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Symmetric Hochschild extension algebras and normalized 2-cocycles

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Abstract. In this paper, for finite dimensional, basic, and connected algebras over a field, we give a sufficient condition, related to 2-cocycles, for Hochschild extension algebras to be symmetric. Moreover, we define the normalized 2-cocycle associated with a complete set of primitive orthogonal idempotents, and we show that for every 2-cocycle there exists a normalized 2-cocycle such that their cohomology classes coincide.

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1. Introduction. Hochschild extensions of algebras give many self-injective algebras. For a finite dimensional algebra A over a field K, the trivial extension algebra $T(A) := A \ltimes \mathrm{Hom}_K(A, K)$ of a K-algebra A by the standard duality module $\mathrm{Hom}_K(A, K)$ is very important in the representation theory of self-injective algebras. This is also one of the Hochschild extension algebras of A. In particular, trivial extension algebras correspond to the zero cocycle in the second Hochschild cohomology groups $\mathrm{H}^2(A, \mathrm{Hom}_K(A, K))$. It is well known that the trivial extension algebra $T(A)$ of a K-algebra A is symmetric by the symmetric regular K-linear map $\mu : T(A) \to K, \mu(a, f) = f(1)$, where $a \in A$ and $f \in \mathrm{Hom}_K(A, K)$. However, it is known that Hochschild extension algebras by duality bimodules are always self-injective [5] but they are not symmetric in general [4].

This paper has two aims:

(1) We will give a sufficient condition, related to 2-cocycles, for Hochschild extension algebras to be symmetric.
(2) For any 2-cocycle α we define normalized 2-cocycles related to a complete set of primitive orthogonal idempotents and construct such a normalized

2-cocycle whose cohomology class coincides with the cohomology class of α.

In [4], Ohnuki, Takeda, and Yamagata gave a sufficient condition, related to 2-cocycles, for Hochschild extension algebras to be symmetric by giving a symmetric regular linear map. They already provided an example (Example 3.3) showing that their condition is not necessary. We give another sufficient condition (Theorem 3.1) which this example satisfies. We also give another proof of [4, Theorem 2.2]. More precisely, we show the following: If 2-cocycles satisfy the sufficient condition in [4], then the corresponding Hochschild extension algebras have a symmetric regular linear map. Then, this map is equal to the symmetric regular linear map μ of the trivial extension algebra as linear maps. In order to show that, we define a normalized 2-cocycle related to a complete set E of primitive orthogonal idempotents, and we call it E-normalized 2-cocycle. Such a normalized 2-cocycle has similar properties as E_0-normalized projective resolutions in [2], where E_0 is a subalgebra of an algebra A such that $A = E_0 \oplus \operatorname{rad} A$.

This paper is organized as follows: In Section 2, we recall the definition and the notation for the Hochschild extension algebras. In Section 3, we give a sufficient condition, related to 2-cocycles, for Hochschild extension algebras to be symmetric (Theorem 3.1), and we also give several examples. In Section 4, for an algebra A and a complete set E of primitive orthogonal idempotents in A, we define E-normalized 2-cocycles of A (Definition 4.1) and we show that for every 2-cocycle α there exists an E-normalized 2-cocycle $\overline{\alpha}$ such that the cohomology classes of α and $\overline{\alpha}$ coincide. By means of normalized 2-cocycles, we also give another proof of [4, Theorem 2.2]. Finally, we show that a complete set of primitive orthogonal idempotents of a Hochschild extension algebra by an E-normalized 2-cocycle is given by the formula of a complete set of primitive orthogonal idempotents of trivial extension algebras (Corollary 4.10).

Throughout this paper, we denote $\otimes_K$ and the n-fold tensor product of A by $\otimes$ and $A^{\otimes n}$, respectively, for the sake of simplicity.

2. Preliminaries. Let K be a field and A a finite dimensional K-algebra. In this section, by following [1] and [5], we recall the definition, the notation, and several properties of Hochschild extensions of A by a duality bimodule.

Let D be a duality between A-mod and A^{op}-mod. Then, there is an A-bimodule M such that $D \cong \operatorname{Hom}_A(-, M)$. In particular, $M \cong DA$ as A-bimodules. Such a module DA is called a duality module.

