Arch. Math. 112 (2019), 407–416

0003-889X/19/040407-10
published online December 13, 2018
https://doi.org/10.1007/s00013-018-1268-8

Archiv der Mathematik

The polynomial Daugavetian index of a complex Banach space

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Abstract. We introduce the *polynomial Daugavetian index* of an infinite-dimensional complex Banach space. This index generalizes to polynomials the Daugavetian index defined for operators by M. Martín in 2003. We also present some results about the introduced index.

Mathematics Subject Classification. 46B20, 46E40, 46G25.

Keywords. Daugavet equation, Daugavetian index, Polynomial Daugavet property.

1. Introduction. Let X be a Banach space. We denote by X^* the dual space of X, by $K(X)$ the Banach space of all compact linear operators on X, and by $\mathcal{P}_K(X;X)$ the normed space of all compact polynomials on X. By B_X, S_X, and S_{X^*} we denote the closed unit ball of X, the unit sphere of X, and the unit sphere of X^*, respectively. We write $\Pi(X)$ to denote the subset of $X \times X^*$ given by

$$\Pi(X) = \big\{(x, x^*) : \ x \in S_X, \ x^* \in S_{X^*}, \ x^*(x) = 1\big\}.$$

For each bounded function $\Phi : S_X \to X$, the *numerical range* of Φ is the set

$$V(\Phi) = \big\{x^*(\Phi(x)) : \ (x, x^*) \in \Pi(X)\big\}$$

and the *numerical radius* of Φ is the number

$$v(\Phi) = \sup\big\{|\lambda| : \ \lambda \in V(\Phi)\big\}.$$

If X is infinite-dimensional, then the compact operators on X are non-invertible, consequently, $\|Id + T\| \geq 1$ for all $T \in K(X)$. This allowed Martín [9] to define the concept of the *Daugavetian index* for an infinite-dimensional Banach space in the following way

$$\mathrm{daug}(X) = \max\{m \geq 0 : \|Id + T\| \geq 1 + m\|T\| \ \text{ for all } \ T \in K(X)\}.$$

Supported by Fapemig Grant APQ-00522-14.

Clearly $0 \leq \text{daug}(X) \leq 1$. When $\text{daug}(X) = 1$, the space X has the *Daugavet property* (DP) [8], that is, every weakly compact operator T on X satisfies

$$\|Id + T\| = 1 + \|T\|.$$

Writing $\omega(T) = \sup \text{Re}V(T)$, Martín [9] proved that

$$\text{daug}(X) = \inf \{\omega(T) : T \in K(X), \|T\| = 1\}.$$

He also obtained several properties about this index, among them, we emphasize the following stability property. Given an arbitrary family of Banach spaces $(X_\lambda)_{\lambda\in\Lambda}$ and denoting by $\left[\bigoplus_{\lambda\in\Lambda} X_\lambda\right]_{c_0}$ (resp. $\left[\bigoplus_{\lambda\in\Lambda} X_\lambda\right]_{\ell_1}$, $\left[\bigoplus_{\lambda\in\Lambda} X_\lambda\right]_{\ell_\infty}$) the c_0-sum (resp. ℓ_1-sum, ℓ_∞-sum) of the family, we have that $\text{daug}\left(\left[\bigoplus_{\lambda\in\Lambda} X_\lambda\right]_{c_0}\right) = \text{daug}\left(\left[\bigoplus_{\lambda\in\Lambda} X_\lambda\right]_{\ell_1}\right) = \text{daug}\left(\left[\bigoplus_{\lambda\in\Lambda} X_\lambda\right]_{\ell_\infty}\right) = \inf \{\text{daug}(X_\lambda) : \lambda \in \Lambda\}$.

