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On the second boundary value problem for a class of fully nonlinear flows II

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Abstract. This article is a continuation of an earlier work (Huang and Ye in Int Math Res Not, 2017. https://doi.org/10.1093/imrn/rnx278), where the long time existence and convergence for some special cases of parabolic type special Lagrangian equations were given. The long time existence and convergence of the flow are obtained for all cases in this article. In particular, we can prescribe the second boundary value problems for a family of special Lagrangian graphs.

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1. Introduction. To find special Lagrangian surfaces is an interesting subject in geometry which attracts attention of many mathematicians. In this article, we are concerned with the existence of a family of special Lagrangian graphs by solving the corresponding special Lagrangian equations with second boundary conditions. Let Ω, $\tilde{\Omega}$ be two uniformly convex bounded domains with smooth boundary in $\mathbb{R}^n$. For convenience of the notation, we introduce two constants $a = \cot\theta$ and $b = \sqrt{|\cot^2\theta - 1|}$ for $\theta \in (0, \frac{\pi}{4})\bigcup(\frac{\pi}{4}, \frac{\pi}{2})$. We consider the minimal Lagrangian diffeomorphism problem ([9]) which is equivalent to the following fully nonlinear elliptic equations with second boundary conditions (cf. [1], [5], and [12]):

$$\begin{cases} F_\theta(D^2v) = c, & x \in \Omega, \\ Dv(\Omega) = \tilde{\Omega}, \end{cases} \tag{1.1}$$

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where

$$F_\theta(A)=\begin{cases}\sum_i \ln \lambda_i, & \theta=0,\\ \sum_i \ln\left(\frac{\lambda_i+a-b}{\lambda_i+a+b}\right) & 0<\theta<\frac{\pi}{4},\\ -\sum_i \frac{1}{1+\lambda_i}, & \theta=\frac{\pi}{4},\\ \sum_i \arctan(\frac{\lambda_i+a-b}{\lambda_i+a+b}), & \frac{\pi}{4}<\theta<\frac{\pi}{2},\\ \sum_i \arctan \lambda_i, & \theta=\frac{\pi}{2},\end{cases}$$

with λ_i $(1\leq i\leq n)$ being the eigenvalues of the $n\times n$ symmetric matrix A, D^2v being the Hessian matrix of v, and Dv being the diffeomorphism from Ω to $\tilde{\Omega}$.

Our motivation for studying equations (1.1) is that they have interesting geometric meanings which were studied by many mathematicians, such as Brendle and Warren, etc. To see this, let us recall the definitions of Lagrangian and special Lagrangian graphs as in [12] or [6]. The graph $\Sigma=\{(x,f(x)):x\in\Omega\}$ is Lagrangian if and only if there exists a function $v:\Omega\to\mathbb{R}$ such that $f(x)=Dv(x)$. Let δ_0 be the standard Euclidean metric on $\mathbb{C}^n\cong\mathbb{R}^n\times\mathbb{R}^n$ and g_0 be the metric defined by

$$dxdy=\frac{1}{2}\sum_i(dx_i\bigotimes dy_i+dy_i\bigotimes dx_i).$$

By taking linear combinations of the metrics δ_0 and g_0, Warren [12] constructed a family of metrics on $\mathbb{R}^n\times\mathbb{R}^n$ for $0\leq\theta\leq\frac{\pi}{2}$:

$$g_\theta=\cos\theta g_0+\sin\theta\delta_0.$$

With this family of metrics, Warren ([12]) derived that the solutions of special Lagrangian equations (1.1) correspond to a family of extremal Lagrangian surfaces. Warren also investigated in [12] that these families of special Lagrangian graphs have extremal volume properties. For $\theta<\frac{\pi}{4}$, $M_\theta=(\mathbb{R}^n\times\mathbb{R}^n,g_\theta)$ is a pseudo-Euclidean space of index n. For $\theta>\frac{\pi}{4}$, M_θ is a Euclidean space. For $\theta=\frac{\pi}{4}$, M_θ carries a degenerate metric of rank n.

We have the following definition of a special Lagrangian graph with respect to the metrics g_θ as in [12].

