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Height estimates for constant mean curvature graphs in Nil_3 and $\widetilde{PSL}_2(\mathbb{R})$

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Abstract. In this paper we obtain height estimates for compact, constant mean curvature vertical graphs in the homogeneous spaces Nil_3 and $\widetilde{PSL}_2(\mathbb{R})$. As a straightforward consequence, we announce a structure-type result for complete graphs defined on relatively compact domains.

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1. Introduction. In the last decades, height estimates have become a powerful tool when studying the global behavior of a certain class of immersed surfaces in some ambient space, see for instance [3,9–13,15,17]. Heinz [9] proved that if M is a *compact graph* in the euclidean space $\mathbb{R}^3$ with positive constant mean curvature H (H-surface in the following) and boundary ∂M lying on a plane Π, then the *maximum height* that M can reach from Π is $1/H$. This estimate is optimal, since it is attained by the H-hemisphere intersecting orthogonally Π. Applying the *Alexandrov reflection technique* yields that a compact embedded H-surface in $\mathbb{R}^3$ with boundary on Π has height from Π at most $2/H$.

These height estimates for H-surfaces in $\mathbb{R}^3$ were the cornerstone for Meeks [13] in his global study of H-surfaces in $\mathbb{R}^3$; for example, he showed that there do not exist properly embedded H-surfaces with one end in $\mathbb{R}^3$, and if a properly embedded H-surface has two ends, then the surface stays at bounded distance from a straight line. Later, Korevaar, Kusner, and Solomon [12] proved that a properly embedded H-surface lying inside a solid cylinder must be rotationally symmetric and hence a cylinder or an onduloid.

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In [11] Korevaar, Kusner, Meeks, and Solomon obtained optimal bounds for the height of H-graphs and of compact, embedded H-surfaces when $H > 1$ in the *hyperbolic space* $\mathbb{H}^3$ with boundary lying on a totally geodesic plane. In the formulation of the problem in both *space forms* $\mathbb{R}^3$ and $\mathbb{H}^3$, the plane where ∂M lies can be chosen without specifying its orthogonal direction, as $\mathbb{R}^3$ and $\mathbb{H}^3$ are *isotropic*; in general, a riemannian manifold is isotropic if its isometry group acts transitively on the tangent bundle.

In this context, the product spaces $\Sigma^2 \times \mathbb{R}$ defined as the riemannian product of a complete riemannian surface Σ^2 and the real line $\mathbb{R}$ are closely related to the space forms in the sense that they are highly symmetric. In [10] Hoffman, de Lira, and Rosenberg obtained height estimates for compact embedded H-surfaces with boundary contained in a slice $\Sigma^2 \times \{t_0\}$. This result was improved by Aledo, Espinar, and Gálvez [3], exhibiting *sharp* bounds for the height of compact, embedded H-surfaces in $\Sigma^2 \times \mathbb{R}$ with boundary in a slice, and characterizing when equality holds. As happened in $\mathbb{R}^3$ and $\mathbb{H}^3$, the H-graph in $\Sigma^2 \times \mathbb{R}$ with boundary in a slice $\Sigma^2 \times \{t_0\}$ attaining the maximum height over $\Sigma^2 \times \{t_0\}$ corresponds to the rotational H-hemisphere intersecting orthogonally $\Sigma^2 \times \{t_0\}$.

For the particular case when the base Σ^2 is a complete, simply connected surface with constant curvature κ, the spaces arising are the product spaces $\mathbb{M}^2(\kappa) \times \mathbb{R}$. Such product spaces belong to a two parameter family of homogeneous, simply connected 3-dimensional manifolds, the $\mathbb{E}(\kappa, \tau)$ spaces. In Section 2, we will introduce these spaces and give a geometric sense to the constants κ and τ. For instance, the product spaces correspond to the case $\tau = 0$ in the $\mathbb{E}(\kappa, \tau)$ family. In the last decade, the theory of immersed surfaces in the $\mathbb{E}(\kappa, \tau)$ spaces, and more specifically constant mean curvature and minimal surfaces, have become a fruitful theory focusing the attention of many geometers. See [1,2,5–7] and references therein for an outline of the development of this theory.