Now we introduce the polynomial Daugavetian index, generalizing the Daugavetian index in the complex case. Let X be an infinite-dimensional complex Banach space and let $P \in \mathcal{P}_K(X; X)$ given by $P = P_0 + P_1 + \cdots + P_n$ where P_j is a j-homogeneous polynomial for $j = 0, \ldots, n$. From [1, Proposition 3.4], we have that $P_1 \in K(X)$. And it follows from the Cauchy inequality that

$$\|Id + P_1\| \leq \|Id + P\|.$$

Since P_1 is compact and X is infinite-dimensional, $\|Id + P_1\| \geq 1$ and consequently $\|Id + P\| \geq 1$. This allows us to define the *polynomial Daugavetian index* of X as

$$\text{daug}_p(X) = \max \{m \geq 0 : \|Id + P\| \geq 1 + m\|P\| \text{ for all } P \in \mathcal{P}_K(X; X)\},$$

generalizing the ideas of the Daugavetian index defined by Martín [9]. Observe that $0 \leq \text{daug}_p(X) \leq 1$. When $\text{daug}_p(X) = 1$, the space X has the *polynomial Daugavet property* (PDP), that is, every weakly compact polynomial P on X satisfies the *Daugavet equation*:

$$\|Id + P\| = 1 + \|P\|.$$

We also have $\text{daug}_p(X) \leq \text{daug}(X)$ for every infinite-dimensional complex Banach space X. Besides that, defining $\omega(P) = \sup \text{Re}V(P)$, we have

$$\omega(P) = \lim_{\alpha\to 0^+} \frac{\|Id + \alpha P\| - 1}{\alpha}$$

by [7, Theorem 2]. Since $\|Id + \alpha P\| \geq 1$ for all $P \in \mathcal{P}_K(X; X)$, we obtain $\omega(P) \geq 0$. We will prove in the next section that

$$\text{daug}_p(X) = \inf \{\omega(P) : P \in \mathcal{P}_K(X; X), \|P\| = 1\}.$$

Let us present some examples of spaces with polynomial Daugavetian index 1 and 0. As we commented in the last paragraph, infinite-dimensional complex Banach spaces with the polynomial Daugavet property have polynomial Daugavetian index 1. This is the case for the space $C_b(\Omega, X)$ of bounded X-valued continuous functions on a perfect completely regular space Ω (see [5]), the space $L_\infty(\mu, X)$ of essentially bounded Bochner-measurable functions with values in X where μ is an atomless σ-finite measure (see [6]), the space

$L_1(\mu, X)$ of Bochner-integrable functions with values in X where μ is an atomless positive measure (see [10]), the disk algebra $\mathcal{A}(\mathbb{D})$ of functions continuous on the closed unit complex disk $\overline{\mathbb{D}}$ and holomorphic in the open disk $\mathbb{D}$ of $\mathbb{C}$ (see [4]), and the representable Banach spaces (see [2]). On the other hand, when an infinite-dimensional complex Banach space has a finite-rank projection P such that $\|P\| = \|Id - P\| = 1$, the polynomial Daugavetian index of the space is 0. This is the case of $C_b(\Omega, X)$ for non-perfect Ω and $L_\infty(\mu, X)$, $L_1(\mu, X)$ when μ has atoms.

The purpose of this note is to extend some results of Martín [9] to the polynomial Daugavetian index.

2. Main results. We start this section by proving that the polynomial Daugavetian index can be calculated using the numerical range of polynomials.

Proposition 2.1. *Let X be an infinite-dimensional complex Banach space. Then*

$$\begin{aligned}\operatorname{daug}_p(X) &= \inf\{\omega(Q) : Q \in \mathcal{P}_K(X;X), \|Q\| = 1\} \\ &= \max\{n \geq 0 : \omega(P) \geq n\|P\| \text{ for all } P \in \mathcal{P}_K(X;X)\}.\end{aligned}$$

Proof. It is easy to see that

$$\begin{aligned}&\inf\{\omega(Q) : Q \in \mathcal{P}_K(X;X), \|Q\| = 1\} \\ &\qquad = \max\{n \geq 0 : \omega(P) \geq n\|P\| \text{ for all } P \in \mathcal{P}_K(X;X)\}.\end{aligned}$$

