_n \text{ and } m - \mu_n \text{ is odd.} \end{cases}$$

Then

$$x = \tilde{x}_0 = (\tilde{x}_{n,0})_{n \in \mathbb{N}}, \tilde{x}_1 = (\tilde{x}_{n,1})_{n \in \mathbb{N}}, \dots, y = \tilde{x}_m = (\tilde{x}_{n,m})_{n \in \mathbb{N}}$$

is a path joining x and y in the graph $\Delta(G)$, so $y \in \Omega_x$. □

Proposition 8. $\Delta(G)$ has 2^{\aleph_0} different connected components.

Proof. Fix $x = (x_n)_{n \in \mathbb{N}} \in \Delta(G)$. Let τ be a real number with $\tau > 1$. Since

$$\text{diam}(\Delta(G_{p_n})) \geq 2^n \geq 1 + \lfloor n/\tau \rfloor$$

for every $n \in \mathbb{N}$, there exists $y_{\tau,n} \in G_{p_n}$ such that $\text{dist}_{\Delta(G_{p_n})}(x_n, y_{\tau,n}) = 1 + \lfloor n/\tau \rfloor$. If $\tau_2 > \tau_1$, then

$$\text{dist}(y_{\tau_2,n}, y_{\tau_1,n}) \geq \text{dist}(x_n, y_{\tau_1,n}) - \text{dist}(x_n, y_{\tau_2,n}) = \lfloor n/\tau_1 \rfloor - \lfloor n/\tau_2 \rfloor$$

tends to infinity with n , so, by Lemma 7, $\Omega_{y_{\tau_1}} \neq \Omega_{y_{\tau_2}}$. □

Proposition 9. Let Ω_x be the connected component of $\Delta(G)$ containing x . Then Ω_x is a dense subset of $V(G)$, and consequently it is not a closed subset of $V(G)$.

Proof. Let $y = (y_n)_{n \in \mathbb{N}} \in V(G)$. A base of open neighbourhoods of y consists of the subsets $\Delta_{y,m} = \{(z_n)_{n \in \mathbb{N}} \mid z_n = y_n \text{ for every } m \leq n\}$ with $m \in \mathbb{N}$. In particular, set $z = (z_n)_{n \in \mathbb{N}}$, with $z_n = y_n$ if $n \leq m$, $z_n = x_n$ otherwise. We have $z \in \Delta_{y,m}$. Moreover, since

$$\sup_{n \in \mathbb{N}} \text{dist}_{\Delta(G_{p_n})}(x_n, z_n) = \sup_{n \leq m} \text{dist}_{\Delta(G_{p_n})}(x_n, y_n) < \infty,$$

it follows from Lemma 7 that $z \in \Omega_x$. Hence $z \in \Omega_x \cap \Delta_{y,m}$. □

Lemma 10. If (x_1, y_1) and (x_2, y_2) are in the same connected component of $\Sigma_2(G)$, then x_1, y_1, x_2, y_2 are in the same connected component of $\Delta(G)$.

Proof. It suffices to prove that if

$$(\alpha_1, \beta_1), \dots, (\alpha_u, \beta_u)$$

is a path in $\Sigma_2(G)$, then the vertices $\alpha_1, \beta_1, \alpha_2, \beta_2, \dots, \alpha_u, \beta_u$ belong to the same connected component of the graph $\Delta(G)$. We prove this claim by induction on u . The statement is clearly true when $u = 1$. Assume $u \geq 2$. By induction, the vertices $\alpha_2, \beta_2, \dots, \alpha_u, \beta_u$ belong to the same connected component of $\Delta(G)$; so it is enough to show that $\alpha_1, \beta_1, \alpha_2, \beta_2$ belong to the same connected component. Since (α_1, β_1) and (α_2, β_2) differ only by one entry, either $\alpha_1 = \alpha_2$ or $\beta_1 = \beta_2$. The graph $\Delta(G)$ contains the path $\beta_1, \alpha_1 = \alpha_2, \beta_2$ in the first case and the path $\alpha_1, \beta_1 = \beta_2, \alpha_2$ in the second case. □

4. Degrees in the generating graph. Before proving Theorem 1, we briefly recall some necessary definitions and results. Given a subset X of a finite group G , we will denote by $d_X(G)$ the smallest cardinality of a set of elements of G generating G together with the elements of X . The following generalizes a result originally obtained by W. Gaschütz [3] for $X = \emptyset$.

