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A second alternative approach for the study of the Muckenhoupt class $A_1(\mathbb{R})$

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Abstract. We find the exact best possible range of those $p > 1$ for which any $\varphi \in A_1(\mathbb{R})$, with A_1 constant equal to c, must also belong to L^p. In this way, we provide an alternative proof of the corresponding result in Bojarski and Sbordone (Studia Math 101(2):155–163, 1992) and Nikolidakis (Ann Acad Scient Fenn Math 40:949–955, 2015).

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1. Introduction. The study of Muckenhoupt weights has been proved to be important in analysis. One of the most important facts about these is their self improving property. A way to express this is through the so-called reverse Hölder inequalities (see [3,4,6]).

For an interval $\mathcal{J}$ on $\mathbb{R}$, we define the class $A_1(\mathcal{J})$ to be the set of all those $\varphi : \mathcal{J} \to \mathbb{R}^+$ for which there exists a constant $c \geq 1$, such that the following inequality is satisfied:

$$\frac{1}{|\mathcal{I}|} \int_{\mathcal{I}} \varphi(x) dx \leq c \cdot \operatorname*{essinf}_{\mathcal{I}}(\varphi) \tag{1.1}$$

for every subinterval $\mathcal{I}$ of $\mathcal{J}$, where $|\cdot|$ is the Lebesque measure on $\mathbb{R}$. The least constant c for which (1.1) holds, is called the A_1-constant of φ and is denoted by $[\varphi]_1$. We will then say that φ belongs to the class $A_1(\mathcal{J})$ with constant c, and we will write $\varphi \in A_1(\mathcal{J}, c)$.

The class $A_1(\mathcal{J}, c)$ has been studied for the first time in [2]. In the present paper, we work on such weights by using the notion of the non-increasing rearrangement of φ, denoted by φ^*, which is a non-negative and non-increasing function defined on $(0, |\mathcal{J}|]$. It is characterized by the following two additional properties. It is equimeasurable to φ (in the sense that $|\{\varphi > \lambda\}| = |\{\varphi^* > \lambda\}|$ for every $\lambda > 0$) and is also left continuous. All these properties uniquely define φ^* as can be seen in [1,5], or [8]. Nevertheless, an equivalent definition of φ^* can be given by the following formula

$$\varphi^*(t) = \sup_{\substack{E\subseteq \mathcal{J}\\ |E|=t}} [\inf_{x\in E} \varphi(x)], \text{for } t \in (0, |\mathcal{J}|],$$

as can be seen in [8].

In [2], it is proved the following

Theorem 1. *Let $\varphi \in A_1(\mathcal{J}, c)$. φ^* satisfies*

$$\frac{1}{t}\int_0^t \varphi^*(y)dy \le c\varphi^*(t), \textit{for } t \in (0, |\mathcal{J}|]. \tag{1.2}$$

That is φ^* belongs to the class $A_1(\mathcal{J})$, with A_1-constant not more than c.

The above theorem describes the A_1-properties of φ^*, in terms of those of φ. It was used effectively by the authors in [2] in order to prove the following:

Theorem 2. *Let $\varphi \in A_1(\mathcal{J}, c)$. Then $\varphi \in L^p$ for every $p \in [1, \frac{c}{c-1})$. Moreover, the following inequality must hold for every subinterval $\mathcal{I}$ of $\mathcal{J}$, and every p in the above range,*

$$\frac{1}{|\mathcal{I}|}\int_{\mathcal{I}} \varphi^p(x)dx \le \frac{1}{c^{p-1}(c+p-pc)}\Big(\frac{1}{|\mathcal{I}|}\int_{\mathcal{I}} \varphi(x)dx\Big)^p. \tag{1.3}$$

Additionally, the above inequality is sharp, that is the constant appearing in the right side of (1.3) cannot be decreased.

The above two theorems have been proved in [2] for the first time and in [10] alternatively. Our aim in this paper is to give a second alternative proof of Theorem 2 by using Theorem 1 and certain techniques involving the well known Hardy operator on $\mathbb{R}$. Additionally, we need to mention that in [7] and [9] related problems for estimates for the respective range of p in higher dimensions have been treated. At last one can consult [11] for further reading.

The paper is organized as follows: In Section 2, we give a brief discussion of the proof of the Theorem 1, as is presented in [2], and in Section 3, we provide the proof of Theorem 2.

2. φ^* as an A_1 weight on $\mathbb{R}$. A similar lemma as the one that is presented below is proved in [2]. It's proof is essentially the same and for this reason we omit it.

