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On principal indecomposable degrees and Sylow subgroups

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Abstract. We conjectured in Malle and Navarro (J Algebra 370:402–406, 2012) that a Sylow p-subgroup P of a finite group G is normal if and only if whenever p does not divide the multiplicity of $\chi \in \mathrm{Irr}(G)$ in the permutation character $(1_P)^G$, then p does not divide the degree $\chi(1)$. In this note, we prove an analogue of this for p-Brauer characters.

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1. Introduction. In [3], we proved that if G is a finite group and $P \in \mathrm{Syl}_p(G)$, then $P \trianglelefteq G$ if and only if p does not divide the degrees of the irreducible constituents of the permutation character $(1_P)^G$. Furthermore, we conjectured that this happens if and only if the irreducible constituents of $(1_P)^G$ with multiplicity not divisible by p had degree not divisible by p. This has turned out to be a difficult problem for symmetric groups (in the cases where $p = 2$ or $p = 3$). In this short note, we aim for an analogous result on modular characters. By using the induction formula and [5, Theorem 2.13], notice that the permutation p-Brauer character $((1_P)^G)^0$ decomposes as

$$((1_P)^G)^0 = \sum_{\varphi \in \mathrm{IBr}(G)} \frac{\Phi_\varphi(1)}{|G|_p}\, \varphi\,,$$

where Φ_φ is the projective indecomposable character associated with the irreducible Brauer character φ.

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We consider the following modular analogue of our conjecture in [3]:

Theorem A. *Let p be a prime, let G be a finite group, and let $P \in \mathrm{Syl}_p(G)$. Then $P \trianglelefteq G$ if and only if whenever $\varphi \in \mathrm{IBr}(G)$ is such that $\Phi_\varphi(1)/|P|$ is not divisible by p, then $\varphi(1)$ is not divisible by p.*

This generalizes the well-known result of G. Michler [4, Theorem 5.5] that G has a normal Sylow p-subgroup if and only if all irreducible Brauer characters of G have degree prime to p.

Our proof (as in [4]) relies on the classification of finite simple groups for p odd only.

2. Proof of Theorem A. We use the notation in [5]. Let p be a fixed prime. We choose a maximal ideal of the ring of algebraic integers in $\mathbb{C}$ containing p, with respect to which we calculate the set $\mathrm{IBr}(G)$ of irreducible (p-)Brauer characters for every finite group G. If $N \trianglelefteq G$ and $\theta \in \mathrm{IBr}(N)$, then $\mathrm{IBr}(G|\theta)$ is the set of irreducible constituents of the induced character θ^G, which, by [5, Corollary 8.7], is the set of irreducible Brauer characters $\varphi \in \mathrm{IBr}(G)$ such that θ is a constituent of the restriction φ_N. For $\varphi \in \mathrm{IBr}(G)$, we write $c_\varphi = \Phi_\varphi(1)/|G|_p$. (This is an integer by Dickson's theorem [5, Corollary 2.14].)

We start with the following observation:

Proposition 2.1. *Let B be a p-block of a finite group G of non-maximal defect. Then B contains an irreducible Brauer character θ of degree divisible by p but with c_θ not divisible by p.*

Proof. By a result of Brauer [5, Theorem 3.28], the p-part of the dimension of B is precisely p^{2a-d}, where p^a is the order of a Sylow p-subgroup of G, and d is the defect of B. Now

$$\dim B = \sum_{\theta \in \mathrm{IBr}(B)} \theta(1)\Phi_\theta(1)$$

and all $\theta(1)$ are divisible by p^{a-d} as all $\chi(1)$, for $\chi \in \mathrm{Irr}(B)$, are divisible by p^{a-d}. It follows that there must be some $\theta \in \mathrm{IBr}(B)$ with $\Phi_\theta(1)/p^a$ not divisible by p. □

We shall need the first two parts of the following lemma. We keep part (c) for future use.

Lemma 2.2. *Let $N \trianglelefteq G$, and let $\theta \in \mathrm{IBr}(N)$. Write*

$$\theta^G = \sum_{\varphi \in \mathrm{IBr}(G|\theta)} a_\varphi \varphi\,,$$

where a_φ are non-negative integers. Let T be the stabilizer of θ in G, and let $t = |G : T|$. Then for all $\varphi \in \mathrm{IBr}(G|\theta)$, we have

(a)

$$a_\varphi t c_\theta = c_\varphi |G : N|_p\,;$$

(b) *if G/N is a p-group, then $c_\varphi = c_\theta$; and*

(c) *c_θ divides c_φ and thus, $|T : N|_p$ divides a_φ.*

Proof. By [5, Corollary 8.8], we have $\Phi_\varphi(1) = a_\varphi t \Phi_\theta(1)$. Hence, (a) is clear.

To prove (b), notice that in this case, $\mathrm{IBr}(G|\theta) = \{\varphi\}$ and $\varphi(1) = t\theta(1)$ by [5, Theorem 8.11]. Now, $\theta^G = a_\varphi \varphi$ (using [5, Corollary 8.7]) and by part (a), we have $c_\theta = c_\varphi$.