Lemma 11 ([1, Lemma 6]). *Let X be a subset of G and N a normal subgroup of G and suppose that $\langle g_1, \dots, g_k, X \rangle N = G$. If $k \geq d_X(G)$, then there exist $n_1, \dots, n_k \in N$, so that $\langle g_1 n_1, \dots, g_k n_k, X \rangle = G$.*

It follows from Lemma 11 that the number $\phi_{G,N}(X, k)$ of k -tuples $(g_1 n_1, \dots, g_k n_k)$ generating G with X is independent of the choice of (g_1, \dots, g_k) . In particular,

$$\phi_{G,N}(X, k) = |N|^k P_{G,N}(X, k)$$

where $P_{G,N}(X, k)$ is the conditional probability that k elements of G generate G with X , given that they generate G with XN . In particular, if

$$1 = N_0 < N_1 < \dots < N_t = G$$

is a chief series of G , then

$$\phi_{G,N}(X, k) = \prod_{1 \leq i \leq t} \phi_{G/N_{i-1}, N_i/N_{i-1}}(XN_{i-1}, k).$$

We apply the previous observations in the particular case when $X = \{g\}$ with $g \in V(G)$ and $k = 1$. In this case, setting $\delta_{G/N_{i-1}, N_i/N_{i-1}}(g) = \phi_{G/N_{i-1}, N_i/N_{i-1}}(gN_{i-1}, 1)$, we get

$$\delta_G(g) = \prod_{1 \leq i \leq t} \delta_{G/N_{i-1}, N_i/N_{i-1}}(g). \tag{4.1}$$

Lemma 12. *Let N be a minimal normal subgroup of a finite group G and let $g \in V(G)$. If either $N \not\leq \text{Frat}(G)$ or $|N| > 2$, then $\delta_{G,N}(g) \geq 2$.*

Proof. If $N \leq \text{Frat}(G)$, then we have $\langle g, xn \rangle = G$ whenever $\langle g, x \rangle = G$ and $n \in N$, so $\delta_{G,N}(g) = |N|$. We may assume $N \not\leq \text{Frat}(G)$. We distinguish two cases:

- (1) N is abelian. Let $q = |\text{End}_G(N)|$ and $r = \dim_{\text{End}_G(N)} N$. By [5, Corollary 7], $P_{G,N}(g, 1) \geq \frac{q-1}{q}$, so

$$\delta_{G,N}(g) \geq \frac{|N|(q-1)}{q} = \frac{q^r(q-1)}{q} = q^{r-1}(q-1) \geq 2,$$

except in the case $q = 2$ and $r = 1$.

- (2) N is non-abelian. Choose x such that $\langle g, x \rangle = G$. Since N is non-abelian, by the main theorem in [7], there exists $1 \neq n \in N$ such that $[g, n] = 1$. Since $N \cap Z(G) = 1$, it must be $x^n = x[x, n] \neq x$. On the other hand, $G = \langle g, x \rangle = \langle g^n, x^n \rangle = \langle g, x[x, n] \rangle$, so the coset xN contains two different elements x and $x[x, n]$ adjacent to g in the graph $\Delta(G)$. This implies $\delta_{G,N}(g) \geq 2$.

