



On conjectures regarding the Nekrasov–Okounkov hook length formula

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Abstract. The Nekrasov–Okounkov hook length formula provides a fundamental link between the theory of partitions and the coefficients of powers of the Dedekind eta function. In this paper we examine three conjectures presented by Amdeberhan. The first conjecture is a refined Nekrasov–Okounkov formula involving hooks with trivial legs. We give a proof of the conjecture. The second conjecture is on properties of the roots of the underlying D’Arcais polynomials. We give a counterexample and present a new conjecture. The third conjecture is on the unimodality of the coefficients of the involved polynomials. We confirm the conjecture up to the polynomial degree 1000.

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1. Introduction. In 2006 Nekrasov and Okounkov [10, 19, 24] published a new type of hook length formula. Based on their work on random partitions and the Seiberg–Witten theory, they obtained an unexpected identity relating the sum over products of partition hook lengths [6, 18] to the coefficients of complex powers of Euler products [14, 20, 22], which is essentially a power of the Dedekind eta function. This paper is devoted to three open conjectures stated by Amdeberhan [2, Section 2].

Let λ be a partition of n denoted by $\lambda \vdash n$ with weight $|\lambda| = n$. We denote by $\mathcal{H}(\lambda)$ the multiset of hook lengths associated to λ and \mathcal{P} be the set of all partitions. The Nekrasov–Okounkov hook length formula is given by

$$\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z}{h^2}\right) = \prod_{m=1}^{\infty} (1 - q^m)^{z-1}. \quad (1)$$

Let $q := e^{2\pi i\tau}$ and τ be in the upper complex half-plane. The identity (1) is valid for all $z \in \mathbb{C}$. The Dedekind eta function $\eta(\tau)$ is given by $q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 - q^m)$ (see [21]).

The first conjecture is a refinement of the Nekrasov–Okounkov hook length formula [2].

Conjecture 1. *Let $\mathcal{H}(\lambda)^\diamond$ be the multiset of hook lengths with trivial legs. Then*

$$\sum_{\lambda \vdash n} \prod_{h \in \mathcal{H}(\lambda)^\diamond} \binom{h+z}{h} = \sum_{\lambda \vdash n} \prod_{h \in \mathcal{H}(\lambda)} \binom{h^2+z}{h^2}. \tag{2}$$

The second conjecture is on the roots of the polynomials given in (2). Note that this is also related to the Lehmer conjecture [14, 17].

Conjecture 2. *Let n be a positive integer. Then the polynomial*

$$Q_n(z) := \sum_{\lambda \vdash n} \prod_{h \in \mathcal{H}(\lambda)} \binom{h^2+z}{h^2} \in \mathbb{C}[z] \tag{3}$$

has (i) only simple roots, (ii) only real roots, (iii) only negative roots.

The third conjecture is on the coefficients of the polynomials defined in (3).

Conjecture 3. *Let n be a positive integer. Then $Q_n(z)$ is unimodal.*

In 1913 D’Arcais [5] studied a sequence of polynomials $P_n(x)$:

$$\sum_{n=0}^{\infty} P_n(z) q^n = \prod_{n=1}^{\infty} (1 - q^n)^{-z}. \tag{4}$$

The coefficients are called D’Arcais numbers [4]. Independently from D’Arcais, Newman and Serre [20, 22] studied the polynomials in the context of modular forms. Serre proved a famous theorem on lacunary modular forms, utilizing the factorization of $P_n(x)$ for $1 \leq n \leq 10$ over \mathbb{Q} . All authors mentioned so far introduced slightly different normalized polynomials: D’Arcais [5], Newman [20], Serre [22], and Amdeberhan [2]. D’Arcais definition fits best to the new Conjecture 2.

In this paper we prove Conjecture 1. We use the Nekrasov–Okounkov hook length formula and combinatorial arguments. It is sufficient to show the formula evaluated at negative integer values. This follows from the Lagrange interpolation formula. We have been alerted¹ that Keith [15, page 4] proved an identity equivalent to Conjecture 1. After the substitution $z \rightarrow -b$, noting that his $M_e = H(\lambda')^\diamond$, and expanding $\sum_{\lambda \vdash n} \prod_{h \in \mathcal{H}(\lambda)^\diamond} \binom{h+z}{h}$ with respect to the variable z , the identity (2) is proven (we followed the notation of [15]). The proof is based on the calculation of the coefficients of the involved polynomials, which utilizes an infinite sum (generalized binomial formula).