Lemma 2.1. *Let E be a measurable bounded subset of $\mathbb{R}$ and $\epsilon > 0$. More precisely, suppose that $E \subseteq I$ for a certain bounded interval I of $\mathbb{R}$ for which $|I - E| > 0$. Then there exists a sequence $(I_\nu)_{\nu=1}^\infty$ of subintervals of I with disjoint interiors and a subset E_1 of E with the properties that $|E_1| = |E|$ and*

(i) $E_1 \subseteq \bigcup_{\nu=1}^\infty I_\nu$,
(ii) $(1-\epsilon)|I_\nu| \le |I_\nu \cap E| < |I_\nu|$ *for every* ν.

We now proceed to the

Proof of Theorem 1. Suppose without loss of generality that $\mathcal{J} = (0,1)$ and that φ satisfies (1.1) for every subinterval $\mathcal{I}$ of $\mathcal{J}$. Let $t \in (0,1)$ and $\epsilon > 0$. Let E_t be a subset of $(0,1)$ such that $|E_t| = t$ and $\varphi(x) \leq \varphi^*(t)$ for any $x \notin E_t$. Obviously $|J - E_t| > 0$. Using Lemma 2.1, we produce a subset $E_{t,1}$ of E_t such that $|E_{t,1}| = t$ and $E_{t,1} \subseteq \bigcup_{\nu=1}^{\infty} I_\nu$, where for every $\nu = 1, 2, \ldots$, the following holds:

$$(1-\epsilon)|I_\nu| \leq |I_\nu \cap E_t| < |I_\nu| \tag{2.1}$$

for a suitable family $(I_\nu)_{\nu=1}^{\infty}$ of subintervals of $(0,1)$ with disjoint interiors. By the strict inequality in (2.1), we conclude that I_ν contains a set of positive measure in the complement of E_t, therefore we must have that

$$\operatorname*{essinf}_{x \in I_\nu} \varphi(x) \leq \varphi^*(t),$$

so using (1.1) and (2.1), we have as a consequence that

$$\begin{aligned} \int_0^t \varphi^*(y)dy &= \int_{E_t} \varphi(x)dx = \int_{E_{t,1}} \varphi(x)dx \leq \sum_{\nu=1}^{\infty} \int_{I_\nu} \varphi(x)dx \leq c \sum_{\nu=1}^{\infty} |I_\nu| \cdot \varphi^*(t) \\ &\leq \frac{c}{1-\epsilon}\Big(\sum_{\nu=1}^{\infty} |I_\nu \cap E_t|\Big) \cdot \varphi^*(t) = \frac{c}{1-\epsilon} \cdot t \cdot \varphi^*(t) \\ &\Rightarrow \frac{1}{t}\int_0^t \varphi^*(y)dy \leq \frac{c}{1-\epsilon}\varphi^*(t) \end{aligned}$$

for every $\epsilon > 0$. Letting $\epsilon \to 0^+$, we conclude (1.2) for any $t \in (0,1)$. The case $t = 1$ is handled by letting $t \to 1^-$ in (1.2) and noting that φ^* is left continuous on $(0,1]$. □

3. L^p integrability of A_1 weights on $\mathbb{R}$. We will now prove the following

Lemma 3.1. *Let $g : (0,1] \to \mathbb{R}^+$ be a non-increasing, left continuous function which satisfies the following inequality:*

$$\frac{1}{t}\int_0^t g(y)dy \leq c \cdot g(t), \quad \forall t \in (0,1], \tag{3.1}$$

for a fixed $c > 1$. Then for any $p \in [1, \frac{c}{c-1})$, the following is true:

$$\int_0^1 g^p(y)dy \leq \frac{1}{c^{p-1}(c+p-pc)}\Big(\int_0^1 g(y)dy\Big)^p. \tag{3.2}$$

Moreover, inequality (3.2) is best possible.

Proof. Fix a p such that $1 \le p < \frac{c}{c-1}$ and let $F = \int_0^1 g^p(y)dy$ and $f = \int_0^1 g(y)dy$. Then by Hölder's inequality, $f^p \le F$. We need to prove that

$$F \le \frac{1}{c^{p-1}(c+p-pc)} \cdot f^p. \tag{3.3}$$

We define the function

$$H_p : \left[1, \frac{p}{p-1}\right] \to [0,1]$$

by $H_p(z) = pz^{p-1} - (p-1)z^p$. Then we easily see that H_p is one to one and onto. We denote it's inverse function by ω_p defined on $[0,1]$, which is decreasing as H_p also is. We shall prove that (3.3) holds, equivalently, $H_p(c) \le \frac{f^p}{F} \Leftrightarrow c \ge \omega_p\left(\frac{f^p}{F}\right) =: \tau$.

Suppose on the contrary that $c < \tau$. We are going to reach a contradiction.

