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The values of the Riemann zeta-function on generalized arithmetic progressions

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Abstract. We study the mean of the values of the zeta-function on a generalized arithmetic progression on the critical line.

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1. Introduction and statement of the main results. Neither the horizontal distribution of the nontrivial (non-real) zeros of the Riemann zeta-function $\zeta(s)$ is well understood nor their vertical distribution. In 1942, Albert Edward Ingham [1] observed that the truth of the Mertens conjecture in the form $\sum_{n\leq x}\mu(n)=O(x^{1/2})$ with the Möbius μ-function would not only imply that all nontrivial zeros lie on the critical line $1/2+i\mathbb{R}$ (the Riemann hypothesis) and are simple (the simplicity hypothesis)—consequences well known at his time—but that in addition "*the imaginary parts of the zeros above the real axis must be linearly dependent (with rational integral multipliers)*" too. The stronger original form of the Mertens conjecture, $|\sum_{n\leq x}\mu(n)|\leq x^{1/2}$, has been disproved by Andrew Odlyzko and Herman te Riele [5] in 1985; on the other hand, no linear relation for zeros has been found so far. A slightly weaker open problem than proving linear independence is to show that *there are no three or more nontrivial zeros in arithmetic progression.* In that direction Putnam [7,8] showed that there is no infinite arithmetic progression of nontrivial zeros. A different approach was found by Lapidus and van Frankenhuijsen [2]. In a later paper [11] van Frankenhuijsen obtained an explicit bound for the length of any hypothetical arithmetic progression, in particular proving $\zeta(\frac{1}{2}+im\delta)=0$ for $1\leq m<M$ with positive real $\delta>44{,}000$ is possible only for $M<13\delta$ by use of the explicit formula. More recently, Elias Wegert and the second author [9] as well as Martin and Ng [4] found another method using discrete moments in combination with estimates for exponential sums. Very recently, Li and Radziwiłł [3] showed by a similar approach that at least one third of the values on arithmetic progression is different from zero; moreover, they proved that among these values there are infinitely many extremely large (resp. small) in absolute value. Here we continue these investigations (and our previous related work [6]) by studying the values of the zeta-function on generalized arithmetic progressions.

In [6] it was observed that the mean of the values of the zeta-function taken on a vertical arithmetic progression $s_0 + im\delta$ with $m = 0, 1, 2, \ldots, M$ inside the critical strip exists and is equal to

$$\lim_{M\to\infty} \frac{1}{M} \sum_{0\leq m<M} \zeta(s_0 + im\delta) = \begin{cases} (1-\ell^{-s_0})^{-1} & \text{if} \quad \delta = \frac{2\pi q}{\log \ell}, q\in\mathbb{N}, 2\leq \ell\in\mathbb{N}, \\ 1 & \text{otherwise.} \end{cases} \tag{1}$$

Here s_0 may be any complex number with real part in $(0, 1)$ and if δ is of the form $\delta = \frac{2\pi q}{\log \ell}$ with $2 \leq \ell \in \mathbb{N}$ and $q \in \mathbb{N}$, then q is supposed to be the smallest integer for which such a value ℓ exists. This condition on q is necessary to determine ℓ uniquely since $\frac{2\pi q}{\log \ell} = \frac{2\pi q b}{\log \ell^b}$ for any $b \in \mathbb{N}$. Any such δ with least possible q is called a *resonance value of order* ℓ, and all other δ are said to be *generic*.

In this note we consider the distribution of values of the zeta-function on generalized arithmetic progressions of the form $\frac{1}{2} + i(\gamma + \delta_1 m_1 + \ldots + \delta_r m_r)$, where γ is a real number, $\delta_1, \ldots, \delta_r$ are positive real numbers, and $m_1, \ldots, m_r$ are positive integers. It appears that the mean of the values of the zeta-function exists also on generalized arithmetic progressions. The case of resonance here is when all δ_j's oscillate with the same logarithm ℓ in the denominators. More precisely, we say that $\delta_1, \ldots, \delta_r$ with some $r \geq 2$ are *in resonance* if there exists some $\ell \in \mathbb{N}$ such that one of the δ_j's is a resonance value of order ℓ and all other δ_j satisfy $\delta_j \log \ell \in 2\pi\mathbb{Z}$ (which implies $\delta_j = \frac{2\pi q_j}{\log \ell}$ with positive integers q_j for $j = 1, \ldots, r$); notice that this does not imply that all δ_j are resonance values. It is easy to see that in the case of existence, the value of ℓ is uniquely determined. Otherwise, when there is no such integer ℓ, the values $\delta_1, \ldots, \delta_r$ are said to be *not in resonance*.