The proof given in this paper is different. It is based on two varying interpretations of the contribution of the multiplicities $k_j(\lambda)$ for $1 \leq j \leq l(\lambda)$

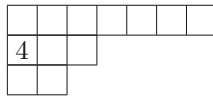
¹The reviewer kindly put our attention on the paper of William Keith.

to the final formula. We refer to the proof in Section 2 and Corollary 2. Only finite sums and short calculations are needed.

We give a counterexample to Conjecture 2. In the degree 10 case, $P_{10}(x)$, non-real complex roots exist. The simplicity of the roots was already studied in [12]. Finally we give evidence for Conjecture 3.

2. New partition Hook length formula. A partition of n (for an introduction we refer to [3, 9, 18, 23]) is a finite decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ of positive integers such that $|\lambda| := \sum_j \lambda_j = n$. We write $\lambda \vdash n$. The set of all partitions of n is denoted $\mathcal{P}(n)$ and the set of all partitions for all $n \in \mathbb{N}$ is denoted \mathcal{P} .

The integers λ_j are called the parts of λ and $l = l(\lambda)$ the length of the partition. Partitions are presented by their Young diagram. Let $\lambda = (7, 3, 2)$. Then $l(\lambda) = 3$ and $n = |\lambda| = 12$.



We attach to each cell u of the diagram the arm $a_u(\lambda)$, the amount of cells in the same row of u to the right of u . Further we have the leg $\ell_u(\lambda)$, the number of cells in the same column of u below of u . The hook length $h_u(\lambda)$ of the cell u is given by $h_u(\lambda) := a_u(\lambda) + \ell_u(\lambda) + 1$. The hook length multiset $\mathcal{H}(\lambda)$ is the multiset of all hook length of λ . Our example gives

$$\mathcal{H}(\lambda) = \{9, 8, 6, 4, 3, 2, 1, 4, 3, 1, 2, 1\}.$$

The list is given from the left to the right and from the top to the bottom in the Young diagram. Cells have the coordinates (i, j) following the same procedure, hence 4 is in the cell $(2, 1)$ with $\ell = 1$ and $a = 2$ (we usually simplify the notation). Let f_λ denote the number of standard Young tableaux of shape λ . These are all possible combinations of filling a Young diagram with the numbers $\{1, 2, \dots, n\}$ for $\lambda \vdash n$, such that each number occurs once and in each row and column (from the left to the right and from the top to the bottom) the numbers are strictly increasing. The famous classical hook formula (Frame, Robinson, Thrall) states

$$f_\lambda = \frac{n!}{\prod_{h \in \mathcal{H}(\lambda)} h}. \tag{5}$$

In the following we prove Conjecture 1. Let $\mathcal{H}(\lambda)^\diamond$ be the multiset of hook lengths with trivial legs. Then Conjecture 1 states

$$Q_n(z) := \sum_{\lambda \vdash n} \prod_{h \in \mathcal{H}(\lambda)^\diamond} \binom{h+z}{h} = P_n(z+1). \tag{6}$$

It can easily be verified for $z = -1$ and $z = 0$. Since there always exists a hook length $h = 1$ for λ , the product in $Q_n(-1)$ always has a factor which is zero. Hence $Q_n(-1) = P_n(0) = 0$ for $n \geq 1$. Let $z = 0$. Then

$$Q_n(0) = \sum_{\lambda \vdash n} 1,$$

which is equal to the number of partitions of n . By Euler it is known that this is the coefficient of q^n in the product $\prod_{m=1}^{\infty} (1 - q^m)^{-1}$, hence equal to $P_n(1)$.