Definition 1.1. We say that $\Sigma=\{(x,f(x))|x\in\Omega\}$ is a special Lagrangian graph in $(\mathbb{R}^n\times\mathbb{R}^n,g_\theta)$ if

$$f=Dv$$

and v satisfies

$$F_\theta(D^2v(x))=c,\quad x\in\Omega.$$

Special Lagrangian graphs have attracted considerable interest in recent years and we recall some work concerning equation (1.1) with second boundary conditions. For the special case $\theta=0$, the equation (1.1) corresponds to the Monge–Ampère equation with second boundary condition. In 1991, Delanoë [3] studied first the problem where the dimension of the domain is 2, and he obtained a unique smooth solution. Later Caffarelli ([2]) and Urbas ([11]), gave the generalizations of Delanoë's theorem to higher dimensional cases. In 2003, Schnürer and Smoczyk ([10]) also obtained the existence of solutions to (1.1) for the case $\theta = 0$ by using parabolic methods. As far as the case $\theta = \frac{\pi}{2}$ is concerned, Brendle and Warren ([1]) proved the existence and uniqueness of a solution by elliptic methods in 2010 and the second author in [4] obtained the existence of a solution by parabolic methods. For the case $\theta = \frac{\pi}{4}$, the existence result of the above problem (1.1) was established by the second author and his coauthors by using both elliptic methods ([5]) and parabolic methods ([6]).

Since the special cases for $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}$ of equations (1.1) have been solved by using both elliptic and parabolic methods, a question can be asked naturally: what about the intermediate cases for the equations (1.1)? The present article is devoted to study the equations (1.1) for all the intermediate cases, that is,

$$\theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right).$$

As in [4,6], we consider the corresponding parabolic type special Lagrangian equations (1.1) and use a parabolic framework to solve them. We settle the longtime existence and convergence of smooth solutions for the following second boundary value problem to parabolic type special Lagrangian equations

$$\begin{cases} \frac{\partial v}{\partial t} = F_\theta(D^2 v), \ t > 0, \ x \in \Omega, \\ Dv(\Omega) = \tilde{\Omega}, \qquad t > 0, \\ v = v_0, \qquad\qquad t = 0, \ x \in \Omega. \end{cases} \tag{1.2}$$

We will prove that the solutions of the above special Lagrangian equations (1.2) converge to translating solutions as $t \to \infty$. The translating solutions are intimately related to the solutions of the minimal Lagrangian diffeomorphism problem (1.1). In general, evolution equations often have special solutions called solitons which keep their shape during the evolution. For example, two very important classes of solitons in mean curvature flow are self-shrinker and translating solutions which evolve from a homothety or a translation, respectively. Translating solutions are interesting examples of solutions of evolution equations since they are precise solutions in the sense that their evolution is known, which is very hard to determine in general. Our main result concerning the asymptotic behavior of special Lagrangian equations (1.2) can be summarized as follows.

Theorem 1.2. *Assume Ω and $\tilde{\Omega}$ are two bounded and uniformly convex domains with smooth boundary in $\mathbb{R}^n$. We also assume $0 < \alpha_0 < 1$ and $\theta \in (0, \frac{\pi}{4}) \cup (\frac{\pi}{4}, \frac{\pi}{2})$. Then for any given initial function $v_0 \in C^{2+\alpha_0}(\bar{\Omega})$ which is uniformly convex and satisfies $Dv_0(\Omega) = \tilde{\Omega}$, the strictly convex solution of (1.2) exists for all $t \geq 0$ and the solution $v(x,t)$ converges to a function $v^\infty(x,t) = \tilde{v}^\infty(x) + C_\infty \cdot t$ in $C^{1+\zeta}(\bar{\Omega}) \cap C^{4+\alpha}(\bar{\Omega}')$ as $t \to \infty$ for any $\Omega' \subset\subset \Omega$, $\zeta < 1$, and $0 < \alpha < \alpha_0$, that is, we have*

$$\lim_{t \to +\infty} \|v(\cdot,t) - v^\infty(\cdot,t)\|_{C^{1+\zeta}(\bar{\Omega})} = 0$$
$$\textit{and} \quad \lim_{t \to +\infty} \|v(\cdot,t) - v^\infty(\cdot,t)\|_{C^{4+\alpha}(\bar{\Omega}')} = 0,$$

and the function $\tilde{v}^{\infty}(x) \in C^{\infty}(\bar{\Omega})$ *is a solution of*

$$\begin{cases} F_\theta(D^2 v) = C_\infty, \; x \in \Omega, \\ Dv(\Omega) = \tilde{\Omega}, \end{cases} \tag{1.3}$$

where the constant C_∞ *depends only on the geometries of* Ω, $\tilde{\Omega}$ *and* F. *The solution to (1.3) is unique up to addition of constants.*