To prove that

$$\operatorname{daug}_p(X) = \inf\{\omega(Q) : Q \in \mathcal{P}_K(X;X), \|Q\| = 1\},$$

it is enough to show that

$$\begin{aligned}&\{n \geq 0 : \omega(P) \geq n\|P\| \text{ for all } P \in \mathcal{P}_K(X;X)\} \\ &\qquad = \{m \geq 0 : \|Id + P\| \geq 1 + m\|P\| \text{ for all } P \in \mathcal{P}_K(X;X)\}.\end{aligned}$$

Let $n \geq 0$ be a constant such that $\omega(P) \geq n\|P\|$ for all $P \in \mathcal{P}_K(X;X)$. Given $Q \in \mathcal{P}_K(X;X)$ and $(x, x^*) \in \Pi(X)$, we have

$$\|Id + Q\| \geq \|x + Q(x)\| \geq |x^*(x + Q(x))| = |1 + x^*(Q(x))| \geq 1 + \operatorname{Re} x^*(Q(x)).$$

Taking the supremum over all $(x, x^*) \in \Pi(X)$, we obtain

$$\|Id + Q\| \geq 1 + \omega(Q) \geq 1 + n\|Q\|.$$

Since Q is arbitrary, we get

$$n \in \{m : \|Id + P\| \geq 1 + m\|P\| \text{ for all } P \in \mathcal{P}_K(X;X)\}.$$

On the other hand, let $m \geq 0$ be a constant such that $\|Id + P\| \geq 1 + m\|P\|$ for all $P \in \mathcal{P}_K(X;X)$. Fix $Q \in \mathcal{P}_K(X;X)$ and observe that

$$\|Id + \alpha Q\| \geq 1 + m\|\alpha Q\| = 1 + m\alpha\|Q\| \text{ for all } \alpha > 0.$$

Thus,

$$\omega(Q) = \lim_{\alpha \to 0^+} \frac{\|Id + \alpha Q\| - 1}{\alpha} \geq m\|Q\|.$$

Therefore,

$$m \in \{n \geq 0 : \omega(P) \geq n\|P\| \ \text{ for all } \ P \in \mathcal{P}_K(X;X)\}.$$

□

This characterization allows us to prove some stability properties that extend those given in [9, Theorem 10]. The proof of the following proposition is based on the proofs of [3, Proposition 2.8] and [11, Proposition 2.3].

Proposition 2.2. *Let* $(X_\lambda)_{\lambda\in\Lambda}$ *be a family of infinite-dimensional complex Banach spaces. Then*

(i) $\operatorname{daug}_p\left(\left[\bigoplus_{\lambda\in\Lambda} X_\lambda\right]_{c_0}\right) \leq \inf\left\{\operatorname{daug}_p(X_\lambda) : \lambda \in \Lambda\right\}$;

(ii) $\operatorname{daug}_p\left(\left[\bigoplus_{\lambda\in\Lambda} X_\lambda\right]_{\ell_1}\right) \leq \inf\left\{\operatorname{daug}_p(X_\lambda) : \lambda \in \Lambda\right\}$;

(iii) $\operatorname{daug}_p\left(\left[\bigoplus_{\lambda\in\Lambda} X_\lambda\right]_{\ell_\infty}\right) \leq \inf\left\{\operatorname{daug}_p(X_\lambda) : \lambda \in \Lambda\right\}$.

Proof. Let Z denote $X \oplus_1 Y$ for any infinite-dimensional complex Banach spaces X and Y. We will prove that $\operatorname{daug}_p(Z) \leq \operatorname{daug}_p(X)$. Given $P \in \mathcal{P}_K(X;X)$ with $\|P\| = 1$, define $Q : Z \to Z$ by

$$Q(x,y) = (P(x), 0).$$

Clearly $Q \in \mathcal{P}_K(Z;Z)$ and $\|Q\| = 1$. If $\omega(Q) = 0$, then

$$\operatorname{daug}_p(Z) = 0 \leq \operatorname{daug}_p(X).$$

Let us suppose that $\omega(Q) > 0$. Then, given $0 < \varepsilon < \omega(Q)$, there exist $(x,y) \in S_Z$ and $(x^*,y^*) \in S_{Z^*} = S_{X^*\oplus_\infty Y^*}$ with $\|x\|\|x^*\| \neq 0$ such that