Proof of Conjecture 1. The Nekrasov–Okounkov hook length formula states that

$$\sum_{\lambda \vdash n} \prod_{h \in \mathcal{H}(\lambda)} \left(\frac{h^2 + z}{h^2} \right) \tag{7}$$

is equal to $P_n(z + 1)$. Hence it is sufficient to prove that

$$\sum_{\lambda \vdash n} \prod_{h \in \mathcal{H}(\lambda)^\diamond} \left(\frac{h - (m + 1)}{h} \right) = P_n(-m) \tag{8}$$

for all $m \in \mathbb{N}$. Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a partition of n . We count all parts of λ with value j and put

$$a_j(\lambda) := \#\{i \mid \lambda_i = j\}.$$

This leads to the bijection

$$\psi : \{\lambda \vdash n\} \longrightarrow \left\{ (a_1, \dots, a_n) \in \mathbb{N}_0 \mid \sum_{k=1}^n k a_k = n \right\}, \tag{9}$$

where λ maps to $a(\lambda) = (a_1(\lambda), \dots, a_n(\lambda))$. We collect all the terms in

$$\prod_{d=1}^{\infty} (1 - q^d)^m$$

contributing to $P_n(-m)$. Then $\psi(\lambda)$ contributes with multiplicity

$$\binom{m}{a_1(\lambda)} \cdots \binom{m}{a_n(\lambda)}.$$

Hence we obtain

$$P_n(-m) = \sum_{\lambda \vdash n} (-1)^{l(\lambda)} \binom{m}{a_1(\lambda)} \cdots \binom{m}{a_n(\lambda)}. \tag{10}$$

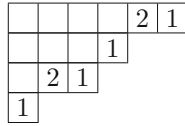
Next we study the hook length term

$$C_m(n) := \sum_{\lambda \vdash n} \prod_{h \in \mathcal{H}(\lambda)^\diamond} \left(\frac{h - (m + 1)}{h} \right). \tag{11}$$

Note that if $h \in \mathcal{H}(\lambda)^\diamond$, then $1, \dots, h - 1$ are also elements of $\mathcal{H}(\lambda)^\diamond$ for $h > 1$. Let λ^c denote the conjugate of λ , which is also a partition of n . Since $\{\lambda \vdash n\} = \{\lambda^c \mid \lambda \vdash n\}$, we have

$$\sum_{\lambda \vdash n} \prod_{h \in \mathcal{H}(\lambda)^\diamond} \left(\frac{h - (m + 1)}{h} \right) = \sum_{\lambda^c \vdash n} \prod_{h \in \mathcal{H}(\lambda)^\diamond} \left(\frac{h - (m + 1)}{h} \right). \tag{12}$$

We consider the Young diagram of λ^c . For example let $\lambda = (4, 3, 3, 2, 1, 1)$. Then we have



The numbers denote the hook length for all cells with no leg. Let

$$b_j(\lambda) := \#\{i \mid \lambda_i = j\}.$$

Then $b_1(\lambda), \dots, b_n(\lambda)$ contributes to $C_m(n)$ with multiplicity

$$\binom{m}{b_1(\lambda)} \cdots \binom{m}{b_n(\lambda)}.$$

Here we used the simple identity

$$\prod_{k=1}^h \binom{k - (m + 1)}{k} = (-1)^h \binom{m}{h}.$$

Hence we obtain

$$C_m(n) = \sum_{\lambda \vdash n} (-1)^{l(\lambda)} \binom{m}{b_1(\lambda)} \cdots \binom{m}{b_n(\lambda)}. \tag{13}$$

Comparing (10) and (13) proves Conjecture 1. □

Corollary 1. *Let $\mathcal{H}(\lambda)^{\circ\circ}$ be the multiset of hook lengths with trivial arms. Then*

$$\sum_{\lambda \vdash n} \prod_{h \in \mathcal{H}(\lambda)^\circ} \binom{h+z}{h} = \sum_{\lambda \vdash n} \prod_{h \in \mathcal{H}(\lambda)^{\circ\circ}} \binom{h+z}{h}. \tag{14}$$

Let $\lambda \vdash n$. Let $k_j := k_j(\lambda) = a_j(\lambda) = b_j(\lambda)$. Since

$$\binom{k+z}{k} = (-1)^k \binom{-z-1}{k} \tag{15}$$

for $k \in \mathbb{N}_0$ and $z \in \mathbb{C}$, we obtain (put $z = -(m + 1)$):

Corollary 2. *Let $\mathcal{H}(\lambda)^\circ$ be the multiset of hook lengths with trivial legs. Then*

$$\sum_{\lambda \vdash n} \prod_{h \in \mathcal{H}(\lambda)^\circ} \binom{h+z}{h} = \sum_{\lambda \vdash n} \prod_{j=1}^n \binom{k_j+z}{k_j} = \sum_{\lambda \vdash n} \prod_{j=1}^n (-1)^{l(\lambda)} \binom{-z-1}{k_j}.$$

This completes the proof of Conjecture 2.1 given in [2]. For the convenience of the reader and recommended by the reviewer of this article, we recall Conjecture 2.1 as stated in [2, Section 2] and [2, Section 13].