Our objective in this paper is to obtain height estimates for vertical H-graphs in the Heisenberg space Nil_3 and in the space $\widetilde{PSL_2}(\mathbb{R})$, the universal cover of the positively oriented isometries of the hyperbolic plane $\mathbb{H}^2$, which correspond to the particular choices in the $\mathbb{E}(\kappa, \tau)$ family of $\kappa = 0, \tau > 0$ and $\kappa < 0, \tau > 0$, respectively. To obtain the height estimates we will use the fact that H-graphs (or more generally, H-surfaces transverse to a Killing vector field) are stable, hence have bounded curvature at any fixed positive distance from their boundary. This behavior of the stability of H-surfaces has been exploited widely in the literature; see the proof of the Main Theorem in [18] for a global understanding of this technique in arbitrary complete 3-manifolds with bounded sectional curvature.

2. Homogeneous 3-dimensional spaces with 4-dimensional isometry group. Let $\mathbb{M}^2(\kappa)$ be the complete, simply connected surface of constant curvature $\kappa \in \mathbb{R}$. The family of homogeneous, simply connected 3-dimensional manifolds $\mathbb{E}$ with a 4-dimensional isometry group can be defined as a family of riemannian submersions $\pi : \mathbb{E} \longrightarrow \mathbb{M}^2(\kappa)$. The *fibre* that passes through a point $p \in \mathbb{M}^2(\kappa)$

is defined as $\pi^{-1}(p)$, and translations along these fibres are ambient isometries generated by the flow of a unitary Killing vector field, ξ. The Killing vector field is related to the Levi-Civita connection $\overline{\nabla}$ of $\mathbb{E}$ and the cross product by the formula

$$\overline{\nabla}_X \xi = \tau X \times \xi,$$

where τ is a constant named the *bundle curvature*. Both κ and τ satisfy $\kappa - 4\tau^2 \neq 0$, and after a change of orientation of $\mathbb{E}$ we can suppose that $\tau > 0$. These spaces are denoted by $\mathbb{E}(\kappa, \tau)$, where κ, τ are the constants defined above. Depending on the value of κ and τ, we obtain the following geometries:

- If $\tau = 0$, then we have the product spaces $\mathbb{M}^2(\kappa) \times \mathbb{R}$, i.e. the space $\mathbb{S}^2(\kappa) \times \mathbb{R}$ if $\kappa > 0$, and the space $\mathbb{H}^2(\kappa) \times \mathbb{R}$ if $\kappa < 0$.
- If $\tau > 0$ and $\kappa = 0$, the $\mathbb{E}(\kappa, \tau)$ space arising is the Heisenberg group Nil_3, the Lie group of matrices
$$\left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}; \ a, b, c \in \mathbb{R} \right\},$$
endowed with a one-parameter family of left-invariant metrics.
- When $\tau > 0$ and $\kappa < 0$, we obtain the space $\widetilde{PSL}_2(\mathbb{R})$, the universal cover of the positively oriented isometries of the hyperbolic plane $\mathbb{H}^2$, endowed with a two-parameter family of left-invariant metrics. Up to a homothetic change of coordinates, the family of left-invariant metrics turns out to depend on one parameter.
- When $\tau > 0$ and $\kappa > 0$, the $\mathbb{E}(\kappa, \tau)$ spaces are the Berger spheres. These spaces can be realized as the 3-dimensional sphere $\mathbb{S}^3$ endowed with a one-parameter family (again, after a homothetic change) of metrics, which are obtained in such a way that the Hopf fibration is still a riemannian fibration, but the length of the fibres is modified.

We can give a unified model for the $\mathbb{E}(\kappa, \tau)$ spaces; when $\kappa \leq 0$ the model is global and when $\kappa > 0$ we get the universal cover of $\mathbb{E}(\kappa, \tau)$ minus one fibre. We endow $\mathbb{R}^3$ (if $\kappa \geq 0$) and $\big(\mathbb{D}(2/\sqrt{-\kappa}) \times \mathbb{R}\big)$ (if $\kappa < 0$) with the metric

$$ds^2 = \lambda^2\big(dx^2 + dy^2\big) + \big(dz + \tau\lambda(ydx - xdy)\big)^2,$$

where λ is defined as

$$\lambda = \frac{4}{4 + \kappa(x^2 + y^2)}.$$

The riemannian submersion is given by the projection onto the first two coordinates. The vector field ∂_z is the unitary Killing vector field whose flow generates the *vertical translations*. The integral curves of this flow are the fibres of the submersion, and they are complete geodesics. The fields given by

$$E_1 = \frac{1}{\lambda}\partial_x - \tau y \partial_z, \quad E_2 = \frac{1}{\lambda}\partial_y + \tau x \partial_z, \quad E_3 = \partial_z,$$

are an orthonormal basis at each point. In this framework, the angle function of an immersed, orientable surface M is defined as $\nu = \langle \eta, \partial_z \rangle$, where η is a unit normal vector field defined on M.