Combining Definition 1.1 and Theorem 1.2 with the results for $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}$, we can extend Brendle-Warren's theorem ([1]) to the following:

Corollary 1.3. *Let* Ω *and* $\tilde{\Omega}$ *be two bounded and uniformly convex domains with smooth boundary in* $\mathbb{R}^n$. *Then for any* $0 \le \theta \le \frac{\pi}{2}$, *there exists a diffeomorphism* $f \colon \Omega \to \tilde{\Omega}$ *such that*

$$\Sigma = \{(x, f(x)) | x \in \Omega\}$$

is a special Lagrangian graph in $(\mathbb{R}^n \times \mathbb{R}^n, g_\theta)$.

The rest of this article is organized as follows. In Section 2, we present a preliminary result for the convergence of general uniformly parabolic operators which has been proved in [6]. The result will be used to give the proof of the main Theorem 1.2. In Section 3, we prove that the geometric evolution equations (1.2) satisfy all the hypotheses in Proposition 2.2. Therefore we are able to characterize the long time behavior of parabolic type special Lagrangian equations (1.2) and give the proof of Theorem 1.2.

2. Preliminaries. Consider a class of fully-nonlinear flows with second boundary condition

$$\begin{cases} \frac{\partial v}{\partial t} = F(D^2 v), \; t > 0, \; x \in \Omega, \\ Dv(\Omega) = \tilde{\Omega}, \quad t > 0, \\ v = v_0, \quad t = 0, \; x \in \Omega, \end{cases} \tag{2.1}$$

where F is a $C^{2+\alpha_0}$ function for some given $0 < \alpha_0 < 1$ defined on the cone Γ_+ of positive definite symmetric matrices. The function F is monotonically increasing and satisfies

$$\begin{cases} F[A] := F(\lambda_1, \lambda_2, \ldots, \lambda_n), \\ F(\ldots, \lambda_i, \ldots, \lambda_j, \ldots) = F(\ldots, \lambda_j, \ldots, \lambda_i, \ldots), \quad \text{for any } \; 1 \le i < j \le n, \end{cases} \tag{2.2}$$

with

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$$

being the eigenvalues of the $n \times n$ symmetric matrix A.

For any given constants $\mu_1 > 0$ and $\mu_2 > 0$, we define

$$\Gamma^+_{]\mu_1, \mu_2[} = \{(\lambda_1, \lambda_2, \ldots, \lambda_n) | 0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n, \lambda_1 \le \mu_1, \lambda_n \ge \mu_2\}.$$

We assume that there exist two positive constants λ and Λ depending only on μ_1 and μ_2, such that for any $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Gamma^+_{]\mu_1, \mu_2[}$, the function F satisfies

$$\lambda \le \sum_{i=1}^{n} \frac{\partial F}{\partial \lambda_i} \le \Lambda \tag{2.3}$$

and

$$\lambda \leq \sum_{i=1}^{n} \frac{\partial F}{\partial \lambda_i} \lambda_i^2 \leq \Lambda. \tag{2.4}$$

In addition,

$$F(A) \ \text{ and } \ F^*(A) \triangleq -F(A^{-1}) \ \text{ are both concave functions on the cone } \Gamma_+. \tag{2.5}$$

Furthermore, we assume that there exist two functions g_1 and g_2 which are monotonically increasing in the interval $(0, +\infty)$ and satisfy

$$g_1(\lambda_1) \leq F(\lambda_1, \lambda_2, \ldots, \lambda_n) \leq g_2(\lambda_n) \quad (\forall\ 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n), \tag{2.6}$$

and for any $\Phi_1, \Phi_2 \in \Delta$,

$$\begin{cases} g_1(t) \leq \Phi_1 \Rightarrow \exists t_1 > 0,\ t \leq t_1, \\ g_2(t) \geq \Phi_2 \Rightarrow \exists t_2 > 0,\ t \geq t_2, \end{cases} \tag{2.7}$$

where

$$\Delta = \{\Upsilon | \exists (\lambda_1, \lambda_2, \ldots, \lambda_n), 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n, \Upsilon = F(\lambda_1, \lambda_2, \ldots, \lambda_n)\}.$$

We give some remarks on the structure conditions (2.3)–(2.7). As illustrated in our previous work ([6]), we cannot expect that F satisfies (2.3) and (2.4) for the universal constants λ and Λ on the cone Γ_+ of positive definite symmetric matrices. The reason is the following: for any $\epsilon > 0$, by taking