An extension of an algebra A is an epimorphism $\rho : T \to A$ of K-algebras. Throughout this paper, we assume that the kernel of ρ is isomorphic to a duality module DA as T-bimodule. When K is a commutative ring, an extension of a K-algebra A with kernel DA is called a Hochschild extension of A by the duality module DA if the extension is K-split. Then, T is called a Hochschild extension algebra of A by DA. In our situation, all extensions of algebras are Hochschild extensions.

The Hochschild extension algebra T is defined by a 2-cocycle. A 2-*cocycle* $\alpha : A \times A \to \mathrm{Hom}_K(A, DA)$ is a K-bilinear map with the 2-cocycle condition

$$(a, b, c)_\alpha := a\alpha(b, c) - \alpha(ab, c) + \alpha(a, bc) - \alpha(a, b)c = 0$$

for $a, b, c \in A$. The Hochschild extension algebra $T \cong A \oplus DA$ as K-modules and the multiplication is defined by

$$(a, f)(b, g) = (ab, ag + fb + \alpha(a, b))$$

for $a, b \in A$ and $f, g \in DA$. We denote such a Hochschild extension algebra T by $T_\alpha(A, DA)$. The identity of $T_\alpha(A, DA)$ has the form $(1, -\alpha(1, 1))$. In particular, the trivial extension of A by DA is the Hochschild extension $T_0(A, DA)$ of A by DA for zero-map.

Hochschild extension algebras of A are related to the second Hochschild cohomology $\mathrm{H}^2(A, DA)$ of A with coefficient in DA, which is the cohomology of the complex

$$\mathrm{Hom}_K(A, DA) \xrightarrow{\delta^1} \mathrm{Hom}_K(A^{\otimes 2}, DA) \xrightarrow{\delta^2} \mathrm{Hom}_K(A^{\otimes 3}, DA),$$

where δ^1 and δ^2 are given by

$$[\delta^1(f)](a \otimes b) = af(b) - f(ab) + f(a)b$$
$$[\delta^2(\alpha)](a \otimes b \otimes c) = a\alpha(b \otimes c) - \alpha(ab \otimes c) + \alpha(a \otimes bc) - \alpha(a \otimes b)c$$

for $a, b, c \in A$, $f \in \mathrm{Hom}_K(A, DA)$ and $\alpha \in \mathrm{Hom}_K(A^{\otimes 2}, DA)$. Hochschild extensions $(T) : 0 \to DA \to T \to A \to 0$ and $(T') : 0 \to DA \to T' \to A \to 0$ are called equivalent if there exists a homomorphism $\iota : T \to T'$ as K-algebras such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & DA & \longrightarrow & T & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle 1} & & \\ 0 & \longrightarrow & DA & \longrightarrow & T' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

In particular, if Hochschild extension algebras T, T' are equivalent, then $T \cong T'$ as K-algebras. It is well known that there exists a one-to-one correspondence between the set of all equivalent classes of Hochschild extensions of A by DA and $\mathrm{H}^2(A, DA)$ (cf. [3]).

The K-linear map $\alpha : A^{\otimes 2} \to DA$ which belongs to $\mathrm{Z}^2(A, DA)$ is induced by a 2-cocycle, so we also call the K-linear map α 2-cocycle if there is no risk of confusion. For $f \in \mathrm{Hom}_K(A, DA)$ we define a 2-cocycle $\delta(f)$ by

$$[\delta(f)](a, b) = af(b) - f(ab) + f(a)b.$$

Then for a 2-cocycle $\alpha : A \times A \to DA$, for any $f \in \mathrm{Hom}_K(A, DA)$ the Hochschild extension of A by DA for α and the one for $\alpha - \delta(f)$ are equivalent. In particular, their Hochschild extension algebras are isomorphic.

Let Q be a finite quiver and $A = KQ/I$, where I is an admissible ideal. We denote by Q_0 and Q_1 the set of all vertices in Q and the set of all arrows in Q, respectively. Let $Q_0 = \{1, 2, \ldots, n\}$ and e_i the primitive idempotent corresponding to $i \in Q_0$. Then it is well known that $\{e_i \mid i \in Q_0\}$ is a complete set of primitive orthogonal idempotents of A. For a nonzero element $a \in A$, with $a = e_i a e_j$ for some i, j, we denote e_i and e_j by $s(a)$ and $t(a)$, respectively. For a path p in KQ we denote by p again the image of p under the canonical map $KQ \to A$ if there is no risk of confusion.