Conjecture 2.1 [2]. *For a partition $\lambda \vdash n$ and its Young diagram, let h_u, a_u, l_u denote the hook-length, arm and legs of cell $u \in \lambda$. Note $h_u = a_u + l_u + 1$. Let $k_j(\lambda) = \#\{i : \lambda_i = j\}$. Then we have variants linked to Nekrasov–Okounkov hook formula [19]*

$$\sum_{\lambda \vdash n} \prod_{\substack{u \in \lambda \\ a_u=0}} \frac{h_u+t}{h_u} = \sum_{\lambda \vdash n} \prod_{j \geq 1} \binom{k_j+t}{k_j} = \sum_{\lambda \vdash n} \prod_{u \in \lambda} \frac{h_u^2+t}{h_u^2} = \sum_{\lambda \vdash n} \prod_{\substack{u \in \lambda \\ l_u=0}} \frac{h_u+t}{h_u}.$$

We also quote Amdeberhan [2, Section 13]: “In Conjecture 2.1, the middle two are immediate from [8] that $\sum_n x^n \prod_{u \in \lambda} \frac{t+h_u^2}{h_u^2} = \prod_j (1-x^j)^{-t-1}$; and the extreme-left and right sides are equal by duality. The main wish here is to prove the rest; a combinatorial argument is strongly desired.”

It would be also interesting to study the conjecture in terms of t -cores [7] to finally obtain a new proof of the Nekrasov–Okounkov hook length formula, but this seems to be very difficult and beyond the scope of this paper.

3. D’Arcais type polynomials. The Nekrasov–Okounkov hook length formula [10, 19] states

$$\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z}{h^2}\right) = \sum_{n=0}^{\infty} P_n(1-z) q^n. \tag{16}$$

The polynomials $P_n(x)$ can also be recursively defined by

$$P_n(x) := \frac{x}{n} \sum_{k=1}^n \sigma(k) P_{n-k}(x).$$

Here $P_0(x) := 1$ and $\sigma(n) := \sum_{d|n} d$. This makes it possible to calculate the coefficients of the polynomials directly. The first 20 polynomials can be found in [13]. Note that the first 10 were published in 1955 by Newman [20] (see also [22]). We claim that $P_{10}(x)$ does not only have real but also non-real roots. Numerically this was already shown in [13]. There we were interested in good numerical approximations for the values of the roots of the polynomials and it turned out that two of them had a (numerically) non-zero imaginary part. We want to note that we also had been not aware of [2]. In this paper we use a theorem of Aissen–Schoenberg–Whitney and Edrai [1] to give a rigorous algebraic proof. This leads to a counterexample for Conjecture 2. We also hope that this approach will give to the reader some deeper insight into the topic. Finally a third proof is given by involving derivatives that Conjecture 2 in the present form is not correct.

Let us first recall the definition of a totally nonnegative matrix. Then we give a bijection between the set of infinite real sequences and certain Toeplitz matrices. We define (finite) Polya frequency sequences and state a well-known characterization of polynomials with real roots.

Let $A = (a_{i,j}^{\infty})_{i,j=0}$ be a two way infinite matrix with real entries. The matrix A is called totally nonnegative (TN) if all minors are nonnegative.

Definition 1. Let a_0, a_1, \dots be an infinite sequence of real numbers. We attach the matrix $A = (a_{i,j})_{i,j=0}^{\infty}$ given by $a_{i,j} := a_{i-j}$ for $0 \leq j \leq i$ and $a_{i,j} = 0$ otherwise.