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n = \epsilon,$$

we obtain

$$\epsilon^2 \Lambda \geq \epsilon^2 \sum_{i=1}^{n} \frac{\partial F}{\partial \lambda_i} = \sum_{i=1}^{n} \frac{\partial F}{\partial \lambda_i} \lambda_i^2 \geq \lambda.$$

In view of the above fact, we introduce the domain $\Gamma^+_{]\mu_1,\mu_2[}$ such that the two conditions (2.3) and (2.4) are compatible. As can be seen in [1], the range of c should be limited for the solvability of the equation (1.1) and the condition (2.6) reflects this issue to some extent. However, we can show that there exist universal constants μ_1 and μ_2 depending on $F_\theta(D^2 v_0)$, such that $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ are always in $\Gamma^+_{]\mu_1,\mu_2[}$ along the flow. So F satisfies the structure conditions (2.3) and (2.4) for the constants λ and Λ along the flow. Although we have proved these facts for the flow equation (2.1) with a general parabolic operator in our previous work ([6]), we can seek out μ_1 and μ_2 explicitly by direct calculation for the special case of parabolic type special Lagrangian equation (1.2). For completeness of the article and convenience of the readers, we state the result as a proposition and give a self-contained proof here.

Proposition 2.1. (Huang and Ye, see Lemma 2.4 and Lemma 3.1–Lemma 3.3 in [6]) *Let (x, t) be an arbitrary point of $\Omega_T = \Omega \times (0, T)$ for any $T > 0$, and let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of $D^2 v$ at (x, t). For any $v_0 \in C^2 + \alpha_0)(\Omega)$ which is uniformly convex and satisfies $D_{v_0} = \tilde{\Omega}$, as long as the convex solution to (1.2) exists, then there exists $\mu_1 > 0, \mu_2 > 0$ depending only on*

$F_\theta(D^2v_0)$, such that $(\lambda - 1 \le \mu_1, \lambda_n \ge \mu_2$, and then the points $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ are always in $\Gamma^+_{]\mu_1,\mu_2[}$.

Proof. We divide the proof in three steps.

Step 1 Using the methods on the second boundary value problems for equations of Monge–Ampère type ([11]), the parabolic second boundary condition in (1.2) can be reformulated as

$$h(Dv) = 0, \qquad x \in \partial\Omega, \quad t > 0, \tag{2.8}$$

where h is a smooth defining function on $\bar{\tilde{\Omega}}$ satisfying $\tilde{\Omega} = \{p \in \mathbb{R}^n | h(p) > 0\}$, $|Dh|_{\partial\tilde{\Omega}} = 1$, and h is strictly concave. By (2.9), [9, (2.9), p. 118], we know that the boundary condition is strictly oblique on uniformly convex solutions v, that is,

$$v \in C^2(\bar{\Omega}) \quad \text{with} \quad D^2v > 0 \Longrightarrow \inf_{\partial\Omega} h_{p_k}(Dv)\nu_k > 0, \tag{2.9}$$

where $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ is the unit inward normal vector of $\partial\Omega$.

Step 2 We claim that the following $\dot{v}$-estimates hold

$$\Psi_0 \triangleq \min_{\bar{\Omega}} F_\theta(D^2v_0) \le \dot{v} \triangleq \frac{\partial v}{\partial t} \le \max_{\bar{\Omega}} F_\theta(D^2v_0) \triangleq \Psi_1. \tag{2.10}$$

In fact, we use a method known from [10, Lemma 2.1]. Differentiating the first equation of (1.2) with respect to t yields

$$\frac{\partial(\dot{v})}{\partial t} - F_\theta^{ij}\partial_{ij}(\dot{v}) = 0,$$

where $F_\theta^{ij}(D^2v) = \frac{\partial F_\theta}{\partial v_{ij}}$. Using the maximum principle, we see that

$$\min_{\bar{\Omega}_T}(\dot{v}) = \min_{\partial\bar{\Omega}_T}(\dot{v})$$

if $\dot{v} \ne$ *constant* and $\exists x_0 \in \partial\Omega, t_0 > 0$, such that $\dot{v}(x_0,t_0) = \min_{\bar{\Omega}_T}(\dot{v})$. On the one hand, $h_{p_k}\nu_k \triangleq \langle\beta,\nu\rangle > 0$ by (2.9), hence $\beta = (h_{p_1}, \ldots, h_{p_n})$ is not tangential to the parabolic boundary. Then we can use the Hopf lemma (cf. [7] or [8]) for parabolic equations to deduce that

$$\dot{v}_\beta(x_0,t_0) \ne 0.$$

On the other hand, by differentiating the reformulated boundary condition (2.8) with respect to t, we obtain

$$\dot{v}_\beta = h_{p_k}(Dv)\frac{\partial \dot{v}}{\partial x_k} = \frac{\partial h(Dv)}{\partial t} = 0.$$