The following theorem is convenience.

Theorem 2.1 ([4, Theorem 2.2]). *Let Q be a finite quiver, $A = KQ/I$ a bound quiver algebra, $DA = \mathrm{Hom}_K(A, K)$, and $\alpha : A \times A \to DA$ a 2-cocycle. If α satisfies $[\alpha(p, q)](t(q)) = [\alpha(q, p)](t(p))$ for all paths p, q which pq is a cycle in Q and $p, q \notin Q_0$, then the Hochschild extension algebra of A for α is symmetric.*

3. Symmetric Hochschild extension algebras. In this section, for connected bound quiver algebras over a field, we give a sufficient condition, related to 2-cocycles, for Hochschild extension algebras to be symmetric.

Let K be a field and $A = KQ/I$ a bound quiver algebra. The algebra A is called symmetric if A is isomorphic to $\mathrm{Hom}_K(A, K)$ as A-bimodules or, equivalently, there exists a K-bilinear map $\mu : A \to K$ such that the following holds:

(S1) μ is regular, that is, $\mu(Ax) \neq 0$ for any $x \in A$.
(S2) μ is symmetric, that is, $\mu(xy) = \mu(yx)$ for any $x, y \in A$.

In [5], it is shown that every Hochschild extension algebra T of A by the duality module DA is self-injective. In particular, the Nakayama permutation of T and the Nakayama permutation by ${}_A(DA)_A$ coincide. However, Hochschild extension algebras are not symmetric in general. It is shown that there is a Hochschild extension algebra which is symmetric if and only if $DA \cong \mathrm{Hom}_K(A, K)$ by [6, Proposition 2.2]. Thus, in this section, a duality module DA means a standard duality module $\mathrm{Hom}_K(A, K)$. In particular, Hochschild extension algebras are always weakly symmetric, that is, their Nakayama permutations are the identity.

In order to describe our assertion, we explain some notation. For a basis element $b \in \mathcal{B}$ of a K-algebra A, we denote the dual basis element of b by b^*. For a 2-cocycle $\alpha : A \times A \to DA$, we denote by η_α a K-bilinear map $A \times A \to DA$ given by $\eta_\alpha(x, y) = \alpha(x, y) - \alpha(y, x)$, where $x, y \in A$. Let $V_\alpha = \{a \in \mathrm{Z}(A) \mid f(a) = 0 \text{ for any } f \in \eta_\alpha(A \times A)\}$.

Theorem 3.1. *Let $A = KQ/I$ be a connected bound quiver algebra and $\alpha : A \times A \to DA$ a 2-cocycle. If there exists $x_0 \in V_\alpha$ such that $e_i^*(x_0) \neq 0$ for all $i (1 \leq i \leq n)$, then the Hochschild extension algebra $T_\alpha(A, DA)$ of A defined by α is symmetric.*

Proof. We define a K-linear map $\lambda : T_\alpha(A, DA) \to K$ by $\lambda(a, f) = f(x_0)$. First, we show that λ is symmetric. Since $x_0 \in V_\alpha$, for all $(a, f), (b, g) \in T_\alpha(A, DA)$, we have

$$
\begin{aligned}
&\lambda((a, f)(b, g) - (b, g)(a, f)) \\
&\quad = \lambda((ab - ba, ag - ga + fb - bf)) + \lambda(0, \alpha(a, b) - \alpha(b, a)) \\
&\quad = (ag - ga + fb - bf)(x_0) + (\alpha(a, b) - \alpha(b, a))(x_0) \\
&\quad = 0.
\end{aligned}
$$

Thus λ is symmetric.

Finally, we show that λ is regular. Let $\mathcal{B}$ be a basis of A whose elements are paths. Since $e_i^*(x_0) \neq 0$ for all $i (1 \leq i \leq n)$, there exists $k \in K$ such that $e_i^*(x_0) = k$ for all $i (1 \leq i \leq n)$. For every $(a, f) \in T_\alpha(A) \backslash \{(0, 0)\}$, we divide into the following two cases: $a = 0$ and $a \neq 0$.