Example 1. Let the sequence $2, 2, 1, 0, 0, \dots$ be given, then the attached matrix A is given by

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & \dots \\ 2 & 2 & 0 & 0 & 0 & \dots \\ 1 & 2 & 2 & 0 & 0 & \dots \\ 0 & 1 & 2 & 2 & 0 & \dots \\ 0 & 0 & 1 & 2 & 2 & \dots \\ \vdots & \vdots & & \ddots & \ddots & \ddots \end{pmatrix}.$$

We consider finite sequences as infinite sequences. Note that A is not a TN since the minor, the determinant of

$$\begin{pmatrix} 2 & 2 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix},$$

is negative.

Definition 2. Let a_0, a_1, \dots, a_n be a finite sequence of nonnegative real numbers. Then the sequence is called Polya frequency sequence if the attached (infinite) matrix A is a totally nonnegative matrix.

The following result is well known [16]. It gives a sufficient and necessary criterion for polynomials with nonnegative coefficients to have only real roots. Note that this can be applied to the D’Arcais polynomials $P_n(x)$. We recall from [11] that $P_n(x) = \frac{x}{n!} \sum_{k=0}^{n-1} a_k^{(n)} x^k$, where $a_k^{(n)} \in \mathbb{N}$. Here $a_{n-1}^{(n)} = 1$.

Theorem 1 (Aissen–Schoenberg–Whitney, Edrai). *A finite sequence a_0, \dots, a_n of nonnegative real numbers is a Polya frequency sequence if and only if the attached polynomial $\sum_{k=0}^n a_k x^k$ has only real roots.*

We have shown that the example $(2, 2, 1)$ is not a Polya frequency sequence, which directly reflects that the polynomial $2 + 2x + x^2$ has a non-real root.

Now we apply the theorem to the polynomial $P_{10}(x)$. Actually [13, 20]

$$P_{10}(x) = \frac{x}{10!} (x + 1) R(x).$$

Here $R(x) = \sum_{k=0}^8 a_k x^k$ with $a_k \in \mathbb{N}$. Hence it is sufficient to show that the coefficients of $R(x)$ are not a Polya frequency sequence.

$$\begin{aligned} a_0 &= 6531840, a_1 = 29758896, a_2 = 28014804, a_3 = 10035116, \\ a_4 &= 1709659, a_5 = 147854, a_6 = 6496, a_7 = 134, a_8 = 1. \end{aligned}$$

Then the attached matrix A (infinite rows and columns) is given by

$$\begin{bmatrix} 6531840 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 29758896 & 6531840 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 28014804 & 29758896 & 6531840 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 10035116 & 28014804 & 29758896 & 6531840 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1709659 & 10035116 & 28014804 & 29758896 & 6531840 & 0 & 0 & 0 & 0 & \dots \\ 147854 & 1709659 & 10035116 & 28014804 & 29758896 & 6531840 & 0 & 0 & 0 & \dots \\ 6496 & 147854 & 1709659 & 10035116 & 28014804 & 29758896 & 6531840 & 0 & 0 & \dots \\ 134 & 6496 & 147854 & 1709659 & 10035116 & 28014804 & 29758896 & 6531840 & 0 & \dots \\ 1 & 134 & 6496 & 147854 & 1709659 & 10035116 & 28014804 & 29758896 & 6531840 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 1 & 134 & 6496 & 147854 & 1709659 & 10035116 & 28014804 & 29758896 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

A matrix that violates the condition of the theorem (i.e. has negative determinant) is the following 26×26 matrix:

$$\begin{bmatrix} 10035116 & 28014804 & 29758896 & 6531840 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1709659 & 10035116 & 28014804 & 29758896 & 6531840 & 0 & 0 & 0 & 0 & \dots \\ 147854 & 1709659 & 10035116 & 28014804 & 29758896 & 6531840 & 0 & 0 & 0 & \dots \\ 6496 & 147854 & 1709659 & 10035116 & 28014804 & 29758896 & 6531840 & 0 & 0 & \dots \\ 134 & 6496 & 147854 & 1709659 & 10035116 & 28014804 & 29758896 & 6531840 & 0 & \dots \\ 1 & 134 & 6496 & 147854 & 1709659 & 10035116 & 28014804 & 29758896 & 6531840 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 1 & 134 & 6496 & 147854 & 1709659 & 10035116 & 28014804 & 29758896 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 1 & 134 & 6496 & 147854 & 1709659 & 10035116 & 28014804 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & 1 & 134 & 6496 & 147854 & 1709659 & 10035116 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

It contains the rows 4–29 and columns 1–26 of the infinite Toeplitz matrix A . Hence $P_{10}(x)$ has non-real roots.