$$x^*(x) + y^*(y) = \|x^*\|\|x\| + \|y^*\|\|y\| = 1 \tag{1}$$

and

$$\omega(Q) - \varepsilon \leq \operatorname{Re}(x^*,y^*)Q(x,y) = \operatorname{Re} x^*(P(x))$$

follows from (1) that $x^*(x) = \|x^*\|\|x\|$. Now, write $P = P_0 + P_1 + \cdots + P_n$, where P_k is a k-homogeneous polynomial on X. Thus

$$\begin{aligned}
\omega(Q) - \varepsilon &\leq \operatorname{Re} x^*(P(x)) \\
&= \operatorname{Re} x^*\big(P_0(x)\big) + \operatorname{Re} x^*\big(P_1(x)\big) + \cdots + \operatorname{Re} x^*\big(P_n(x)\big) \\
&\leq \frac{\operatorname{Re} x^*\big(P_0(x)\big)}{\|x^*\|} + \frac{\operatorname{Re} x^*\big(P_1(x)\big)}{\|x^*\|\|x\|} + \cdots + \frac{\operatorname{Re} x^*\big(P_n(x)\big)}{\|x^*\|\|x\|^n} \\
&= \operatorname{Re} \frac{x^*}{\|x^*\|}\left(P_0\left(\frac{x}{\|x\|}\right)\right) + \cdots + \operatorname{Re} \frac{x^*}{\|x^*\|}\left(P_n\left(\frac{x}{\|x\|}\right)\right) \\
&= \operatorname{Re} \frac{x^*}{\|x^*\|}\left(P\left(\frac{x}{\|x\|}\right)\right) \\
&\leq \sup \operatorname{Re} \mathrm{V}(P) = \omega(P),
\end{aligned}$$

because $\|x^*\| \leq 1$, $\|x\| \leq 1$ and $\frac{x^*}{\|x^*\|}\left(\frac{x}{\|x\|}\right) = 1$. Then $\operatorname{daug}_p(Z) - \varepsilon \leq \omega(Q) - \varepsilon \leq \omega(P)$. Hence $\operatorname{daug}_p(Z) \leq \omega(P)$ for all $P \in \mathcal{P}_K(X;X)$ with $\|P\| = 1$. Thus $\operatorname{daug}_p(Z) \leq \operatorname{daug}_p(X)$.

For any $\mu \in \Lambda$ we have

$$\left[\bigoplus_{\lambda\in\Lambda} X_\lambda\right]_{\ell_1} = \left[\bigoplus_{\lambda\neq\mu} X_\lambda\right]_{\ell_1} \oplus_1 X_\mu,$$

so

$$\operatorname{daug}_p\left(\left[\bigoplus_{\lambda\in\Lambda} X_\lambda\right]_{\ell_1}\right) \leq \operatorname{daug}_p\left(X_\mu\right).$$

Therefore

$$\operatorname{daug}_p\left(\left[\bigoplus_{\lambda\in\Lambda} X_\lambda\right]_{\ell_1}\right) \leq \inf\left\{\operatorname{daug}_p(X_\lambda) : \lambda\in\Lambda\right\}.$$

The argument for the c_0-sum and ℓ_∞-sum is the same. □

Using the ideas of [6, Proposition 2.3] we can prove the following result. The statement for the ℓ_1-case remains open.

Proposition 2.3. *Let $(X_\lambda)_{\lambda\in\Lambda}$ be a family of infinite-dimensional complex Banach spaces. Then*

(i) $\operatorname{daug}_p\left(\left[\bigoplus_{\lambda\in\Lambda} X_\lambda\right]_{c_0}\right) \geq \inf\left\{\operatorname{daug}_p(X_\lambda) : \lambda\in\Lambda\right\}$;

(ii) $\operatorname{daug}_p\left(\left[\bigoplus_{\lambda\in\Lambda} X_\lambda\right]_{\ell_\infty}\right) \geq \inf\left\{\operatorname{daug}_p(X_\lambda) : \lambda\in\Lambda\right\}$.