If $a = 0$, then $f \neq 0$. Hence, there exists a minimal set $\mathcal{B}'(\subset \mathcal{B})$ such that the equation $f = \sum_{r \in \mathcal{B}'} k_r r^*$ holds for some $k_r \neq 0$. We can take a path of maximal length in $\mathcal{B}'$ without paths including a path in a minimal relation of A if there exists such a path. If there is no such a path, then for each $p \in \mathcal{B}'$ we set

$$l_p := \min \left\{ (\operatorname{length} p) - (\operatorname{length} q) \,\middle|\, \begin{array}{l} q \in \mathcal{B},\ q \text{ is a subpath of } p \text{ and there} \\ \text{exists a minimal relation } \sum_{l=1}^{t} k_l q_l = 0 \\ \text{such that } q_j = q \text{ for some } j \end{array} \right\}$$

and we can take a path $p \in \mathcal{B}'$ such that $l_p = \max \{ l_q \mid q \in \mathcal{B}' \}$. So, there exists $p \in \mathcal{B}'$ such that $pr^* = 0$ for any $r \in \mathcal{B}' \backslash \{p\}$. Therefore, $\lambda((p, 0)(0, f)) = \lambda((0, k_p s(p)^*) = k_p s(p)^*(x_0) \neq 0$.

If $a \neq 0$, then there exists $\mathcal{B}' \subset \mathcal{B}$ such that $a = \sum_{q \in \mathcal{B}'} k_q q$ for some $k_q \in K \backslash \{0\}$. Then, by taking a path of minimal length in $\mathcal{B}'$ without paths including a path in a minimal relation of A (if there is no such a path, then we take a path p with $l_p = \min \{ l_q \mid q \in \mathcal{B}' \}$), there exists a path p in $\mathcal{B}'$ such that $p^* q = 0$ for any $q \in \mathcal{B}' \backslash \{p\}$. Hence we have $\lambda((0, p^*)(a, f)) = \lambda((0, k_p p^* p)) = \lambda((0, k_p t(p)^*)) = k_p t(p)^*(x_0) \neq 0$.

Therefore, λ is regular. □

Remark 3.2. If $\alpha = 0$, then $T_0(A, DA)$ is trivial extension and $V_0 = \mathrm{Z}(A)$. In particular, $1 \in V_0$, so the K-linear map λ in this proof is a well-known symmetric regular K-linear map.

Example 3.3 ([4]). Let Q be a quiver with a vertice and three loops x, y, z. Let $A = KQ/R_Q^2$, where R_Q is the arrow ideal of KQ, $\mathcal{B} = \{1, x, y, z\}$ a basis of A and $\alpha : A \times A \to DA$ a 2-cocycle given by

$$\alpha(x, y) = 1^* - z^*,\ \alpha(y, z) = 1^* - x^*,\ \alpha(z, x) = 1^* - y^*,\ \alpha(a, b) = 0$$

for $(a, b) \in \mathcal{B} \times \mathcal{B} \backslash \{(x, y), (y, z), (z, x)\}$. Then, by direct computation, we have $V_\alpha = \langle 1 + x + y + z \rangle_K$. Since $1^*(1 + x + y + z) = 1$, the Hochschild extension algebra $T_\alpha(A, DA)$ of A by DA for α is symmetric by Theorem 3.1.

Moreover, $T_\alpha(A, DA) \cong KQ/I$, where I is an ideal of KQ generated by

$$x^2, y^2, z^2, xz, yx, zy, xyz - yzx, yzx - zxy.$$

4. Normalized 2-cocycles and their applications. Let A be a finite dimensional algebra over a field, E a complete set of primitive orthogonal idempotents of A, M an A-bimodule, and $\alpha : A \times A \to M$ a 2-cocycle. If α satisfies that $\alpha(1, a) = \alpha(a, 1) = 0$ for all $a \in A$, then α is called a normalized 2-cocycle. If $M = DA$, then α is normalized if and only if $(1, 0)$ is the identity of $T_\alpha(A, DA)$. Moreover, for every 2-cocycle α, $\alpha - \delta f_\alpha$ is a normalized 2-cocycle whose cohomology class coincides the cohomology class of α, where f_α is given by $f_\alpha(a) = \alpha(a, 1)$ for $a \in A$.