Henceforth, we will denote simply by $\big(\mathbb{E},\langle\cdot,\cdot\rangle\big)$ any of the $\mathbb{E}(\kappa,\tau)$ spaces with the model given above. A *section*[1] of $\mathbb{E}$ is a subset of the form $\{z = z_0;\ z_0\in\mathbb{R}\}$, where z_0 is called the height of the section. Every such a section is a minimal surface, and when $\tau=0$, they are totally geodesic copies of $\mathbb{M}^2(\kappa)$ that differ one from the other by a vertical translation. A *vertical graph* in $\mathbb{E}$ is a surface with the property that it intersects each fibre of the submersion at most once. As a matter of fact, each vertical graph in $\mathbb{E}$ can be parametrized as

$$\big\{(x,y,f(x,y));\ (x,y)\in\Omega\big\},$$

for a certain smooth function f defined in a domain Ω contained in some section $\{z=z_0\},\ z_0\in\mathbb{R}$. Note that after a vertical translation, the domain of a vertical graph can be contained in a section with any height. A graph is compact if Ω is compact and f extends to $\partial\Omega$ continuously. The *boundary* of a compact graph is defined as $f(\partial\Omega)$. A compact graph has boundary in a section if its boundary has constant height. This is equivalent to the fact that f restricted to $\partial\Omega$ is a constant function.

2.1. Stability of H-surfaces in the $\mathbb{E}(\kappa,\tau)$ spaces. It is a well-known fact that an H-surface M immersed in an arbitrary riemannian 3-manifold is a critical point for the area functional associated to compactly supported variations of the surface that preserve the enclosed volume. Equivalently, M is an H-surface if and only if it is a critical point for the functional Area-$2H$Vol [4]. The second variation of this functional is given by the quadratic form

$$\mathcal{Q}(f,f) = -\int_M \big(\Delta_M f + f(|\sigma|^2+\mathrm{Ric}(\eta)\big) f\,dA, \qquad \forall f\in C_0^\infty(M), \tag{2.1}$$

where Δ_M is the Laplace-Beltrami operator of the surface M, $|\sigma|^2$ is the squared length of the second fundamental form of M, η is the unit normal of M, and $\mathrm{Ric}(\eta)$ is the *Ricci curvature* along the direction η. Equation (2.1) can be rewritten by defining the elliptic operator

$$\mathcal{L} = \Delta_M + |\sigma|^2 + \mathrm{Ric}(\eta) \tag{2.2}$$

and thus (2.1) is equivalent to

$$\mathcal{Q}(f,f) = -\int_M f\mathcal{L}f\,dA, \quad \forall f\in C_0^\infty(M). \tag{2.3}$$

The operator $\mathcal{L}$ is the *Jacobi operator*, or *stability operator* of M. An orientable immersion M in an $\mathbb{E}(\kappa,\tau)$ space is said to be *stable* if and only if

$$-\int_M f\mathcal{L}f\,dA \geq 0, \qquad \forall f\in C_0^\infty(M).$$

The non-vanishing functions $f\in C^\infty(M)$ lying in the kernel of $\mathcal{L}$ are called *Jacobi functions*. If M is an orientable immersed surface in an $\mathbb{E}(\kappa,\tau)$ space and ν denotes the angle function of M, then ν is a Jacobi function for the

[1] Abresch and Rosenberg also call these surfaces *umbrellas*, see [2].

stability operator $\mathcal{L}$ [5], i.e. the elliptic equation $\mathcal{L}\nu = 0$ holds. This equation reads as

$$\Delta_M \nu + \nu\big((1-\nu^2)(\kappa - 4\tau^2) + |\sigma|^2 + 2\tau^2\big) = 0. \tag{2.4}$$

A classical theorem due to Fischer-Colbrie [8] asserts that the existence of a positive Jacobi function defined on a surface M is equivalent to the stability of the surface.