□

Proof of Theorem 2. Let r be the number of non-Frattini factors of order 2 in a chief series of G . Since C_2^r is an epimorphic image of G and G can be generated by 2 elements, it must be $r \leq 2$. So the conclusion follows combining (4.1) and Lemma 12. \square

Lemma 13. *Let M be a closed subgroup of a 2-generated profinite group G . If $g \in V(G)$ and $\langle g, x \rangle M = G$, then there exists $m \in M$ such that $\langle g, xm \rangle = G$.*

Proof. Let \mathcal{N} be the set of the open normal subgroups of G . Given $N \in \mathcal{N}$, let $\Omega_N = \{m \in M \mid \langle g, xm \rangle N = G\}$. It follows from Lemma 11 that $\Omega_N \neq \emptyset$. Moreover, if $m \in \Omega_N$, then $m(N \cap M) \subseteq \Omega_N$ and consequently Ω_N is a closed subset of M . If $N_1, \dots, N_t \in \mathcal{N}$, then $\emptyset \neq \Omega_{N_1 \cap \dots \cap N_t} \subseteq \Omega_{N_1} \cap \dots \cap \Omega_{N_t}$. Since M is compact, $\bigcap_{N \in \mathcal{N}} \Omega_N \neq \emptyset$. Let $m \in \bigcap_{N \in \mathcal{N}} \Omega_N$. Since $\langle g, xm \rangle N = G$ for every $N \in \mathcal{N}$, we have $\langle g, xm \rangle = G$. \square

Proof of Theorem 1. Let $g \in V(G)$ and assume, by contradiction, that $\delta_G(g)$ is finite. Set $u = \delta_G(g)$. Since G is infinite, there exists an open normal subgroup N of G with the property that the length of a chief series of G is equal to $\lceil \log_2 u \rceil + 3$. By Theorem 2, $\delta_{G/N}(gN) \geq 2^{t-2} = 2^{\lceil \log_2 u \rceil + 1} \geq 2u$. This means that there exist $x_1, \dots, x_{2m} \in G$ such that $x_1 N \neq \dots \neq x_{2u} N$ and $\langle x_1, g \rangle N = \dots = \langle x_{2u}, g \rangle N = G$. By Lemma 13, there exist $n_1, \dots, n_{2u} \in N$ such that $G = \langle g, x_1 n_1 \rangle = \dots = \langle g, x_{2u} n_{2u} \rangle$. This implies $u = \delta_G(g) \geq 2u$, a contradiction. \square

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

- [1] Crestani, E., Lucchini, A.: The generating graph of finite soluble groups. *Israel J. Math.* **198**(1), 63–74 (2013)
- [2] Crestani, E., Lucchini, A.: The non-isolated vertices in the generating graph of a direct power of simple groups. *J. Algebraic Combin.* **37**(2), 249–263 (2013)
- [3] Gaschütz, W.: Zu einem von B.H. und H. Neumann gestellten Problem. *Math. Nachr.* **14**, 249–252 (1955)
- [4] Harper, S., Lucchini, A.: Connectivity of generating graphs of nilpotent groups. [arXiv: 2002.03330](https://arxiv.org/abs/2002.03330) (2020)
- [5] Lucchini, A.: The diameter of the generating graph of a finite soluble group. *J. Algebra* **492**, 28–43 (2017)
- [6] Roman'kov, V.A.: The Tennant-Turner swap conjecture. *Algebra Logic* **34**(4), 249–257 (1995)
- [7] Rowley, P.: Finite groups admitting a fixed-point-free automorphism group. *J. Algebra* **174**(2), 724–727 (1995)
- [8] Tennant, R.F., Turner, E.C.: The swap conjecture. *Rocky Mountain J. Math.* **22**(3), 1083–1095 (1992)

ANDREA LUCCHINI
Dipartimento di Matematica “Tullio Levi-Civita”
Università degli Studi di Padova
Via Trieste 63
35121 Padua
Italy
e-mail: lucchini@math.unipd.it

Received: 15 February 2020