In this section, we define E-normalized 2-cocycles and we show that for every 2-cocycle there exists an E-normalized 2-cocycle such that their cohomology classes coincide. By means of that construction of E-normalized 2-cocycles, for bound quiver algebras we give another proof of a result by Ohnuki, Takeda and Yamagata in [4] by means of Theorem 3.1.

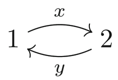
Definition 4.1. Let A be a basic connected finite dimensional algebra over a field, E a complete set of primitive orthogonal idempotents of A, M an A-bimodule, and $\alpha : A \times A \to M$ a 2-cocycle. If α satisfies

$$\alpha(e, A) = \alpha(A, e) = 0$$

for all $e \in E$, then α is called an E-normalized 2-cocycle.

Remark 4.2. If α is an E-normalized 2-cocycle, then α is a normalized 2-cocycle.

Example 4.3. Let K be a field with characteristic 0 and Q the following quiver:



Moreover, let $A = KQ/(x, y)^2$, $E = \{e_1, e_2\}$ a complete set of primitive orthogonal idempotents of A, $\mathcal{B} = \{e_1, e_2, x, y\}$ a K-basis, and $\{e_1^*, e_2^*, x^*, y^*\}$ the dual basis of $\mathcal{B}$ and $\alpha : A \times A \to DA = \mathrm{Hom}_K(A, K)$ a 2-cocycle given by

α	e_1	e_2	x	y
e_1	e_1^*	e_1^*	e_2^*	0
e_2	$e_1^* + 2x^*$	$-e_1^*$	e_2^*	0
x	e_1^*	e_1^*	0	e_1^*
y	0	0	0	0

Then, $\alpha - \delta f_\alpha$ is given by

$\alpha - \delta f_\alpha$	e_1	e_2	x	y
e_1	$-e_1^*$	e_1^*	e_2^*	0
e_2	e_1^*	$-e_1^*$	$-e_2^*$	0
x	$-e_1^*$	e_1^*	0	e_1^*
y	0	0	0	0

Therefore, the 2-cocycle $\alpha - \delta f_\alpha$ is normalized, however, $\alpha - \delta f_\alpha$ is not E-normalized.

From now on, for every 2-cocycle α we will construct an E-normalized 2-cocycle whose cohomology class coincides with the cohomology class of α.

We will define some notation. For a 2-cocycle $\alpha : A \times A \to M$, we define $h_{\mathrm{R}}(\alpha) \in \mathrm{Hom}_K(A, M)$ by $[h_{\mathrm{R}}(\alpha)](a) = \sum_{k=1}^{n} \alpha(a, e_k)e_k$ for $a \in A$. Similarly, we define $h_{\mathrm{L}}(\alpha) \in \mathrm{Hom}_K(A, M)$ by $[h_{\mathrm{L}}(\alpha)](a) = \sum_{k=1}^{n} e_k\alpha(e_k, a)$ for $a \in A$.

Moreover, we put $H_{\mathrm{R}}(\alpha) = \alpha - \delta(h_{\mathrm{R}}(\alpha))$ and $H_{\mathrm{L}}(\alpha) = \alpha - \delta(h_{\mathrm{L}}(\alpha))$ which belong to $\mathrm{Z}^2(A, M)$.

Proposition 4.4. *The following statements hold*:

(1) $[H_{\mathrm{R}}(\alpha)](A, e_i) = 0$ *for every* $i (1 \leq i \leq n)$.
(2) $[H_{\mathrm{L}}(\alpha)](e_i, A) = 0$ *for every* $i (1 \leq i \leq n)$.
(3) $H_{\mathrm{R}}^2(\alpha) = H_{\mathrm{R}}(\alpha)$.
(4) $H_{\mathrm{L}}^2(\alpha) = H_{\mathrm{L}}(\alpha)$.
(5) $H_{\mathrm{L}} H_{\mathrm{R}}(\alpha) = H_{\mathrm{R}} H_{\mathrm{L}}(\alpha)$.