Consider now a vertical graph M in $\mathbb{E}(\kappa,\tau)$. As by definition M intersects each fibre of the space $\mathbb{E}(\kappa,\tau)$ at most once, then M is transversal to the vertical Killing vector field ∂_z at every interior point. This is equivalent to the fact that the angle function $\nu = \langle \eta, \partial_z \rangle$ has no zeros at any interior point of the graph. As a matter of fact, each vertical H-graph in an $\mathbb{E}(\kappa,\tau)$ space is a stable surface, since either the function ν or $-\nu$ is positive.

3. Height estimates. In this section, H will denote a positive constant and $\big(\mathbb{E}, \langle\cdot,\cdot\rangle\big)$ will be either the space Nil_3 or $\widetilde{PSL_2}(\mathbb{R})$ with the corresponding metric. In particular, as in both spaces we have $\kappa \le 0$, $\mathbb{E}$ is given by the global model defined in Section 2. The theorem that we prove is the following:

Theorem 1. *Let H be a positive constant and suppose that*

$$4H^2 + \kappa > 0.$$

Then, there exists a constant $C = C(H,\kappa,\tau) > 0$, such that for every vertical H-graph M in $\mathbb{E}$ whose positive height is realized and with boundary contained in a section, the height that M reaches over that section is at most C.

In the space $\widetilde{PSL_2}(\mathbb{R})$, the hypothesis $4H^2 + \kappa > 0$ has a relevant geometric sense, since this condition for H and κ ensures the existence of a rotationally symmetric H-sphere. In general, in an $\mathbb{E}(\kappa,\tau)$ space the quantity $\sqrt{-\kappa}/2$ is known as the *critical mean curvature*. There exists an H-sphere in an $\mathbb{E}(\kappa,\tau)$ space if and only if $H > \sqrt{-\kappa}/2$.

Before proving Theorem 1 we recall a technical lemma that guarantees a uniform bound of the second fundamental form for H-graphs in $\mathbb{E}$. See [18] for a detailed proof.

Lemma 2. *Let M be a vertical H-graph in $\mathbb{E}$, with boundary of M in $\{z = 0\}$. If $d > 0$, there is a constant K, depending on d and $\mathbb{E}$, such that $|\sigma(p)| < K$ for all p in M with $d(p, \partial M) > d$.*

Now, we stand in position to prove Theorem 1.

Proof. Arguing by contradiction, suppose that the height estimate in the statement of the theorem does not hold. Then, there exists a sequence of compact vertical H-graphs M_n, whose boundaries are contained in sections of the form $\{z = z_n\}$, and such that if we denote by h_n the height of each M_n from $\{z = z_n\}$, then $\{h_n\} \to \infty$. After a vertical translation we can suppose that all the boundaries are contained in the section $\Pi = \{z = 0\}$. By the mean curvature comparison principle, each graph is contained in one of the half-spaces $\{z \ge 0\}$ or $\{z \le 0\}$. Passing to a subsequence we can suppose that all the graphs lie above Π, i.e. they lie in the half-space $\{z \ge 0\}$. Let η_n be the unit normal to each M_n such that the mean curvature with respect to η_n is H. In particular, each M_n is downwards oriented as a consequence again of the mean curvature comparison principle, and thus every angle function $\nu_n = \langle \eta_n, \partial_z \rangle$ is a negative function on M_n. Fix some positive number d and let us now denote $M_n^* := \{p \in M_n;\ d(p, \partial M_n) > 2d\}$. As the heights of M_n from Π tend to infinity, it is clear that M_n^* is a non-empty, possibly non-connected, graph over Π for n large enough. In this situation, Lemma 2 ensures us that there exists a positive constant Λ in such a way that the second fundamental form $\sigma_{M_n^*}$ of each surface M_n^* satisfies $|\sigma_{M_n^*}| < \Lambda$.