For the convenience of the reader we give a second proof using the derivative $R'(x)$. Inserting the limiting values of the following intervals into $R(x)$ we observe that $R(x)$ has (at least) one root $z_1 \in (-59, -58)$, $z_2 \in (-33, -32)$, $z_3 \in (-18, -17)$, $z_4 \in (-14, -13)$, $z_7 \in (-2, -1)$, and $z_8 \in (-1, 0)$. Similarly for the derivative $R'(x)$ we find a root $z'_1 \in (-53, -52)$, $z'_2 \in (-29, -28)$, $z'_3 \in (-16, -15)$, $z'_4 \in (-11, -10)$, $z'_5 \in (-6, -5)$, $z'_6 \in (-4, -3)$, and $z'_7 \in (-1, 0)$. The degree of $R'(x)$ is 7. Hence each of these intervals contains exactly one (simple) root.

Firstly we note that in particular the root $z'_7 \in (-1, 0)$ of $R'(x)$ is unique, $R'(x)$ does not have a root in $(-2, -1]$, and there is a root $z_7 \in (-2, -1)$ of $R(x)$. Since the roots of $R'(x)$ are simple, z'_7 could be a double root of $R(x)$. But this would contradict the opposite signs of the values of $R(x)$ for the limits of $(-1, 0)$. (Note that there is only *one* root of $R'(x)$ in $(-1, 0)$.) Hence we must have $z_7 < z'_7 < z_8$. Thus a root smaller than z_1 and larger than z_8 of $R(x)$ is not possible since this would imply a root of $R'(x)$ for $x < z_1$ or $x > z_8$ and we have found that there are no such roots.

From the distribution of the roots of $R'(x)$ we can observe that $R(x)$ is increasing on (z'_3, z'_4) , decreasing on (z'_4, z'_5) , and increasing on (z'_5, z'_6) . Since $R(-6) > 0$ and $R(-5) > 0$ with a minimum at z'_5 in between, the only chance for the ‘missing’ zeros of $R(x)$ would be to be in the interval $(-6, -5)$.

We show next that $R(x)$ is strictly positive on $(-6, -5)$. Thus the second derivative $R''(x)$ has exactly one root in $(z'_4, z'_5) \supset (-10, -6)$ and $(z'_5, z'_6) \supset$

