Arch. Math. 113 (2019), 37–41

0003-889X/19/010037-5
published online March 9, 2019
https://doi.org/10.1007/s00013-019-01312-x

Archiv der Mathematik

The maximal discrete extension of $SL_2(\mathcal{O}_K)$ for an imaginary quadratic number field K

Aloys Krieg, Joana Rodriguez, and Annalena Wernz

Abstract. Let $\mathcal{O}_K$ be the ring of integers of an imaginary quadratic number field K. In this paper we give a new description of the maximal discrete extension of the group $SL_2(\mathcal{O}_K)$ inside $SL_2(\mathbb{C})$, which uses generalized Atkin–Lehner involutions. Moreover we find a natural characterization of this group in $SO(1,3)$.

Mathematics Subject Classification. 11F06.

Keywords. Discrete groups, Maximal discrete extension, Atkin–Lehner involution.

Throughout this paper let

$$K = \mathbb{Q}(\sqrt{-m}) \subset \mathbb{C},\ m \in \mathbb{N} \text{ squarefree},$$

be an imaginary quadratic number field. Its discriminant and ring of integers are

$$d_K = \begin{cases} -m \\ -4m \end{cases} \quad \text{and} \quad \mathcal{O}_K = \begin{cases} \mathbb{Z} + \mathbb{Z}(1+\sqrt{-m})/2, & \text{if } m \equiv 3 \pmod 4, \\ \mathbb{Z} + \mathbb{Z}\sqrt{-m}, & \text{if } m \equiv 1,2 \pmod 4. \end{cases}$$

Denote by

$$\Gamma_K := SL_2(\mathcal{O}_K)$$

the group of integral 2×2 matrices of determinant 1 and by

$$\Gamma_K^* \leqslant SL_2(\mathbb{C})$$

its maximal discrete extension in $SL_2(\mathbb{C})$ according to [3, Chap. 7.4]. If $\langle \alpha_1, \ldots, \alpha_n \rangle$ denotes the $\mathcal{O}_K$-module generated by $\alpha_1, \ldots, \alpha_n$ and $\sqrt{\cdot}$ stands for an arbitrary (fixed) branch of the square root, we obtain a description of Γ_K^* from [3, Chap. 7.4].

Birkhäuser

Proposition. *For the imaginary quadratic number field K, one has*

$$\Gamma_K^* = \left\{ \frac{1}{\sqrt{\alpha\delta - \beta\gamma}} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix};\ \alpha, \beta, \gamma, \delta \in \mathcal{O}_K,\ \langle \alpha\delta - \beta\gamma \rangle = \langle \alpha, \beta, \gamma, \delta \rangle \neq \{0\} \right\}$$

satisfying

$$[\Gamma_K^* : \Gamma_K] = 2^\nu,\ \nu = \sharp\{p;\ p \text{ prime},\ p \mid d_K\}.$$

The main aim of this note ist to give an alternative description of Γ_K^* in terms of generalized Atkin–Lehner involutions, which are more familiar in the theory of modular forms, as well as a characterization of Γ_K^* inside the orthogonal group $SO(1,3)$.

Let $\omega := m + \sqrt{-m}$. Then we see that $\gcd(d, \omega\overline{\omega}/d) = 1$ for each squarefree divisor d of d_K. Hence there are $u, v \in \mathbb{Z}$ such that

$$ud - v\omega\overline{\omega}/d = 1, \quad \text{i.e.} \quad V_d := \frac{1}{\sqrt{d}} \begin{pmatrix} ud & v\omega \\ \overline{\omega} & d \end{pmatrix} \in SL_2(\mathbb{C}). \tag{1}$$

A straightforward verification shows that the coset

$$\Gamma_K V_d = V_d \Gamma_K \subseteq \frac{1}{\sqrt{d}} \mathcal{O}_K^{2\times 2} \cap SL_2(\mathbb{C}) \tag{2}$$

is independent of the particular choice of u and v, hence well-defined by d. Moreover one has for squarefree divisors d and e of d_K

$$V_d^2 \in \Gamma_K, \tag{3}$$
$$V_d \cdot V_e \in \Gamma_K V_f, \quad f = de/\gcd(d,e)^2. \tag{4}$$

Hence V_d can be viewed as a generalization of the Atkin–Lehner involution (cf. [1, Sect. 2]).