To prove (c), using the Clifford correspondence for Brauer characters [5, Theorem 8.9], we may assume that $T = G$. We have $(\Phi_\varphi)_N = a_\varphi \Phi_\theta$ again by [5, Corollary 8.8]. Let $H = PN$, where $P \in \mathrm{Syl}_p(G)$. By [5, Problem 8.7], we have that $(\Phi_\varphi)_H$ is a positive sum of some Φ_τ for suitable $\tau \in \mathrm{IBr}(H)$. Using [5, Corollary 8.8] (and the fact that $\{\Phi_\mu \mid \mu \in \mathrm{IBr}(N)\}$ is linearly independent [5, Theorem 2.13]), we conclude that every such τ necessarily lies over θ. However, there is a unique $\hat{\theta} \in \mathrm{IBr}(H)$ over θ (by [5, Theorem 8.11]). We conclude that

$$\Phi_\varphi = v \Phi_{\hat{\theta}}$$

for some integer v. Then

$$c_\varphi = v c_{\hat{\theta}} = v c_\theta$$

using part (b). The last part now follows from (a). □

We are now ready to prove our main result.

Theorem 2.3. *Let p be a prime, let G be a finite group, and let $P \in \mathrm{Syl}_p(G)$. Then $P \trianglelefteq G$ if and only if whenever $\Phi_\varphi(1)/|P|$ is not divisible by p, then $\varphi(1)$ is not divisible by p.*

Proof. If $P \trianglelefteq G$, then p does not divide $\varphi(1)$ for all $\varphi \in \mathrm{IBr}(G)$ since $P \subseteq \ker \varphi$ ([5, Lemma 2.32]) and $\mathrm{IBr}(G/P) = \mathrm{Irr}(G/P)$ (by [5, Theorem 2.12]).

To prove the converse in the case $p = 2$, we borrow an argument in [2]. First, recall that $\varphi(1)$ is even if $1 \neq \varphi \in \mathrm{IBr}(G)$ is real-valued ([5, Theorem 2.30]) and c_1 is odd ([5, Corollary 3.34]). By hypothesis, and using that $c_\varphi = c_{\bar{\varphi}}$ since $(1_P)^G$ is real-valued, we can write

$$((1_P)^G)^0 = c_1 1 + 2\Delta + \sum_{\varphi \in \Lambda} c_\varphi (\varphi + \bar{\varphi}),$$

where Δ is a Brauer character of G, $\bar{\varphi}$ is the complex conjugate of φ, and Λ is a subset of non-trivial non-real-valued characters in $\mathrm{IBr}(G)$. If P is not normal in G, then there exists a real element $1 \neq x \in G$ of odd order ([1, Proposition 6.4]). Then $0 = (1_P)^G(x)$, and we conclude that $-c_1 = 2\alpha$ for some algebraic integer α. This is impossible since c_1 is odd.

To prove the converse for p odd, we argue by induction on $|G|$. By Proposition 2.1, all blocks of G have maximal defect. Let N be a proper non-trivial normal subgroup of G.

First, we claim that p divides $|G/N|$. Assume the contrary. Let $\theta \in \mathrm{IBr}(N)$ be such that p does not divide c_θ. Let T be the stabilizer of θ in G, and let $t = |G : T|$. Let $\varphi \in \mathrm{IBr}(G)$ be over θ. By [5, Corollary 8.7], we have that $\varphi(1) = et\theta(1)$, where e is the multiplicity of θ in φ_N. Also by [5, Corollary 8.7], this is the multiplicity of φ in θ^G (what we called a_φ in Lemma 2.2). By Lemma 2.2(a), we have $c_\varphi = \frac{\varphi(1)}{\theta(1)} c_\theta$. We conclude that c_φ is not divisible by p, (using Dade's theorem [5, Theorem 8.30]). By hypothesis, we conclude that

$\varphi(1)$ is not divisible by p, and therefore neither is $\theta(1)$. Hence N has a normal Sylow p-subgroup, and we are done in this case.

Suppose now that G/N is a p-group. Let $\theta \in \mathrm{IBr}(N)$ be such that p does not divide c_θ, and let $\varphi \in \mathrm{IBr}(G|\theta)$. By Lemma 2.2(b), we have $c_\theta = c_\varphi$. Using the hypothesis, we conclude that p does not divide $\varphi(1)$, and therefore p does not divide $\theta(1)$. Therefore, N has a normal Sylow p-subgroup, by induction.

Suppose now that $c_{1_N} = 1$. (By Fong's dimensional formula [5, Corollary 10.14], this happens, for instance, if N is p-solvable.) By [6, Lemma 2.6], if $\bar\varphi \in \mathrm{IBr}(G/N)$ is the inflation of $\varphi \in \mathrm{IBr}(G)$, then $c_\varphi = c_{\bar\varphi}$, and by induction, G/N has a normal Sylow p-subgroup. By the two previous paragraphs, we deduce that $G = N$. In particular, G does not have proper non-trivial normal p-solvable subgroups.

Suppose now that N is a non-abelian minimal normal subgroup of G. Let $S \trianglelefteq N$ be simple, so that $N = S^{x_1} \times \cdots \times S^{x_k}$ for some $x_i \in G$. By [4, Theorem 5.3], there exists a block b of S that does not have maximal defect. Then the block $e = b^{x_1} \times \cdots \times b^{x_k}$ does not have maximal defect. However, if B is any block of G covering e, then B has maximal defect, and therefore so does e by Knörr's theorem [5, Theorem 9.26]. □

A possible variation of Theorem A would be to consider $\mathrm{Fix}_P(\varphi)$ instead of c_φ, where $\mathrm{Fix}_P(\varphi)$ is the dimension of the fixed points of P in any G-module affording the Brauer character φ.

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