This is a contradiction. So we deduce that

$$\dot{v} \ge \min_{\bar{\Omega}_T}(\dot{v}) = \min_{\partial\bar{\Omega}_T|_{t=0}}(\dot{v}) = \min_{\bar{\Omega}} F_\theta(D^2v_0) \triangleq \Psi_0.$$

For the same reason, we obtain

$$\frac{\partial v}{\partial t} \le \max_{\bar{\Omega}} F_\theta(D^2v_0) \triangleq \Psi_1.$$

Putting these facts together, the claim (2.10) follows.

Step 3 Let (x,t) be an arbitrary point of Ω_T, and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of D^2v at (x,t). As long as the convex solution to (1.2) exists, using $\dot{v}$-estimates from Step 2 and condition (2.6), we obtain:

Case 1, $\theta \in (0, \frac{\pi}{4})$.

$$\Psi_0 \leq \frac{\partial v}{\partial t} = F_\theta(D^2v) = \sum_i \ln\left(\frac{\lambda_i + a - b}{\lambda_i + a + b}\right) \leq n \ln\left(\frac{\lambda_n + a - b}{\lambda_n + a + b}\right). \quad (2.11)$$

By uniform convexity of the initial value v_0 and the definition of Ψ_0 in Step 2, we know

$$0 > \Psi_0 \triangleq \min_{\bar{\Omega}} F_\theta(D^2 v_0) > n \ln \frac{a-b}{a+b}. \quad (2.12)$$

Combining (2.11) and (2.12) and using the intermediate value theorem for the monotonically increasing and continuous function $g_1(t) = n \ln \frac{t+a-b}{t+a+b}$ on the interval $[0, \lambda_n]$, we know that there exists a unique $\mu_2 \in (0, \lambda_n)$, such that $n \ln \frac{\mu_2+a-b}{\mu_2+a+b} = \Psi_0 < 0$ or equivalently $\mu_2 = \frac{e^{\frac{\Psi_0}{n}}(a+b)-a+b}{1-e^{\frac{\Psi_0}{n}}} > 0$. Since $g_1(t)$ is monotonically increasing, we obtain

$$\lambda_n \geq \frac{e^{\frac{\Psi_0}{n}}(a+b) - a + b}{1 - e^{\frac{\Psi_0}{n}}} = \mu_2 > 0.$$

Similarly, we know that there exists $\mu_1 = \frac{e^{\frac{\Psi_1}{n}}(a+b)-a+b}{1-e^{\frac{\Psi_1}{n}}} > 0$, such that $0 < \lambda_1 \leq \mu_1$.

Case 2, $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$.

$$\Psi_0 \leq \frac{\partial v}{\partial t} = F_\theta(D^2v) = \sum_i \arctan\left(\frac{\lambda_i + a - b}{\lambda_i + a + b}\right) \leq n \arctan\left(\frac{\lambda_n + a - b}{\lambda_n + a + b}\right). \quad (2.13)$$

By uniform convexity of the initial value v_0 and the definition of Ψ_0 in Step 2, we know

$$\Psi_0 \triangleq \min_{\bar{\Omega}} F_\theta(D^2 v_0) > n \arctan \frac{a-b}{a+b}. \quad (2.14)$$

Then similarly to case 1, by combining (2.13) and (2.14) and using the intermediate value theorem for the monotonically increasing function $g_2(t) = n \arctan \frac{t+a-b}{t+a+b}$ on the interval $[0, \lambda_n]$ again, we know that there exists a unique $\mu_2 \in (0, \lambda_n)$, such that $n \arctan \frac{\mu_2+a-b}{\mu_2+a+b} = \Psi_0$ or equivalently $\mu_2 = \frac{(\tan \frac{\Psi_0}{n})(a+b)-a+b}{1-\tan \frac{\Psi_0}{n}} > 0$. Since $g_2(t)$ is monotonically increasing, we obtain

$$\lambda_n \geq \frac{(\tan \frac{\Psi_0}{n})(a+b) - a + b}{1 - \tan \frac{\Psi_0}{n}} = \mu_2 > 0.$$

Similarly, we know that there exists $\mu_1 = \frac{(\tan \frac{\Psi_1}{n})(a+b)-a+b}{1-\tan \frac{\Psi_1}{n}} > 0$, such that $0 < \lambda_1 \leq \mu_1$.