Proof. If $\inf\left\{\operatorname{daug}_p(X_\lambda) : \lambda\in\Lambda\right\} = 0$, there is nothing to show. Let us suppose that $\inf\left\{\operatorname{daug}_p(X_\lambda) : \lambda\in\Lambda\right\} > 0$. Let $X = \left[\bigoplus_{\lambda\in\Lambda} X_\lambda\right]_{\ell_\infty}$ and take $P \in \mathcal{P}_K(X;X)$ with $\|P\| = 1$. We can see P as a family $(P_\lambda)_{\lambda\in\Lambda}$, where $P_\lambda \in \mathcal{P}_K(X;X_\lambda)$. Then

$$\|P\| = \sup_{x\in B_X}\|P(x)\| = \sup_{x\in B_X}\sup_{\lambda\in\Lambda}\|P_\lambda(x)\| = \sup_{\lambda\in\Lambda}\sup_{x\in B_X}\|P_\lambda(x)\| = \sup_{\lambda\in\Lambda}\|P_\lambda\|.$$

So, given $0 < \varepsilon < \inf\left\{\operatorname{daug}_p(X_\lambda) : \lambda\in\Lambda\right\}$, there exists $\mu\in\Lambda$ such that $\|P_\mu\| > 1-\varepsilon$. Write $X = X_\mu \oplus_\infty Y$, where $Y = \left[\bigoplus_{\lambda\neq\mu} X_\lambda\right]_{\ell_\infty}$. Let $(x_0, y_0) \in B_X$ be such that $x_0 \in X_\mu$, $y_0 \in Y$ and

$$\|P_\mu(x_0, y_0)\| > 1-\varepsilon. \tag{2}$$

We may suppose that $\|x_0\| = 1$. Indeed, fix $x_1 \in S_X$ such that $\|x_0\|x_1 = x_0$ and fix $x_\mu^* \in S_{X_\mu^*}$ such that

$$|x_\mu^*(P_\mu(x_0, y_0))| > 1-\varepsilon.$$

Since the function

$$z \longmapsto x_\mu^*\left(P_\mu\big(zx_1, y_0\big)\right)$$

is holomorphic, the maximum modulus theorem ensures the existence of $z_0 \in \mathbb{T}$ such that

$$1-\varepsilon < |x_\mu^*(P_\mu(x_0, y_0))| = |x_\mu^*(P_\mu(\|x_0\|x_1, y_0))| \leq |x_\mu^*(P_\mu(z_0x_1, y_0))|,$$

that is,

$$\|P_\mu(z_0x_1, y_0)\| \geq 1 - \varepsilon,$$

where $\|z_0x_1\| = |z_0|\|x_1\| = 1$. Thus, replacing x_0 by z_0x_1, we obtain the inequality (2). For simplicity, let us consider $\|x_0\| = 1$. Now, let $x_0^* \in S_{X^*}$ be such that $x_0^*(x_0) = 1$. Consider the polynomial $Q : X_\mu \to X_\mu$ defined by

$$Q(u) = P_\mu(u, x_0^*(u)y_0),$$

which is compact and satisfies

$$1 = \|P\| \geq \|P_\mu\| \geq \|Q\| \geq \|Q(x_0)\| = \|P_\mu(x_0, y_0)\| > 1 - \varepsilon.$$

Thus there exists $(u_0, u_0^*) \in \Pi(X_\mu)$ such that

$$\operatorname{Re} u_0^* \left(\frac{Q}{\|Q\|}(u_0) \right) > \omega \left(\frac{Q}{\|Q\|} \right) - \varepsilon \geq \operatorname{daug}_p(X_\mu) - \varepsilon,$$

which implies

$$\operatorname{Re} u_0^* \left(Q(u_0)\right) > (\operatorname{daug}_p(X_\mu) - \varepsilon)\|Q\| > (\operatorname{daug}_p(X_\mu) - \varepsilon)(1 - \varepsilon).$$