Theorem 1. *For the imaginary quadratic number field K, the group Γ_K^* admits the description*

$$\Gamma_K^* = \bigcup_{d \mid d_K,\, d \text{ squarefree}} \Gamma_K V_d.$$

Proof. Denote the right-hand side by G. Given $M, N \in \Gamma_K$ and squarefree divisors d, e of d_K, we obtain

$$MV_d \cdot NV_e \in \Gamma_K (V_d V_e) = \Gamma_K V_f$$

due to (2) and (4). Moreover (3) and (2) imply

$$(MV_d)^{-1} = V_d^{-1} M^{-1} \in V_d \Gamma_K = \Gamma_K V_d.$$

Therefore G is a group, which contains $\Gamma_K = \Gamma_K V_1$. As Γ_K is a discrete subgroup of $SL_2(\mathbb{C})$, this is also true for G. Thus we have $G \subseteq \Gamma_K^*$ as well as

$$[G : \Gamma_K] = [\Gamma_K^* : \Gamma_K] = 2^\nu$$

due to the Proposition, hence $G = \Gamma_K^*$ follows. □

As this is the main result we sketch a direct proof. Start with a matrix M in a discrete subgroup G of $SL_2(\mathbb{C})$ containing Γ_K. We conclude that $M \in r\mathcal{O}_K^{2\times 2}$ for some $r \in \mathbb{R}$, $r > 0$, just as in [3, Chap. 7, Proposition 4.3]. In view of $\det M = 1$ we may choose $r = 1/\sqrt{d}$ for some $d \in \mathbb{N}$. Moreover the minimal possible d turns out to be a squarefree divisor of d_K. Considering MV_d^{-1} we end up with the case that G is a subgroup of $SL_2(K)$, which was covered in [3, Chap.7, Lemma 4.4].

Remark 1. (a) From Theorem 1 it is clear that Γ_K is normal in Γ_K^* and that the factor group Γ_K^*/Γ_K is isomorphic to C_2^ν.

(b) If $d \,|\, d_K$, $d > 1$ is squarefree, then the coefficients of the matrices in $\Gamma_K V_d$ in (2) are algebraic integers of the biquadratic number field $\mathbb{Q}(\sqrt{d}, \sqrt{-m})$.

(c) There is no maximal discrete extension of Γ_K inside $GL_2(\mathbb{C})$ as the series of groups

$$\{e^{2\pi ij/n}M;\ j = 0, \ldots, n-1,\ M \in \Gamma_K\},\quad n \in \mathbb{N},$$

shows.

We want to give a natural characterization of the groups above in $SO(1,3)$. Therefore we consider the 4-dimensional $\mathbb{R}$-vector space

$$V := \{H \in \mathbb{C}^{2\times 2};\ H = \overline{H}^{tr}\}$$

of Hermitian 2×2 matrices over $\mathbb{C}$. Then

$$q : V \to \mathbb{R}, \quad H \mapsto \det H,$$

is a quadratic form on V of signature $(1,3)$. Let $SO(V,q)$ denote the attached special orthogonal group. Moreover let $SO_0(V,q)$ stand for the connected component of the identity element. It can be characterized by the fact that it maps the cone of positive definite matrices in V onto itself. More precisely one has

$$SO_0(V,q) = \{\varphi \in SO(V,q);\ \operatorname{trace}\varphi(E) > 0\}, \quad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

From [5, §4, Sect. 14], we quote the

Lemma. *The mapping*

$$\phi : SL_2(\mathbb{C}) \to SO_0(V,q),\quad M \mapsto \phi_M,$$
$$\phi_M : V \to V,\quad H \mapsto MH\overline{M}^{tr},$$

is a surjective homomorphism of groups with kernel $\{\pm E\}$.

Now we consider the situation over the rational numbers $\mathbb{Q}$. Let $V_K := V \cap K^{2\times 2}$ denote the associated $\mathbb{Q}$-vector space of dimension 4 and

$$\widehat{\Sigma}_K := \left\{\varphi \in SO_0(V,q);\ \varphi(V_K) = V_K\right\},$$
$$\widehat{\Gamma}_K := \left\{\tfrac{1}{\sqrt{d}}M \in SL_2(\mathbb{C});\ d \in \mathbb{N},\ M \in \mathcal{O}_K^{2\times 2}\right\} \supseteq SL_2(K).$$

Corollary 1. *For the imaginary quadratic number field K, one has*

$$\phi(\widehat{\Gamma}_K) = \widehat{\Sigma}_K.$$

Proof. "⊆" This is obvious due to the Lemma.
"⊇" Start with $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in SL_2(\mathbb{C})$ such that $\phi_M \in \widehat{\Sigma}_K$ in accordance with the Lemma. Assume $\alpha \neq 0$, because we may replace M by MJ, $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Calculating $\phi_M(H)$ for a suitable basis of V_K leads to

$$\alpha\overline{\alpha} \in \mathbb{Q}, \quad \alpha^2, \beta/\alpha, \gamma/\alpha \in K.$$

Thus we obtain $\alpha \in \frac{1}{\sqrt{d}}\mathcal{O}_K$ for a suitable $d \in \mathbb{N}$ and

$$M = \begin{pmatrix} 1 & 0 \\ \gamma/\alpha & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix} \begin{pmatrix} 1 & \beta/\alpha \\ 0 & 1 \end{pmatrix} \in \widehat{\Gamma}_K.$$

□

Now consider the lattice $\Lambda_K := V_K \cap \mathcal{O}_K^{2\times 2}$ of integral Hermitian matrices and the group of lattice automorphisms in $SO_0(V,q)$

$$\Sigma_K^* := \{\varphi \in SO_0(V,q);\ \varphi(\Lambda_K) = \Lambda_K\} \leqslant \widehat{\Sigma}_K.$$

Thus we get a very natural description of Γ_K^* in this context.