Define the function g_1 on $(0,1]$ by $g_1(t) = \frac{f}{\tau} t^{-1+\frac{1}{\tau}}$. This is obviously non-increasing and continuous $(0,1]$. Additionally, it satisfies for any $t \in (0,1]$, the following equality:

$$\frac{1}{t}\int_0^t g_1(y)dy = \tau \cdot g_1(t). \tag{3.4}$$

Indeed: $\frac{1}{t}\int_0^t g_1(y)dy = \frac{1}{t}\frac{f}{\tau}\int_0^t y^{-1+\frac{1}{\tau}}dy = \frac{f}{t}\left[y^{\frac{1}{\tau}}\right]_{y=0}^t = \frac{f}{t}\cdot t^{\frac{1}{\tau}} = \tau \cdot \left(\frac{f}{\tau}t^{-1+\frac{1}{\tau}}\right) = \tau g_1(t)$. Moreover, it satisfies $\int_0^1 g_1(y)dy = f$ and $\int_0^1 g_1^p(y)dy = F$. The first equation is obvious, in view of (3.4). As for the second, it is equivalent to $\frac{f^p}{\tau^p}\int_0^1 y^{-p+\frac{p}{\tau}}dy = F \Leftrightarrow \frac{f^p}{\tau^p(1+\frac{p}{\tau}-p)} = F \Leftrightarrow p\tau^{p-1} - (p-1)\tau^p = \frac{f^p}{F} \Leftrightarrow H_p(\tau) = \frac{f^p}{F} \Leftrightarrow \tau = \omega_p(\frac{f^p}{F})$, which is true by the definition of τ.

We are now aiming to prove that the following inequality is satisfied:

$$\int_0^t g(y)dy \le \int_0^t g_1(y)dy, \text{ for any } t \in (0,1]. \tag{3.5}$$

For this reason, we define the following subset of $(0,1)$:

$G = \left\{t \in (0,1) : \int_0^t g(y)dy > \int_0^t g_1(y)dy\right\}$, and we suppose that G is non-empty. By the continuity of the involving integral functions on t, we have as a consequence that G is an open subset of $(0,1)$. Since $G \ne \emptyset \Rightarrow G = \bigcup_\nu I_\nu$, where $(I_\nu)_\nu$ is a (possibly finite) sequence of pairwise disjoint open intervals on $(0,1)$. Let us choose one of them, $I_\nu = (\alpha_\nu, b_\nu)$. Since $\alpha_\nu \notin G$,

$$\int_0^{\alpha_\nu} g(y)dy \le \int_0^{\alpha_\nu} g_1(y)dy. \tag{3.6}$$

Let now $(x_n)_n \subseteq I_\nu$ be a sequence such that $x_n \to \alpha_\nu$, as $n \to \infty$. Since $x_n \in G, \forall n = 1,2,\ldots$, we must have that $\int_0^{x_n} g(y)dy > \int_0^{x_n} g_1(y)dy$, so letting $n \to \infty$, we conclude that

$$\int_0^{\alpha_\nu} g(y)dy \geq \int_0^{\alpha_\nu} g_1(y)dy. \tag{3.7}$$

By (3.6) and (3.7), we see that $\int_0^{\alpha_\nu} g(y)dy = \int_0^{\alpha_\nu} g_1(y)dy$. In the same way, we prove that $\int_0^{b_\nu} g(y)dy = \int_0^{b_\nu} g_1(y)dy$. As a consequence, we must have that

$$\int_{\alpha_\nu}^{b_\nu} g(y)dy = \int_{\alpha_\nu}^{b_\nu} g_1(y)dy. \tag{3.8}$$

Let now $t \in I_\nu = (\alpha_\nu, b_\nu)$. Since $t \in G$ and because of (3.1) and (3.4) and the assumption on τ, we must have the following: $cg(t) \geq \frac{1}{t}\int_0^t g(y)dy > \frac{1}{t}\int_0^t g_1(y)dy = \tau \cdot g_1(t) > cg_1(t)$ thus $g(t) > g_1(t)$ for every $t \in I_\nu$. This is impossible in view of (3.8). Thus we have proved (3.5).

For the following, consult [5, page 88].

Lemma 3.2. *Let $\varphi_1, \varphi_2 : (0,1] \to \mathbb{R}^+$ be integrable functions. Then the following are equivalent*

(i) $\int_0^t \varphi_1^*(y)dy \leq \int_0^t \varphi_2^*(y)dy$ *for every* $t \in (0,1]$.
(ii) $\int_0^1 G(\varphi_1(x))dx \leq \int_0^1 G(\varphi_2(x))dx$

for any convex, non-negative, increasing, and left continuous function G on $[0,+\infty)$.