Let $x = (u_0, x_0^*(u_0)y_0) \in S_X$ and $x^* = (u_0^*, 0) \in S_{X^*}$. Hence $(x, x^*) \in \Pi(X)$ and

$$\begin{aligned}\omega(P) &\geq \operatorname{Re} x^*(P(x)) = \operatorname{Re} u_0^*(P_\mu(u_0, x_0^*(u_0)y_0)) = \operatorname{Re} u_0^*(Q(u_0)) \\ &> (\operatorname{daug}_p(X_\mu) - \varepsilon)(1 - \varepsilon) \geq \left(\inf_{\lambda \in \Lambda} \operatorname{daug}_p(X_\lambda) - \varepsilon \right)(1 - \varepsilon).\end{aligned}$$

Then

$$\omega(P) \geq \inf_{\lambda \in \Lambda} \operatorname{daug}_p(X_\lambda)$$

for all $P \in \mathcal{P}_K(X; X)$ with $\|P\| = 1$. Therefore

$$\operatorname{daug}_p \left(\left[\bigoplus_{\lambda \in \Lambda} X_\lambda \right]_{\ell_\infty} \right) \geq \inf \left\{ \operatorname{daug}_p(X_\lambda) : \lambda \in \Lambda \right\}.$$

The argument for the c_0-sum is the same. □

These stability properties allow us to prove characterizations of the polynomial Daugavetian index for vector-valued essentially bounded function spaces and continuous vector-valued function spaces. First, let us fix some notation. Given a compact Hausdorff space K, we denote by $C(K, X)$ the Banach space of all continuous functions from K into X, endowed with the supremum norm. For a σ-finite measure space (Ω, Σ, μ), we denote by $L_\infty(\mu, X)$ the Banach space of all (equivalence classes of) essentially bounded Bochner-measurable functions from Ω into X with the essential supremum norm. Also, given a positive measure space (Ω, Σ, μ), we denote by $L_1(\mu, X)$ the Banach space of all (equivalence classes of) Bochner-integrable functions from Ω into X with the norm

$$\|f\| = \int_\Omega \|f(t)\| d\mu(t).$$

The proofs of the following three results are based on the proofs of [11, Proposition 3.1, 3.3, and 3.4].

Proposition 2.4. *Let X be a complex Banach space and let K be a compact Hausdorff space. Then*

$$\operatorname{daug}_p(C(K,X)) = \max\{\operatorname{daug}_p(C(K)), \operatorname{daug}_p(X)\}.$$

Proof. First, we will prove that $\operatorname{daug}_p(C(K,X)) \geq \operatorname{daug}_p(X)$. Given $P \in \mathcal{P}_K(C(K,X); C(K,X))$, we need to prove that

$$\|Id + P\| \geq 1 + \operatorname{daug}_p(X)\|P\|.$$

For every $\varepsilon > 0$, there exist $f_0 \in S_{C(K,X)}$ and $t_0 \in K$ such that

$$\|P(f_0)(t_0)\| > \|P\| - \frac{\varepsilon}{2}. \tag{3}$$

Since P is continuous at f_0, there exists $\delta > 0$ such that

$$\|P(f_0) - P(g)\| < \frac{\varepsilon}{2} \text{ if } \|f_0 - g\| < \delta. \tag{4}$$

Consider the set $A = \{t \in K : \|f_0(t) - f_0(t_0)\| \geq \delta\}$. Observe that A is closed and $t_0 \notin A$. Thus, by Urysohn's lemma, we may find a continuous function $\varphi : K \to [0,1]$ such that $\varphi(t_0) = 1$ and $\varphi(A) = \{0\}$. Fix $x_0 \in S_X$ such that $f_0(t_0) = \|f_0(t_0)\|x_0$ and define $\Psi : \mathbb{C} \to C(K,X)$ by

$$\Psi(z) = (1-\varphi)f_0 + \varphi x_0 z.$$

Notice that $\Psi(\|f_0(t_0)\|)(t) - f_0(t) = (1-\varphi(t))f_0(t) + \varphi(t)f_0(t_0) - f_0(t) = \varphi(t)(f_0(t_0) - f_0(t))$. Since $\varphi(A) = \{0\}$, we have

$$\|\Psi(\|f_0(t_0)\|) - f_0\| = \sup_{t\in K} \varphi(t)\|f_0(t_0) - f_0(t)\| < \delta.$$