Proof. (1) For $a \in A$ and $e \in E$, we have

$$\begin{aligned}
&[H_{\mathrm{R}}(\alpha)](a, e) \\
&= \alpha(a, e) - a \sum_{k=1}^{n} \alpha(e, e_k) e_k + \sum_{k=1}^{n} \alpha(ae, e_k) e_k - \sum_{k=1}^{n} \alpha(a, e_k) e_k e \\
&= \alpha(a, e) - a \sum_{k=1}^{n} \alpha(e, e_k) e_k + \sum_{k=1}^{n} (a\alpha(e, e_k) e_k + \alpha(a, ee_k) e_k - \alpha(a, e) e_k) \\
&\quad - \alpha(a, e) e \\
&= 0.
\end{aligned}$$

(2) By the similar computation of (1), we have $[H_{\mathrm{L}}(\alpha)](e, a) = 0$.
(3) By the assertion (1), for $a, b \in A$ we have

$$\begin{aligned}
[H_{\mathrm{R}}^2(\alpha)](a, b) &= [H_{\mathrm{R}}(\alpha)](a, b) - a \sum_{k=1}^{n} [H_{\mathrm{R}}(\alpha)](b, e_k) e_k \\
&\quad + \sum_{k=1}^{n} [H_{\mathrm{R}}(\alpha)](ab, e_k) e_k - \sum_{k=1}^{n} [H_{\mathrm{R}}(\alpha)](a, e_k) e_k b \\
&= [H_{\mathrm{R}}(\alpha)](a, b).
\end{aligned}$$

(4) By the similar way of the proof of (3), we can show the assertion (4).
(5) We note that $A = \oplus_{i=1}^{n} \oplus_{j=1}^{n} e_i A e_j$ as a K-module. For $p \in e_{i_1} A e_{j_1}, q \in e_{i_2} A e_{j_2}$, we have

$$\begin{aligned}
&[H_{\mathrm{R}}(\alpha)](p, q) \\
&= \alpha(p, q) - p \sum_{k=1}^{n} \alpha(q, e_k) e_k + \sum_{k=1}^{n} \alpha(pq, e_k) e_k - \sum_{k=1}^{n} \alpha(p, e_k) e_k q \\
&= \alpha(p, q) - p \sum_{k=1}^{n} \alpha(q, e_k) e_k + \sum_{k=1}^{n} (p\alpha(q, e_k) e_k + \alpha(p, qe_k) e_k - \alpha(p, q) e_k) \\
&\quad - \alpha(p, e_{i_2}) q \\
&= \alpha(p, q) e_{j_2} - \alpha(p, e_{i_2}) q.
\end{aligned}$$

Similarly, we have $[H_{\mathrm{L}}(\alpha)](p, q) = e_{i_1} \alpha(p, q) - p\alpha(e_{j_1}, q)$ for $p \in e_{i_1} A e_{j_1}$, $q \in e_{i_2} A e_{j_2}$. Hence, for $p \in e_{i_1} A e_{j_1}, q \in e_{i_2} A e_{j_2}$ we have

$$
\begin{aligned}
&[H_{\mathrm{L}}H_{\mathrm{R}}(\alpha)](p,q)\\
&\quad= e_{i_1}[H_{\mathrm{R}}(\alpha)](p,q) - p[H_{\mathrm{R}}(\alpha)](e_{j_1},q)\\
&\quad= (e_{i_1}\alpha(p,q) - p\alpha(e_{j_1},q))e_{j_2} - (e_{i_1}\alpha(p,e_{i_2}) - p\alpha(e_{j_1},e_{i_2}))q\\
&\quad= [H_{\mathrm{L}}(\alpha)](p,q)e_{j_2} - [H_{\mathrm{L}}(\alpha)](p,e_{i_2})q\\
&\quad= [H_{\mathrm{R}}H_{\mathrm{L}}(\alpha)](p,q).
\end{aligned}
$$

□

By Proposition 4.4, we put $\overline{\alpha} = H_{\mathrm{L}}H_{\mathrm{R}}(\alpha) \in \mathrm{Z}^2(A, DA)$ for every 2-cocycle α. Then, by direct computation, we have the following properties.