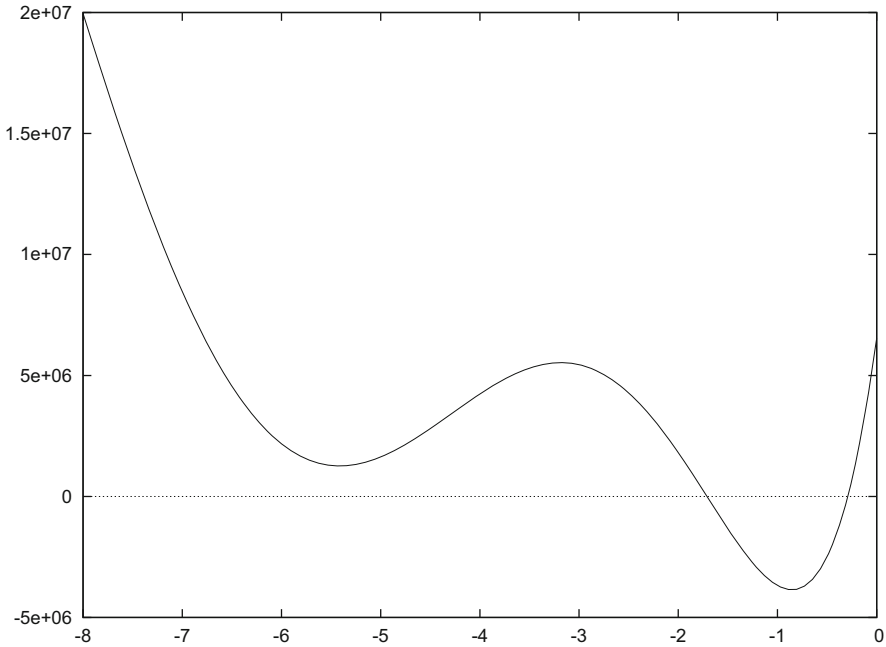


FIGURE 1. Graph of $R(x)$

$(-5, -4)$. Checking the limiting values shows that there is exactly one root of $R''(x)$ in $(-9, -8)$ and $(-5, -4)$. This means for the third derivative $R'''(x)$ that in between these two roots of $R''(x)$ there is exactly one root of $R'''(x)$. Checking the limiting values shows that this root is in the interval $(-7, -6)$, which means that there is no sign change of $R'''(x)$ on $(-6, -5)$, i.e. $R'''(x) < 0$ for $x \in (-6, -5)$. Expanding $R(x)$ around -5 yields $R(x) = 1632960 + 1690056(x + 5) + 1663164(x + 5)^2 + \text{terms of higher degree}$. Since the third derivative is negative on $(-6, -5)$, we obtain $R(x) > 1632960 + 1690056(x + 5) + 1663164(x + 5)^2$ for $x \in (-6, -5)$. The discriminant of this quadratic polynomial is negative so it does not have real roots and the same holds for $R(x)$ on $x \in (-6, -5)$ (Fig. 1).

Similarly it is possible to show for

$$P_{11}(x) = \frac{x}{11!} (x + 1)(x + 2)(x + 3)(x + 8) \tilde{R}(x)$$

with $\tilde{R}(x)$ an irreducible polynomial of degree 6 that it has only real roots. This can be done by checking the limits of the intervals $(-67, -66)$, $(-39, -38)$, $(-22, -21)$, $(-17, -16)$, $(-8, -7)$, and $(-1, 0)$.

Since $Q_n(z) = P_n(1 + z)$, we have found a counterexample to Conjecture 2 (ii). This implies that also (iii) has to be revised. Based on numerical investigations [13, 14] we propose the following revised version of Conjecture 2.

Conjecture 2 (New). *Let n be a positive integer. Then the polynomial*

$$P_n(z) = \sum_{\lambda \vdash n} \prod_{h \in \mathcal{H}(\lambda)} \left(\frac{h^2 + z - 1}{h^2} \right) \in \mathbb{C}[z] \quad (17)$$

has (i) only simple roots and (ii) real part of all non-trivial roots is negative.

Remark 1. (a) The first part of the conjecture has been proven for integral roots and n or $n - 1$ equal to a prime power [12].

(b) The second part of the conjecture has been verified for $n \leq 700$ ([13]). Polynomials satisfying (ii) are called Hurwitz polynomials or stable polynomials. They play an important role in the theory of dynamical systems.

4. Unimodality. Let a_0, a_1, \dots, a_n be a finite sequence of nonnegative real numbers. The sequence is denoted unimodal if $a_0 \leq a_1 \leq \dots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \dots \geq a_n$ for some k . It is denoted log-concave if $a_j^2 \geq a_{j-1} a_{j+1}$ for all $j > 0$. A sequence is called ultra-log-concave if the attached sequence $a_k / \binom{n}{k}$ is log-concave. Due to Newton (1707), a finite sequence a_0, a_1, \dots, a_n of nonnegative real numbers with real roots is log-concave. Actually it is already ultra-log-concave. The $Q_n(x)$ are polynomials with nonnegative real coefficients, but with potentially non-real roots. This makes Conjecture 3 considerably complicated.

Nevertheless, numerical calculations provide the following result.

Theorem 2. *Let $1 \leq n \leq 1000$. Then $Q_n(x)$ is ultra-log-concave. This implies Conjecture 3 (for $n \leq 1000$).*

Note that for a general polynomial $P(x)$ we do *not* have the property: $P(x)$ is unimodal if and only if $P(x + 1)$ is unimodal. For example $x^2 + 2$ is not unimodal as $a_2 = 1 > 0 = a_1 < a_0 = 2$ but $(x + 1)^2 + 1 = x^2 + 2x + 3$ is unimodal.

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