By (4), we obtain

$$\big\|P\big(\Psi(\|f_0(t_0)\|)\big) - P(f_0)\big\| < \frac{\varepsilon}{2}|$$

that implies

$$\big\|P\big(\Psi(\|f_0(t_0)\|)\big)(t_0) - P(f_0)(t_0)\big\| < \frac{\varepsilon}{2}.$$

It follows from (3) that

$$\big\|P\big(\Psi(\|f_0(t_0)\|)\big)(t_0)\big\| > \|P(f_0)(t_0)\| - \frac{\varepsilon}{2} > \|P\| - \varepsilon.$$

Then, by the Hahn-Banach theorem, there exists $x_0^* \in S_{X^*}$ such that

$$x_0^*\left(\big[P\big(\Psi(\|f_0(t_0)\|)\big)\big](t_0)\right) > \|P\| - \varepsilon.$$

Since the function

$$z \longmapsto x_0^*\left(\big[P\big(\Psi(z)\big)\big](t_0)\right)$$

is holomorphic, the maximum modulus theorem ensures the existence of $z_0 \in \mathbb{T}$ such that

$$\begin{aligned}\big\|P\big(\Psi(z_0)\big)(t_0)\big\| &\geq \left|x_0^*\left(\big[P\big(\Psi(z_0)\big)\big](t_0)\right)\right| \\ &\geq x_0^*\left(\big[P\big(\Psi(\|f_0(t_0)\|)\big)\big](t_0)\right) > \|P\| - \varepsilon.\end{aligned}$$

Take $x_1 = z_0 x_0 \in S_X$, fix $x_1^* \in S_{X^*}$ such that $x_1^*(x_1) = 1$, and define $\Phi : X \to C(K, X)$ by

$$\Phi(x) = x_1^*(x)(1 - \varphi) f_0 + \varphi x.$$

Notice that $\|\Phi(x)\| \leq 1$ for all $x \in B_X$ and that $\Phi(x_1) = \Psi(z_0)$. Thus,

$$\big\|P\big(\Phi(x_1)\big)(t_0)\big\| > \|P\| - \varepsilon.$$

Consider the polynomial $Q : X \to X$ defined by

$$Q(x) = \big[P\big(\Phi(x)\big)\big](t_0).$$

Observe that $Q \in \mathcal{P}_K(X; X)$ and satisfies

$$\|Q\| \geq \|Q(x_1)\| = \big\|\big[P\big(\Phi(x_1)\big)\big](t_0)\big\| > \|P\| - \varepsilon.$$

Thus,

$$\|Id + Q\| \geq 1 + \mathrm{daug}_p(X)\|Q\| > 1 + \mathrm{daug}_p(X)(\|P\| - \varepsilon).$$

Let $x_2 \in B_X$ be such that

$$\|x_2 + Q(x_2)\| > 1 + \mathrm{daug}_p(X)(\|P\| - \varepsilon)$$

and define $g = \Phi(x_2) \in C(K, X)$. So, $\|g\| \leq 1$ and

$$\begin{aligned}
\|Id + P\| &\geq \|g + P(g)\| \geq \|g(t_0) + P(g)(t_0)\| \\
&\geq \|x_1^*(x_2)(1 - \varphi(t_0)) f(t_0) + \varphi(t_0) x_2 + Q(x_2)\| \\
&= \|x_2 + Q(x_2)\| > 1 + \mathrm{daug}_p(X)(\|P\| - \varepsilon).
\end{aligned}$$

Letting $\varepsilon \to 0$, we obtain

$$\|Id + P\| \geq 1 + \mathrm{daug}_p(X)\|P\|.$$

Therefore, $\mathrm{daug}_p(C(K, X)) \geq \mathrm{daug}_p(X)$.