Hence the points $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ are always in $\Gamma^+_{]\mu_1,\mu_2[}$ along the geometric evolution equation (1.2). $\square$

The following proposition concerns the asymptotic convergence of general fully-nonlinear flows (2.1) with second boundary condition under certain conditions. This proposition plays a fundamental role in the proof of Theorem 1.2.

Proposition 2.2. (Huang and Ye, see [6, Theorem 1.1]) *Assume that Ω and $\tilde{\Omega}$ are two bounded and uniformly convex domains with smooth boundary in $\mathbb{R}^n$. We also assume $0 < \alpha_0 < 1$ and that the map F satisfies the conditions (2.2), (2.3), (2.4), (2.5), (2.6), (2.7). Then for any given initial function $v_0 \in C^{2+\alpha_0}(\bar{\Omega})$ which is uniformly convex and satisfies the gradient map $Dv_0(\Omega) = \tilde{\Omega}$, the strictly convex solution of (2.1) exists for all $t \geq 0$ and the solution $v(x,t)$ converges to a function $v^\infty(x,t) = \tilde{v}^\infty(x) + C_\infty \cdot t$ in $C^{1+\zeta}(\bar{\Omega}) \cap C^{4+\alpha}(\bar{\Omega}')$ as $t \to \infty$ for any $\Omega' \subset\subset \Omega$, $\zeta < 1$ and $0 < \alpha < \alpha_0$, that is, we have the following estimates*

$$\lim_{t\to+\infty} \|v(\cdot,t) - v^\infty(\cdot,t)\|_{C^{1+\zeta}(\bar{\Omega})} = 0$$
$$\text{and} \quad \lim_{t\to+\infty} \|v(\cdot,t) - v^\infty(\cdot,t)\|_{C^{4+\alpha}(\bar{\Omega}')} = 0,$$

and the function $\tilde{v}^\infty(x) \in C^{1+1}(\bar{\Omega}) \cap C^{4+\alpha}(\Omega)$ is a solution of

$$\begin{cases} F(D^2 v) = C_\infty, \ x \in \Omega, \\ Dv(\Omega) = \tilde{\Omega}, \end{cases} \tag{2.15}$$

where the constant C_∞ depends only on the geometries of Ω, $\tilde{\Omega}$ and F. The solution to (2.15) is unique up to addition of constants.

Proposition 2.2 naturally yields the convergence of solutions to equation (1.2) as $t \to \infty$ by verifying that the structure conditions are satisfied. In [6], Huang and Ye used the inverse function theorem to establish the short time existence of the flow (2.1). Then the authors used the conditions (2.2)–(2.7) to construct suitable auxiliary functions as barriers and finally established the a priori estimates needed to prove the convergence of the flow.

3. Proof of Thoerem 1.2. In this section, we verify that the hypotheses in (2.2)–(2.7) are valid for the geometric evolution equation (1.2) via elementary methods. To that end, we require an elementary result for monotonically increasing functions.

Lemma 3.1. *Let $f(t)$ be a monotonically increasing continuous function on $(0, +\infty)$. Then for any $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, there exists a unique $\lambda \in [\lambda_1, \lambda_n]$ such that*

$$f(\lambda) = \frac{\sum_{i=1}^n f(\lambda_i)}{n}.$$

Proof. Since $f(t)$ is monotonically increasing,

$$f(\lambda_1) \leq \frac{\sum_{i=1}^n f(\lambda_i)}{n} \leq f(\lambda_n).$$

By making use of the intermediate value theorem for continuous functions, we obtain the conclusion. □

We put

$$F_\theta(\lambda_1, \lambda_2, \ldots, \lambda_n) = \begin{cases} \sum_i \ln(\frac{\lambda_i + a - b}{\lambda_i + a + b}), & \text{for} \quad 0 < \theta < \frac{\pi}{4}, \\ \sum_i \arctan(\frac{\lambda_i + a - b}{\lambda_i + a + b}), & \text{for} \quad \frac{\pi}{4} < \theta < \frac{\pi}{2}. \end{cases}$$

Without loss of generality, we always assume that $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

Lemma 3.2. *For any $\theta \in (0, \frac{\pi}{4}) \cup (\frac{\pi}{4}, \frac{\pi}{2})$, the operator F_θ satisfies the hypotheses in (2.2)–(2.7).*

Proof. Case 1, $\theta \in (0, \frac{\pi}{4})$.