For convenience we introduce some simplifying notation: we denote the inner product of $\delta = (\delta_1, \ldots, \delta_r)$ and $\mathbf{m} = (m_1, \ldots, m_r)$ by $\delta \cdot \mathbf{m} = \sum_{j=1}^r \delta_j m_j$. Moreover, $\mathbf{m} \leq \mathbf{M}$ indicates that a summation is taken over all positive integers $m_1, \ldots, m_r$ satisfying $m_j \leq M_j$ for $j = 1, \ldots, r$, where $\mathbf{M} = (M_1, \ldots, M_r)$. Finally, we define $\Pi := M_1 \cdot \ldots \cdot M_r$.

Theorem 1. *Let $M_1, \ldots, M_r$ be quantities tending to infinity. Then,*

$$\frac{1}{\Pi} \sum_{\mathbf{m}\leq\mathbf{M}} \zeta(\tfrac{1}{2} + i(\gamma + \delta \cdot \mathbf{m})) = c_\delta + o(1),$$

where c_δ is a non-zero constant defined by

$$c_\delta = \begin{cases} (1 - \ell^{-(\frac{1}{2}+i\gamma)})^{-1} & \textit{if all} \quad \delta_j = \frac{2\pi q_j}{\log \ell} \quad \textit{are in resonance} \\ 1 & \textit{otherwise.} \end{cases}$$

If the δ_j's are in resonance, the mean value is of similar form as in (1); otherwise the mean equals 1. Here and everywhere the implicit constants may depend on $\gamma, \delta_1, \ldots, \delta_r$.

In the following section we shall give a simplified proof of Formula (1) for s_0 from the critical line, which may also be considered as the induction hypothesis for the proof of Theorem 1 by induction in the final section.

2. The case of an arithmetic progression. We begin with (a very simple form of) the approximate functional equation, namely

$$\zeta(s) = \sum_{n\le x} n^{-s} + \frac{x^{1-s}}{s-1} + O(x^{-\sigma}),$$

valid uniformly for $\sigma \ge \sigma_0 > 0$ and $|t| \ll x$, where we write $s = \sigma + it$ (see [10, §4.11]); writing here and elsewhere the range of summation as $n \le x$ indicates that we sum up overall *positive* integers $n \le x$. This yields for $x \asymp M$ (meaning that $x \ll M \ll x$) that

$$\zeta(\tfrac12 + it) = \sum_{n\le x} n^{-\frac12 - it} + \frac{x^{\frac12 - it}}{-\frac12 + it} + O(M^{-\frac12}),$$

and

$$\sum_{m\le M} \zeta(\tfrac12 + i(\gamma+\delta m))$$
$$= \sum_{m\le M}\sum_{n\le x} n^{-\frac12 - i(\gamma+\delta m)} + \sum_{m\le M} \frac{x^{\frac12 - i(\gamma+\delta m)}}{-\frac12 + i(\gamma+\delta m)} + O(M^{\frac12}).$$

The second sum on the right-hand side can be estimated as

$$\sum_{m\le M} \frac{x^{\frac12 - i(\gamma+\delta m)}}{-\frac12 + i(\gamma+\delta m)} \ll x^{\frac12} \sum_{m\le M} \frac{1}{m} \ll M^{\frac12}\log M.$$

And the double sum on the right can be rewritten as

$$M + \sum_{1<n\le x} n^{-\frac12 - i\gamma} \sum_{m\le M} \exp(-i\delta m \log n).$$

Using the classical bound

$$S_M(\alpha) := \sum_{m\le M} \exp(-im\alpha) \ll \min\{M, \|\tfrac{\alpha}{2\pi}\|^{-1}\}, \tag{2}$$

where, as usual, $\|\Delta\|$ denotes the distance of Δ to the nearest integer, we have

$$\sum_{1<n\le x} n^{-\frac12 - i\gamma} \sum_{m\le M} \exp(-i\delta m \log n)$$
$$= \left\{ \sum_{\substack{1<n\le x\\ \delta\log n \in 2\pi\mathbb{Z}}} + \sum_{\substack{1<n\le x\\ \delta\log n \notin 2\pi\mathbb{Z}}} \right\} n^{-\frac12 - i\gamma} S_M(\delta \log n) = S_1 + S_2,$$

say. If $\delta\log n\in 2\pi\mathbb{Z}$, then $S_M(\delta\log n)=M$; hence

$$S_1=M\sum_{\substack{1<n\le x\\ \delta\log n\in 2\pi\mathbb{Z}}} n^{-\frac12-i\gamma}.$$