Theorem 2. *For the imaginary quadratic number field K, one has*

$$\phi(\Gamma_K^*) = \Sigma_K^*.$$

Proof. "⊆" This is a consequence of the Lemma and Theorem 1, because $\phi_{V_d}(\Lambda_K)$ equals Λ_K for all squarefree divisors d of d_K.

"⊇" $\phi^{-1}(\Sigma_K^*)$ is a discrete subgroup of $SL_2(\mathbb{C})$ containing Γ_K due to the Lemma. As Γ_K^* is maximal discrete, we get

$$\phi^{-1}(\Sigma_K^*) \subseteq \Gamma_K^*.$$

□

Now it is easy to characterize Γ_K as the so-called discriminant kernel. For this purpose consider the dual lattice

$$\begin{aligned} \Lambda_K^\sharp &:= \{S \in V_K;\ \operatorname{trace}(SH) \in \mathbb{Z} \text{ for all } H \in \Lambda_K\} \\ &= \left\{\begin{pmatrix} s_1 & s \\ \overline{s} & s_2 \end{pmatrix};\ s_1, s_2 \in \mathbb{Z},\ s \in \tfrac{1}{\sqrt{d_K}}\mathcal{O}_K\right\} \supseteq \Lambda_K. \end{aligned}$$

Any $\varphi \in \Sigma_K^*$ satisfies $\varphi(\Lambda_K^\sharp) = \Lambda_K^\sharp$. Hence we may define

$$\Sigma_K := \{\varphi \in \Sigma_K^*;\ \varphi \text{ induces id on } \Lambda_K^\sharp/\Lambda_K\}.$$

Corollary 2. *For the imaginary quadratic number field K, one has*

$$\phi(\Gamma_K) = \Sigma_K.$$

Proof. For $M \in \Gamma_K$ it is easy to verify that $\phi_M \in \Sigma_K$. If $d \mid d_K$ is squarefree, a simple calculation yields

$$\phi_{V_d} \in \Sigma_K \Leftrightarrow d = 1.$$

Thus $\phi_M \in \Sigma_K$, $M \in \Gamma_K^*$, holds if and only if $M \in \Gamma_K$, due to the Lemma in combination with Theorem 1. □

Remark 2. (a) Similar results hold for the group

$$\Gamma_0(N) := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z});\ \gamma \equiv 0 \pmod N \right\}$$

instead of Γ_K (cf. [7, Sect. 6]) and moreover in the case of the paramodular group of degree 2 (cf. [4], [6]).

(b) Considering the Hermitian modular group of degree 2 over K (cf. [2]) the group Γ_K^* also appears if one wants to compute its maximal normal extension (cf. [8]).

References

[1] Atkin, A.O.L., Lehner, J.: Hecke operators on $\Gamma_0(m)$. Math. Ann. **185**, 134–160 (1970)

[2] Braun, H.: Hermitian modular functions. Math. Ann. **50**, 827–855 (1949)

[3] Elstrodt, J., Grunewald, F., Mennicke, J.: Groups Acting on Hyperbolic Space. Springer, Berlin (1998)

[4] Gallenkämper, J., Krieg, A.: The Hecke algebras for the orthogonal group $SO(2,3)$ and the paramodular group of degree 2. Int. J. Number Theory **24**, 2409–2423 (2018)

[5] Hein, W.: Einführung in die Struktur- und Darstellungstheorie der klassischen Gruppen. Springer, Berlin (1990)

[6] Köhler, G.: Erweiterungsfähigkeit paramodularer Gruppen. Nachr. Akad. Wiss. Göttingen **1968**, 229–238 (1967)

[7] Krieg, A.: Integral orthogonal groups. In: Hagen, T., et al. (eds.) Dynamical Systems, Number Theory and Applications, pp. 177–195. World Scientific, Hackensack (2016). N.J. edition

[8] Wernz, A.: On Hermitian modular groups and modular forms. PhD thesis, RWTH Aachen (2019)

Aloys Krieg, Joana Rodriguez, and Annalena Wernz
Lehrstuhl A für Mathematik
RWTH Aachen University
52056 Aachen
Germany
e-mail: krieg@rwth-aachen.de

Joana Rodriguez
e-mail: joana.rodriguez@rwth-aachen.de

Annalena Wernz
e-mail: annalena.wernz@rwth-aachen.de

Received: 15 November 2018