It is obvious that $a = \cot\theta > b = \sqrt{|\cot^2\theta - 1|}$. We observe

$$\frac{\partial F_\theta}{\partial \lambda_i} = \frac{1}{\lambda_i + a - b} - \frac{1}{\lambda_i + a + b} = \frac{2b}{(\lambda_i + a - b)(\lambda_i + a + b)} > 0.$$

Then the equation (2.1) is parabolic and F_θ satisfies (2.2). For any $\mu_1 > 0, \mu_2 > 0$, if $\lambda_1 \leq \mu_1, \lambda_n \geq \mu_2$, we obtain

$$\begin{aligned} \frac{2nb}{(a-b)(a+b)} \geq \sum_{i=1}^n \frac{\partial F_\theta}{\partial \lambda_i} &= \sum_{i=1}^n \frac{2b}{(\lambda_i + a - b)(\lambda_i + a + b)} \\ &\geq \frac{2b}{(\lambda_1 + a - b)(\lambda_1 + a + b)} \\ &\geq \frac{2b}{(\mu_1 + a - b)(\mu_1 + a + b)} \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} 2nb \geq \sum_{i=1}^n \frac{\partial F_\theta}{\partial \lambda_i} \lambda_i^2 &= \sum_{i=1}^n \frac{2b\lambda_i^2}{(\lambda_i + a - b)(\lambda_i + a + b)} \\ &\geq \frac{2b\lambda_n^2}{(\lambda_n + a - b)(\lambda_n + a + b)} \\ &\geq \frac{2b\mu_2^2}{(\mu_2 + a - b)(\mu_2 + a + b)}. \end{aligned} \tag{3.2}$$

By (3.1) and (3.2) we deduce that F_θ satisfies (2.3) and (2.4). We calculate directly to obtain:

$$\sum_{i,j=1}^n \frac{\partial^2 F_\theta}{\partial \lambda_i \lambda_j} \xi_i \xi_j = \sum_{i=1}^n \left(\frac{\xi_i^2}{(\lambda_i + a + b)^2} - \frac{\xi_i^2}{(\lambda_i + a - b)^2} \right) \leq 0$$

and

$$\sum_{i,j=1}^n \frac{\partial^2 F_\theta^*}{\partial \lambda_i \lambda_j} \xi_i \xi_j = \sum_{i=1}^n \left(\frac{\xi_i^2}{(\lambda_i + (a-b)^{-1})^2} - \frac{\xi_i^2}{(\lambda_i + (a+b)^{-1})^2} \right) \leq 0.$$

Consequently, F_θ satisfies (2.5). Let

$$g_1(t) = g_2(t) \triangleq n \ln\left(\frac{t + a - b}{t + a + b} \right).$$

It is elementary to check that

$$g_1(\lambda_1) \leq F_\theta(\lambda_1, \lambda_2, \ldots, \lambda_n) \leq g_2(\lambda_n).$$

Given $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, if

$$g_1(t) \leq F_\theta(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

then we have

$$\ln\left(\frac{t+a-b}{t+a+b}\right) \leq \frac{\sum_{i=1}^n \ln\left(\frac{\lambda_i+a-b}{\lambda_i+a+b}\right)}{n}. \tag{3.3}$$

It is easy to see that

$$\ln\left(\frac{t+a-b}{t+a+b}\right)$$

is a monotonically increasing continuous function on $(0, +\infty)$. By Lemma 3.1, there exists a unique $t_1 \in [\lambda_1, \lambda_n]$ such that

$$\frac{\sum_{i=1}^n \ln\left(\frac{\lambda_i+a-b}{\lambda_i+a+b}\right)}{n} = \ln\left(\frac{t_1+a-b}{t_1+a+b}\right).$$

Combining with (3.3), we obtain

$$\ln\left(\frac{t+a-b}{t+a+b}\right) \leq \ln\left(\frac{t_1+a-b}{t_1+a+b}\right).$$

This implies $t \leq t_1$. Using the same methods, if

$$g_2(t) \geq F_\theta(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

then there exists a unique $t_2 \in [\lambda_1, \lambda_n]$ such that $t \geq t_2$. Putting these facts together, we see that F_θ satisfies all the hypotheses in(2.2)–(2.7) for $\theta \in (0, \frac{\pi}{4})$.