The condition $\delta\log n\in 2\pi\mathbb{Z}$ implies that $\delta=\frac{2\pi q}{\log\ell}$ is a resonance value of some order ℓ and n has to be a power of ℓ. Writing $n=\ell^b$, this leads after a short calculation with geometric series to

$$S_1=M\sum_{1\le b\le\frac{\log x}{\log\ell}}\ell^{-b(\frac12+i\gamma)}=M\left((1-\ell^{-(\frac12+i\gamma)})^{-1}-1\right)+O(Mx^{-\frac12}).$$

In view of $x\asymp M$ the error term is $O(M^{\frac12})=o(M)$.

In order to treat S_2, let $\mathcal{N}$ be the minimum of $M^{\frac13}$ and the least positive integer N for which the inequalities

$$\left\|\frac{\delta\log n}{2\pi}\right\|>M^{-\frac12}\qquad\text{for}\quad 1<n<N$$

do not hold. Hence, in combination with (2),

$$\sum_{\substack{1<n<\mathcal{N}\\ \delta\log n\notin 2\pi\mathbb{Z}}} n^{-\frac12-i\gamma}S_M(\delta\log n)\ll\mathcal{N}M^{\frac12}=O(M^{\frac56})=o(M).$$

It remains to estimate

$$\sum_{\substack{\mathcal{N}\le n\le x\\ (\delta\log n\notin 2\pi\mathbb{Z})}} n^{-\frac12-i\gamma}S_M(\delta\log n),$$

where the condition on $\delta\log n$ may be dropped. Of course, this sum can be empty (if $\mathcal{N}>x$). We shall use an elementary (nevertheless tricky) estimate due to Martin and Ng [4, Proposition 4.2], which implies here

$$\sum_{\mathcal{N}\le n\le x} n^{-\frac12}\min\left\{M,\left\|\frac{\delta\log n}{2\pi}\right\|^{-1}\right\}\ll M\mathcal{N}^{-\frac12}+x^{\frac12}\log x.$$

Since $x\asymp M$ and $\mathcal{N}$ tends to infinity as $M\to\infty$, it follows that $S_2=o(M)$.

Combining all estimates yields the formula from the theorem for the case $r=1$ as well as Formula (1) for s_0 from the critical line; it is not difficult to extend the above reasoning to get (1) in its full generality.

3. The general case. The general case is proved by an induction argument. Without loss of generality, we may assume that δ_r is a resonance value of order ℓ if $\delta_1,\ldots,\delta_r$ are in resonance. We write

$$\sum_{\mathbf{m}\le\mathbf{M}}\zeta(\tfrac12+i(\gamma+\delta\cdot\mathbf{m}))=\sum_{m_1\le M_1}\cdots\sum_{m_{r-1}\le M_{r-1}}\sum_{m_r\le M_r}\zeta(\tfrac12+i(\gamma_r+\delta_r m_r)),$$

where $\gamma_r:=\gamma+\delta_1m_1+\ldots+\delta_{r-1}m_{r-1}$. In view of (1) with s_0 from the critical line, resp. the result proven in the previous section (that is, the case $r=1$ of the statement of the theorem),

$$\sum_{m_r\le M_r}\zeta(\tfrac12+i(\gamma_r+\delta_r m_r))=(c_r+o(1))M_r,$$

where

$$c_r=\begin{cases}(1-\ell^{-\frac12-i\gamma_r})^{-1} & \text{if}\quad \delta_r=\frac{2\pi q_r}{\log\ell},q_r\in\mathbb{N},2\le\ell\in\mathbb{N},\\ 1 & \text{otherwise.}\end{cases}$$

This leads to

$$\begin{aligned}&\sum_{\mathbf{m}\le\mathbf{M}}\zeta(\tfrac12+i(\gamma+\delta\cdot\mathbf{m}))\\ &=\sum_{m_1\le M_1}\dots\sum_{m_{r-2}\le M_{r-2}}\sum_{m_{r-1}\le M_{r-1}}(c_r+o(1))\cdot M_r.\end{aligned}\tag{3}$$