Now, suppose that K is perfect. In this case, [5, Corollary 2.5] ensures that $C(K, X)$ and $C(K)$ have the PDP, that is,

$$\mathrm{daug}_p(C(K, X)) = \mathrm{daug}_p(C(K)) = 1.$$

Then

$$\mathrm{daug}_p(C(K, X)) = \max\{\mathrm{daug}_p(C(K)), \mathrm{daug}_p(X)\}.$$

Finally, suppose that K has an isolated point. Then $\mathrm{daug}_p(C(K)) = 0$ and $C(K, X) = X \oplus_\infty Z$ for some appropriate Banach space Z. Hence, by Proposition 2.2 we have that $\mathrm{daug}_p(C(K, X)) \leq \mathrm{daug}_p(X)$. Therefore

$$\mathrm{daug}_p(C(K, X)) = \max\{\mathrm{daug}_p(C(K)), \mathrm{daug}_p(X)\}.$$

□

Proposition 2.5. *Let (Ω, Σ, μ) be a σ-finite measure space and let X be a complex Banach space. Then*

$$\mathrm{daug}_p(L_\infty(\mu, X)) = \max\{\mathrm{daug}_p(L_\infty(\mu)), \mathrm{daug}_p(X)\}.$$

Proof. If μ is atomless, then [6, Theorem 6.5] ensures that $L_\infty(\mu, X)$ and $L_\infty(\mu)$ have the PDP. Thus

$$\mathrm{daug}_p(L_\infty(\mu, X)) = \mathrm{daug}_p(L_\infty(\mu)) = 1.$$

Therefore

$$\mathrm{daug}_p(L_\infty(\mu, X)) = \max\{\mathrm{daug}_p(L_\infty(\mu)), \mathrm{daug}_p(X)\}.$$

Now suppose that μ has an atom. Hence, there exist a non-empty set I and an atomless σ-finite measure ν such that

$$L_\infty(\mu, X) = L_\infty(\nu, X) \oplus_\infty \left[\bigoplus_{i\in I} X\right]_{\ell_\infty}.$$

Thus $\mathrm{daug}_p(L_\infty(\nu, X)) = 1$ and $\mathrm{daug}_p(L_\infty(\mu, X)) = \mathrm{daug}_p(X)$, by Propositions 2.2 and 2.3. Since $\mathrm{daug}_p(L_\infty(\mu)) = 0$, we have

$$\mathrm{daug}_p(L_\infty(\mu, X)) = \max\{\mathrm{daug}_p(L_\infty(\mu)), \mathrm{daug}_p(X)\}.$$

□

Proposition 2.6. *Let (Ω, Σ, μ) be a positive measure space and let X be a complex Banach space. Then*

$$\mathrm{daug}_p(L_1(\mu, X)) \leq \max\{\mathrm{daug}_p(L_1(\mu)), \mathrm{daug}_p(X)\}.$$

Proof. By [10, Theorem 3.3], we know that if μ is an atomless measure then $L_1(\mu, X)$ and $L_1(\mu)$ have the PDP and, in particular,

$$\mathrm{daug}_p(L_1(\mu, X)) = \mathrm{daug}_p(L_1(\mu)) = 1.$$

On the other hand, if μ is a measure with an atom, then there exist a non-empty set I and an atomless positive measure ν such that

$$L_1(\mu, X) = L_1(\nu, X) \oplus_1 \left[\bigoplus_{i\in I} X\right]_{\ell_1}.$$

In this case, by Proposition 2.2 we have that

$$\mathrm{daug}_p(L_1(\mu, X)) \leq \mathrm{daug}_p(X) = \max\{\mathrm{daug}_p(L_1(\mu)), \mathrm{daug}_p(X)\},$$

since $\mathrm{daug}_p(L_1(\mu)) = 0$. □

The reverse inequality in Proposition 2.6 remains open.

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Received: 4 July 2018