Case 2, $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$.

A direct calculation as in Case 1 gives

$$\frac{\partial F_\theta}{\partial \lambda_i} = \frac{2b}{(\lambda_i+a-b)^2 + (\lambda_i+a+b)^2} > 0.$$

Then the equation (2.1) is parabolic and F_θ also satisfies (2.2). For any $\mu_1 > 0$ and $\mu_2 > 0$, if $\lambda_1 \leq \mu_1, \lambda_n \geq \mu_2$, we deduce that

$$\begin{aligned} \frac{2nb}{(a-b)^2 + (a+b)^2} \geq \sum_{i=1}^n \frac{\partial F_\theta}{\partial \lambda_i} &= \sum_{i=1}^n \frac{2b}{(\lambda_i+a-b)^2 + (\lambda_i+a+b)^2} \\ &\geq \frac{2b}{(\lambda_1+a-b)^2 + (\lambda_1+a+b)^2} \\ &\geq \frac{2b}{(\mu_1+a-b)^2 + (\mu_1+a+b)^2} \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} nb &\geq \sum_{i=1}^{n} \frac{\partial F_\theta}{\partial \lambda_i}\lambda_i^2 = \sum_{i=1}^{n} \frac{2b\lambda_i^2}{(\lambda_i+a-b)^2+(\lambda_i+a+b)^2} \\ &\geq \frac{2b\lambda_n^2}{(\lambda_n+a-b)^2+(\lambda_n+a+b)^2} \\ &\geq \frac{2b\mu_2^2}{(\mu_2+a-b)^2+(\mu_2+a+b)^2}. \end{aligned} \tag{3.5}$$

By (3.4) and (3.5), we see that F_θ satisfies (2.3) and(2.4). Clearly, we calculate directly to obtain:

$$\sum_{i,j=1}^{n} \frac{\partial^2 F_\theta}{\partial \lambda_i \lambda_j}\xi_i\xi_j = \sum_{i=1}^{n} -\frac{8(\lambda_i+a)b\xi_i^2}{((\lambda_i+a-b)^2+(\lambda_i+a+b)^2)^2} \leq 0$$

and

$$\sum_{i,j=1}^{n} \frac{\partial^2 F_\theta^*}{\partial \lambda_i \lambda_j}\xi_i\xi_j = \sum_{i=1}^{n} -\frac{8b(a+\lambda_i(a^2+b^2))\xi_i^2}{((1+\lambda_i(a-b))^2+(1+\lambda_i(a+b))^2)^2} \leq 0.$$

Therefore, F_θ satisfies (2.5). We define

$$g_1(t) = g_2(t) \triangleq n \arctan\left(\frac{t+a-b}{t+a+b}\right).$$

Note that the above functions are monotonically increasing and continuous in t. We have the pointwise inequalities

$$g_1(\lambda_1) \leq F_\theta(\lambda_1, \lambda_2, \ldots, \lambda_n) \leq g_2(\lambda_n).$$

Given $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, if

$$g_1(t) \leq F_\theta(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

then we obtain

$$\arctan\left(\frac{t+a-b}{t+a+b}\right) \leq \frac{\sum_{i=1}^{n} \arctan\left(\frac{\lambda_i+a-b}{\lambda_i+a+b}\right)}{n}. \tag{3.6}$$

By Lemma 3.1, there exists a unique $t_1 \in [\lambda_1, \lambda_n]$ such that

$$\frac{\sum_{i=1}^{n} \arctan\left(\frac{\lambda_i+a-b}{\lambda_i+a+b}\right)}{n} = \arctan\left(\frac{t_1+a-b}{t_1+a+b}\right).$$

Combining with (3.6), we conclude that

$$\arctan\left(\frac{t+a-b}{t+a+b}\right) \leq \arctan\left(\frac{t_1+a-b}{t_1+a+b}\right).$$

This implies $t \leq t_1$. Using the same methods, if

$$g_2(t) \geq F_\theta(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

then there exists a unique $t_2 \in [\lambda_1, \lambda_n]$ such that $t \geq t_2$. To summarize, we have also shown that F_θ satisfies (2.2)–(2.7) for $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$. Finally, combining Case 1 with Case 2, we obtain the desired results. □

Finally, we can give the proof of the main theorem.

Proof of Theorem 1.2. Using Lemma 3.2, we obtain that the operator F_θ satisfies all the hypotheses in (2.2)–(2.7). By Proposition 2.2, the assertion follows. □

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