Proposition 4.5. *For a* 2*-cocycle* $\alpha : A \times A \to M$*, the* 2*-cocycle* $\overline{\alpha}$ *satisfies the following*:

(1) *The cohomology class* $[\overline{\alpha}]$ *of* $\overline{\alpha}$ *coincides with the cohomology class* $[\alpha]$ *of* α*.*
(2) *The* 2*-cocycle* $\overline{\alpha}$ *is an* E*-normalized* 2*-cocycle.*
(3) *If* A *is a bound quiver algebra* KQ/I *over a field* K*, then*

$$
\overline{\alpha}(p,q) = \begin{cases} s(p)\alpha(p,q)t(q) - p\alpha(t(p),s(q))q & \text{if } pq \neq 0 \text{ in } KQ \\ 0 & \text{if } pq = 0 \text{ in } KQ \end{cases}
$$

for all paths p, q *in* Q*.*

Proof. The assertions (1) and (2) are trivial by the construction of $\overline{\alpha}$. The proof of the assertion (3) is obtained by the 2-cocycle conditions $(p, t(p), t(p))_\alpha = 0$, $(s(q), s(q), q)_\alpha = 0$ and the following equation:

$$
\overline{\alpha}(p,q) = s(p)\alpha(p,q)t(q) - p\alpha(t(p),q)t(q) - s(p)\alpha(p,s(q))q + p\alpha(t(p),s(q))q
$$

for p, q paths in A. □

Example 4.6. Let α be a 2-cocycle given in Example 4.3. Then, $\overline{\alpha}$ is given by

$\overline{\alpha}$	e_1	e_2	x	y
e_1	0	0	0	0
e_2	0	0	0	0
x	0	0	0	e_1^*
y	0	0	0	0

As an application of E-normalized 2-cocycles, we give another proof of Theorem 2.1 by means of a normalized 2-cocycle.

The proof of Theorem 2.1. Let $h : A \to DA$ be a K-linear map given by

$$
h(p) = \begin{cases} 0 & \text{if } l(p) \geq 1 \text{ and } p \text{ is a cycle,} \\ [h_{\mathrm{R}}(\alpha)](p) & \text{otherwise} \end{cases}
$$

for path p in A. Let $\beta = \alpha - \delta(h)$. Then, the following hold:

(1) If α satisfies the assumption of Theorem 2.1, then β also satisfies the assumption.

(2) $\beta(e_i, e_j) = 0$ for all $i (1 \leq i \leq n)$.

Now we assume that α satisfies the assumption of Theorem 2.1. Then $\overline{\beta}$ also satisfies the assumption of Theorem 2.1 by the above two properties with respect to the 2-cocycle β.

We will show that $T_{\overline{\beta}}(A)$ is symmetric. Let p, q be paths in Q. Suppose that $l(p), l(q) \geq 1$. Then, it is clear that $\overline{\beta}(p,q) = 0$ if $pq = 0$ in KQ and $[\overline{\beta}(p,q)](1) = 0$ if pq is not a cycle. Hence, if $pq = 0$ in KQ or pq is not a cycle, then $[\langle p,q\rangle_{\overline{\beta}}](1) = 0$ by Proposition 4.5. Moreover, if pq is a cycle, then $\langle p,q\rangle_{\overline{\beta}}(1) = [\overline{\beta}(p,q) - \overline{\beta}(q,p)](1) = [\overline{\beta}(p,q)](t(q)) - [\overline{\beta}(q,p)](t(p)) = 0$. Therefore, $1 \in V_{\overline{\beta}}$, so the Hochschild extension of A defined by $\overline{\beta}$ is symmetric by Theorem 3.1.

Since $T_\alpha(A, DA)$ is isomorphic to $T_{\overline{\beta}}(A, DA)$, the Hochschild extension of A defined by α is also symmetric. □

Finally, we show that a complete set of primitive orthogonal idempotents of Hochschild extension algebras by E-normalized 2-cocycles is the same form of a complete set of primitive orthogonal idempotents of the trivial extension algebras. In order to show that assertion, we prepare the following lemma.