Consider for each n the connected component M_n^0 of M_n^* of maximum height from Π. Let $x_n \in M_n^0$ be the point where this maximum height is attained, and consider the isometry Φ_n that sends x_n to the origin. Now, define $M_n^1 = \Phi_n(M_n^0)$. The length of the second fundamental form of each graph M_n^1 is uniformly bounded by $\Lambda > 0$, as all the M_n^1 are obtained by translations of subsets of M_n^*. Moreover, the distances in M_n^1 of the origin to ∂M_n^1 diverge to ∞. Now, by a standard compactness argument for a sequence of surfaces with bounded curvature, we deduce that, up to a subsequence, there are subsets $K_n \subset M_n^1$ that converge uniformly on compact sets in the C^2 topology to a complete, possibly non-connected, H-surface M_∞ that passes through the origin. From now on, we will consider the connected component of M_∞ that passes through the origin, and we will still denote this component by M_∞ . Let $\nu_\infty := \langle \eta_\infty, \partial_z \rangle$ denote the angle function of M_∞, where here η_∞ is the unit normal of M_∞. Since M_∞ is a limit of the downwards-oriented graphs M_n^1, we see that ν_∞ is non-positive. We claim that ν_∞ cannot be bounded away from zero; indeed, assume that $\nu_\infty^2 \geq c > 0$ for some $c > 0$. Consider the projection $\mathfrak{p} : M_\infty \to \mathbb{M}^2(\kappa)$, let $\langle, \rangle_{\text{proj}}$ be the induced metric on M_∞ via $\mathfrak{p}$, and let $\langle, \rangle$ be the induced ambient metric on M_∞.

As $\langle, \rangle$ is complete and it is well known that $\nu_\infty^2 \langle, \rangle \leq \langle, \rangle_{\text{proj}}$, we conclude by $\nu_\infty^2 \geq c > 0$ that $\langle, \rangle_{\text{proj}}$ is also complete. In particular, $\mathfrak{p}$ is a local isometry from $(M_\infty, \langle, \rangle_{\text{proj}})$ onto $\mathbb{M}^2(\kappa)$. In these conditions, $\mathfrak{p}$ is necessarily a (surjective) covering map over the simply connected surface $\mathbb{M}^2(\kappa)$, and thus M_∞ is an entire vertical graph. Let $\mathcal{S}$ be the sphere with constant mean curvature H; the condition $4H^2 + \kappa > 0$ ensures us the existence of such a sphere for the case $\kappa < 0$. Let $\mathcal{S}(0)$ be such a sphere centered at the origin. Translate $\mathcal{S}(0)$ down until it is below the graph of M_∞. Then translate the sphere back up until it touches M_∞ for the first time. By the maximum principle the sphere equals M_∞, which contradicts that M_∞ is not compact. Therefore, there must exist a sequence of $p_n \in M_\infty$ with $\nu_\infty(p_n) \to 0$.

Let Θ_n be an isometry in $\mathbb{E}$ that takes each point p_n to the origin $\mathfrak{o} \in \mathbb{E}$, and define $M_\infty^n = \Theta_n(M_\infty)$, which is a sequence of complete, stable surfaces with constant mean curvature H passing through $\mathfrak{o}$ and whose angle functions satisfy $\nu_\infty^n \leq 0$. Again, standard elliptic theory ensures that, up to a subsequence, the surfaces M_∞^n converge to a stable H-surface M_∞^*, passing through $\mathfrak{o}$. As this convergence is C^2, the angle function ν_∞^* of M_∞^* satisfies $\nu_\infty^* \leq 0$ and $\nu_\infty^*(\mathfrak{o}) = 0$. Also, the stability operators $\mathcal{L}_n$ converge to the stability operator $\mathcal{L}_\infty$ of the limit surface M_∞^*.

The maximum principle for elliptic operators applied to $\mathcal{L}_\infty$ yields that any non-zero solution to (2.4) changes sign around any of its zeros. As $\mathcal{L}_\infty$ also admits the zero function as a solution and ν^*_∞ vanishes at $\mathfrak{o}$, the condition $\nu^*_\infty \leq 0$ implies that ν^*_∞ is identically zero. Therefore the limit surface M^*_∞ is contained in a flat cylinder $\gamma \times \mathbb{R}$, for a planar curve γ in $\mathbb{R}^2$ or $\mathbb{H}^2$ (depending on whether $\kappa = 0$ or $\kappa < 0$, respectively). An analytic prolongation argument yields that the maximal surface containing M^*_∞ has to be the complete flat cylinder $\gamma \times \mathbb{R}$. This cylinder is an H-cylinder as well, and thus the geodesic curvature of γ satisfies $\kappa_\gamma = 2H$. This implies that γ is a closed curve in $\mathbb{R}^2$ or $\mathbb{H}^2$ (depending if $\kappa = 0$ or $\kappa < 0$, respectively). In the cylinder $\gamma \times \mathbb{R}$, the operator $\mathcal{L}_\infty$ has the expression