If $c_r=1$, then the right-hand side equals $(1+o(1))\Pi$ (even if there are resonance values amongst the other δ_j's). Otherwise, when $\delta_r=\frac{2\pi q_r}{\log\ell}$ is a resonance value, then the most inner sum in (3) can be rewritten as

$$\begin{aligned}&\sum_{m_{r-1}\le M_{r-1}}(1-\ell^{-\frac12-i\gamma_r})^{-1}\\ &=\sum_{m_{r-1}\le M_{r-1}}\left(1+\sum_{n\ge1}\ell^{-n(\frac12+i\gamma_{r-1}+i\delta_{r-1}m_{r-1})}\right)\\ &=M_{r-1}+\left\{\sum_{1\le n<N}+\sum_{n\ge N}\right\}\ell^{-n(\frac12+i\gamma_{r-1})}\sum_{m_{r-1}\le M_{r-1}}\exp(-im_{r-1}n\delta_{r-1}\log\ell).\end{aligned}$$

If $\delta_{r-1}\log\ell\in2\pi\mathbb{Z}$, then the most inner sum on the right-hand side equals M_{r-1} and we find

$$\sum_{m_{r-1}\le M_{r-1}}(1-\ell^{-\frac12-i\gamma_r})^{-1}=(1-\ell^{-\frac12-i\gamma_{r-1}})^{-1}M_{r-1};$$

substituting this in (3) yields

$$\begin{aligned}&\sum_{\mathbf{m}\le\mathbf{M}}\zeta(\tfrac12+i(\gamma+\delta\cdot\mathbf{m}))\\ &\quad=\sum_{m_1\le M_1}\dots\sum_{m_{r-2}\le M_{r-2}}(c_{r-1}+o(1))\cdot M_rM_{r-1},\end{aligned}\tag{4}$$

where c_{r-1} equals $(1-\ell^{-\frac12-i\gamma_{r-1}})^{-1}$. Notice that in this case δ_{r-1} and δ_r are in resonance of order ℓ and we may proceed with induction.

However, if $\delta_{r-1}\log\ell\notin2\pi\mathbb{Z}$, then we may argue similar as in the previous section. Let $\mathcal{N}$ be the least positive integer N for which the inequalities

$$\left\|\frac{n\delta_{r-1}\log\ell}{2\pi}\right\|>M_{r-1}^{-\frac12}\qquad\text{for}\quad 1<n<N$$

do not hold. In view of the trivial bound (2), we have, for $n < \mathcal{N}$,

$$\sum_{m_{r-1}\le M_{r-1}} \exp(-im_{r-1}n\delta_{r-1}\log\ell)$$
$$= S_{M_{r-1}}(n\delta_{r-1}\log\ell) \ll \left\|\frac{n\delta_{r-1}\log\ell}{2\pi}\right\|^{-1} \ll M_{r-1}^{\frac{1}{2}}$$

and thus

$$\sum_{1\le n<\mathcal{N}} \ell^{-n(\frac{1}{2}+i\gamma_{r-1})} \sum_{m_{r-1}\le M_{r-1}} \exp(-im_{r-1}n\delta_{r-1}\log\ell) \ll M_{r-1}^{\frac{1}{2}}.$$

Moreover,

$$\sum_{n\ge\mathcal{N}} \ell^{-n(\frac{1}{2}+i\gamma_{r-1})} \sum_{m_{r-1}\le M_{r-1}} \exp(-im_{r-1}n\delta_{r-1}\log\ell) \ll \ell^{-\frac{\mathcal{N}}{2}} M_{r-1}.$$

Since $\mathcal{N}$ tends to infinity as $M_{r-1}\to\infty$, we get

$$\sum_{m_{r-1}\le M_{r-1}} (1-\ell^{-\frac{1}{2}-i\gamma_r})^{-1} = (1+o(1))M_{r-1}.$$

Substituting this in (3) leads to (4) and we may proceed with the induction.

Thus, if all δ_j are in resonance, we finally arrive at

$$\sum_{\mathbf{m}\le\mathbf{M}} \zeta(\tfrac{1}{2}+i(\gamma+\delta\cdot\mathbf{m}))$$
$$= \sum_{m_1\le M_1} (c_2+o(1))\cdot M_r M_{r-1}\cdot\ldots\cdot M_2 = (c_1+o(1))\Pi.$$

Here $c_2 = (1-\ell^{-\frac{1}{2}-i\gamma_2})^{-1}$ and $c_1 = (1-\ell^{-(\frac{1}{2}+i\gamma)})^{-1}$ which equals c_δ in the case that all δ_j are on resonance of order ℓ. This finishes the proof of the theorem.

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