Lemma 4.7. *Let $A = KQ/I$ be a bound quiver algebra, $\alpha : A \times A \to DA$ a 2-cocycle, and $\mathbf{e}_i = (e_i, -\sum_{k=1}^{n} \alpha(e_i, e_k)e_k)$ for $i \in Q_0$. Then the set $\{\mathbf{e}_i \mid i \in Q_0\}$ is a complete set of primitive orthogonal idempotents of $T_\alpha(A, DA)$.*

Proof. First, we check that $\mathbf{e}_i^2 = \mathbf{e}_i$. By the 2-cocycle condition $(e_i, e_i, e_j)_\alpha = 0$, we have

$$\begin{aligned}
\mathbf{e}_i^2 &= \left(e_i, -\sum_{k=1}^{n} \alpha(e_i, e_k)e_k\right)\left(e_i, -\sum_{k'=1}^{n} \alpha(e_i, e_{k'})e_{k'}\right) \\
&= \left(e_i, -\sum_{k=1}^{n} \alpha(e_i, e_k)e_k e_i - \sum_{k'=1}^{n} e_i\alpha(e_i, e_{k'})e_{k'} + \alpha(e_i, e_i)\right) \\
&= \left(e_i, -\alpha(e_i, e_i)e_i - \sum_{k'=1}^{n} \alpha(e_i, e_{k'})e_{k'} + \sum_{k'=1}^{n} \alpha(e_i, e_i e_{k'})e_{k'}\right. \\
&\qquad \left. - \sum_{k'=1}^{n} \alpha(e_i, e_i)e_{k'} + \alpha(e_i, e_i)\right) \\
&= \left(e_i, -\sum_{k'=1}^{n} \alpha(e_i, e_{k'})e_{k'}\right) \\
&= \mathbf{e}_i.
\end{aligned}$$

Next, we show that $\mathbf{e}_i\mathbf{e}_j = 0$. By the 2-cocycle condition $(e_i, e_j, e_k)_\alpha = 0$, we have

$$\begin{aligned}\mathbf{e}_i\mathbf{e}_j &= \left(e_i, -\sum_{k=1}^{n}\alpha(e_i,e_k)e_k\right)\left(e_j, -\sum_{k'=1}^{n}\alpha(e_j,e_{k'})e_{k'}\right)\\ &= \left(0, -\sum_{k=1}^{n}\alpha(e_i,e_k)e_ke_j - \sum_{k'=1}^{n} e_i\alpha(e_j,e_{k'})e_{k'} + \alpha(e_i,e_j)\right)\\ &= \left(e_i, -\alpha(e_i,e_j)e_j + \sum_{k'=1}^{n}\alpha(e_i,e_je_{k'})e_{k'} - \sum_{k'=1}^{n}\alpha(e_i,e_j)e_{k'} + \alpha(e_i,e_j)\right)\\ &= 0.\end{aligned}$$

It is easily shown that $\mathbf{e}_i$ is primitive for $i(1 \le i \le n)$.

Finally, we show that $(1, -\alpha(1,1)) = \sum_{i=1}^{n}\mathbf{e}_i$. By the 2-cocycle condition $(1, e_i, e_j)_\alpha = \alpha(1, e_i)e_j = 0$ for $i \neq j$, we have

$$\sum_{i=1}^{n}\mathbf{e}_i = \sum_{i=1}^{n}\left(e_i, -\sum_{k=1}^{n}\alpha(e_i,e_k)e_k\right) = \left(1, -\sum_{k=1}^{n}\alpha(1,e_k)e_k\right) = (1, -\alpha(1,1)).$$

□

Remark 4.8. Let $\mathbf{e}'_i = (e_i, -\sum_{k=1}^{n} e_k\alpha(e_k,e_i))$ for $i \in Q_0$. Then, the set $\{\mathbf{e}'_i \,|\, i \in Q_0\}$ is a complete set of primitive orthogonal idempotents of $T_\alpha(A, DA)$.

Remark 4.9. For basic Artin algebras, Proposition 4.7 holds.

By Proposition 4.5 and Lemma 4.7, we have the following result.

Corollary 4.10. *Let $A = KQ/I$ be a bound quiver algebra and $\alpha : A \times A \to DA$ a 2-cocycle. Then the set $\{(e_i, 0) \,|\, i \in Q_0\} \in A \oplus DA$ is a complete set of primitive orthogonal idempotents of $T_{\overline{\alpha}}(A, DA)$.*

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