We consider now two cases:

(A) We have equality in (3.5) for every $t \in (0,1]$. That is $\int_0^t g(y)dy = \int_0^t g_1(y)dy$ for every $t \in (0,1]$. This immediately gives as a consequence that $g(t) = g_1(t)$ almost everywhere on $(0,1]$, and since g_1 is continuous on $(0,1]$, we must have that $g(t) = g_1(t)\ \forall t \in (0,1] \Rightarrow g(t) = \frac{f}{\tau}t^{-1+\frac{1}{\tau}}\ \forall t \in (0,1] \Rightarrow \frac{1}{t}\int_0^t g(y)dy = \tau g(t)\ \forall t \in (0,1]$. Then in view of (3.1), we conclude that $c \geq \tau$ which is a contradiction since we have supposed the opposite inequality.

(B) There exists a $t_0 \in (0,1)$ such that

$$\int_0^{t_0} g(y)dy < \int_0^{t_0} g_1(y)dy.$$

Then, by continuity reasons, we have as a consequence that there exists a $\delta > 0$ such that

$$\int_0^t g(y)dy < \int_0^t g_1(y)dy, \text{ for any } t \in (t_0 - \delta, t_0 + \delta) = I_\delta. \tag{3.9}$$

We define now the quantities d_1, d_2 by the following equations:

$$\frac{1}{\delta}\int_{t_0-\delta}^{t_0} g_1(y)dy = d_1 \quad \text{and} \quad \frac{1}{\delta}\int_{t_0}^{t_0+\delta} g_1(y)dy = d_2. \tag{3.10}$$

Then by Hölder's inequality on the interval $(t_0 - \delta, t_0)$ for g_1, we conclude that

$$\frac{1}{\delta} \int\limits_{t_0-\delta}^{t_0} g_1^p(y)dy > d_1^p, \tag{3.11}$$

which is a strict inequality since g_1 is strictly decreasing (therefore not constant) on the interval $(t_0 - \delta, t_0)$. In the same way, we have

$$\frac{1}{\delta} \int\limits_{t_0}^{t_0+\delta} g_1^p(y)dy > d_2^p. \tag{3.12}$$

Then since g_1 is decreasing, we have that $d_2 < d_1$. We define now the following non-increasing (as can easily be seen) function on $(0, 1]$:

$$g_2(t) = \begin{cases} g_1(t), \ t \in (0,1] \setminus (t_0 - \delta, t_0 + \delta), \\ d_1, \quad t \in [t_0 - \delta, t_0), \\ d_2, \quad t \in [t_0, t_0 + \delta]. \end{cases} \tag{3.13}$$

By (3.9) and since g_1 is decreasing, we easily see that we can choose $\delta > 0$ small enough, so that

$$\int\limits_0^t g(y)dy \leq \int\limits_0^t g_2(y)dy, \text{ for any } t \in (0, 1]. \tag{3.14}$$

Additionally, because of (3.11) and (3.12), we must have that

$$\int\limits_0^1 g_2^p(y)dy < \int\limits_0^1 g_1^p(y)dy = F.$$

Since (3.14) holds for any $t \in (0, 1]$ and because of Lemma 3.2, we conclude that $\int_0^1 g^p(y)dy \leq \int_0^1 g_2^p(y)dy < F$ by considering the function $G(t) = t^p$. This is obviously a contradiction according to the way that F is defined. In this way, we derive the proof of our lemma. □

We now proceed to the

Proof of Theorem 2. Without loss of generality, we suppose that $\mathcal{J} = (0, 1)$. Let $p \in [1, \frac{c}{c-1})$ and $\mathcal{I} \subseteq (0, 1)$ and let also $\varphi_{\mathcal{I}} = \varphi/_{\mathcal{I}}$ be the restriction of φ to $\mathcal{I}$. Consider now the function $g : (0, |\mathcal{I}|] \to \mathbb{R}^+$, defined by $g = (\varphi_{\mathcal{I}})^*$. Then since $\varphi_{\mathcal{I}} \in A_1(\mathcal{I})$ with A_1 constant not more than c, we must have, by using Theorem 1, that $\frac{1}{t}\int_0^t g(y)dy \leq cg(t)$ for any $t \in (0, |\mathcal{I}|]$. Thus by Lemma 3.1, it is easy to see that the following is true:

$$\frac{1}{|\mathcal{I}|} \int\limits_0^{|\mathcal{I}|} g^p(y)dy \leq \frac{1}{c^{p-1}(c + p - pc)} \Big(\frac{1}{|\mathcal{I}|} \int\limits_0^{|\mathcal{I}|} g(y)dy\Big)^p,$$

which is

$$\frac{1}{|\mathcal{I}|}\int\limits_{\mathcal{I}} \varphi^p(x)dx \le \frac{1}{c^{p-1}(c+p-pc)}\Big(\frac{1}{|\mathcal{I}|}\int\limits_{\mathcal{I}} \varphi(x)dx\Big)^p.$$

The relation (1.3) is proved. □

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