$$\mathcal{L}_\infty = \Delta_M + \kappa_\gamma^2 + \kappa.$$

As all the surfaces M^n_∞ are stable, the limit cylinder M^*_∞ is also a stable surface. But a complete, vertical H-cylinder in an $\mathbb{E}(\kappa, \tau)$ is stable if and only if [14]

$$\kappa_\gamma^2 + \kappa \leq 0.$$

Thus, the limit cylinder is stable if and only if $4H^2 + \kappa \leq 0$, which is a contradiction with the hypothesis $4H^2 + \kappa > 0$. This contradiction completes the proof of Theorem 1. □

Corollary 3. *If H is a positive constant such that*

$$4H^2 + \kappa > 0,$$

then there do not exist complete vertical H-graphs defined over relatively compact domains $\Omega \subset \{z = z_0\}$ in the spaces Nil_3 and $\widetilde{PSL}_2(\mathbb{R})$.

Proof. Let M be a complete vertical H-graph over a relatively compact domain $\Omega \subset \{z = z_0\}$. Without losing generality we can suppose that M lies in the half-space $\{z \leq 0\}$ and intersects tangentially the section $\{z = 0\}$. Let C be the constant appearing in Theorem 1. Then, as the height of M with respect to the section $\{z = 0\}$ is unbounded, there exists some $d_0 > 0$ such that if we intersect M with the half-space $\{z \geq -d_0\}$, we obtain a compact H-graph with boundary lying in the section $\{z = -d_0\}$ and with height over $\{z = -d_0\}$ greater than C, contradicting Theorem 1. □

We finish this paper with two observations concerning further discussions of height estimates of H-graphs in Nil_3 and $\widetilde{PSL}_2(\mathbb{R})$.

First, although the constant C in Theorem 1 is not explicit, for some values of H we can derive an estimate for it. Let $S : \mathbb{E} \to \mathbb{R}$ denote the *scalar curvature* of $\mathbb{E}$, and suppose that there exists some constant $c > 0$ such that the inequality

$$3H^2 + S(x) \geq c \tag{3.1}$$

holds for every $x \in \mathbb{E}$. It was proved by Rosenberg in [16] that if Σ is a stable H-surface immersed in $\mathbb{E}$, for every $p \in \Sigma$ one has

$$d_\Sigma(p, \partial\Sigma) \leq \frac{2\pi}{\sqrt{3c}}.$$

Recall also that if Σ is an immersed surface in $\mathbb{E}$, the intrinsic distance d_Σ is always less or equal than the ambient distance. Thus, whenever inequality (3.1) holds for some $c > 0$ and every $x \in \mathbb{E}$, the height of an H-graph is less or equal than $2\pi/\sqrt{3c}$. In particular, the constant C in Theorem 1 can be bounded from above by $2\pi/\sqrt{3c}$.

Second we point out that $2\pi/\sqrt{3c}$ is not optimal. Indeed, denote by $\mathcal{S}^{H,\kappa,\tau}$ the rotationally symmetric H-sphere in Nil_3 or $\widetilde{PSL_2}(\mathbb{R})$, and by $\overline{\mathcal{S}_+^{H,\kappa,\tau}}$ the upper, closed H-hemisphere. For H big enough, the height of $\overline{\mathcal{S}_+^{H,\kappa,\tau}}$ tends to zero: see [20,21] for explicit expressions of the height of $\overline{\mathcal{S}_+^{H,\kappa,\tau}}$ in the spaces Nil_3 and $\widetilde{PSL_2}(\mathbb{R})$. But for H large enough inequality (3.1) holds, proving that the estimate $2\pi/\sqrt{3c}$ is not sharp.

Motivated by the discussions made in the Introduction about the height estimates for H-graphs in the space forms $\mathbb{R}^3$ and $\mathbb{H}^3$, and in the product spaces $\mathbb{M}^2(\kappa) \times \mathbb{R}$, we suggest that the maximum height that an H-graph M should attain in both Nil_3 and $\widetilde{PSL_2}(\mathbb{R})$ is the height of the upper, closed H-hemisphere $\overline{\mathcal{S}_+^{H,\kappa,\tau}}$, with equality at some point if and only if M agrees with $\overline{\mathcal{S}_+^{H,\kappa,\tau}}$.

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