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Large abelian normal subgroups

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Abstract. In this paper we study the family of finite groups with the property that every maximal abelian normal subgroup is self-centralizing. It is well known that this family contains all finite supersolvable groups, but it also contains many other groups. In fact, every finite group G is a subgroup of some member Γ of this family, and we show that if G is solvable, then Γ can be chosen so that every abelian normal subgroup of G is contained in some self-centralizing abelian normal subgroup of Γ.

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1. Introduction. Recall that a normal subgroup N of a group G is sometimes referred to as a **large** subgroup of G if N contains its centralizer $\mathbf{C}_G(N)$. In particular, an abelian normal subgroup A of G is large precisely when $A = \mathbf{C}_G(A)$, or in other words, A is self-centralizing in G. A large abelian normal subgroup of G, therefore, is necessarily maximal among abelian normal subgroups of G. Conversely, it is well known that if G is nilpotent, then every maximal abelian normal subgroup of G is large. (See, for instance [4, Lemma 4.16]). In fact, an essentially identical proof shows that the same conclusion holds under the weaker assumption that G is supersolvable.

The purpose of this note is to study the family of groups that share with supersolvable groups the property that every maximal abelian normal subgroup is large. For notational convenience, we refer to the members of this family as MANL groups.

Supersolvable groups are MANL groups, but as the alternating group A_4 shows, not every MANL group is supersolvable. The fact that A_4 is MANL is a special case of something more general than supersolvability, however: if the supersolvable residual of a group G is abelian, then G is MANL. (Recall that by definition, the **supersolvable residual** of a group G is the unique smallest normal subgroup $S = G^{ss}$ such that G/S is supersolvable). Still more general is the following.

Theorem A. *Let S be the supersolvable residual of a solvable group G, and suppose that the Fitting subgroup $\mathbf{F}(S)$ is abelian. Then G is an MANL group.*

The smallest group G such that G^{ss} is not abelian is $SL(2,3)$ of order 24, and it is easy to see that $SL(2,3)$ is not MANL. (The unique maximal abelian normal subgroup of $SL(2,3)$ is its center, which is not self-centralizing.) It follows that $SL(2,3)$ is the smallest non-MANL group.

Suppose that G is solvable and that every Sylow subgroup of the supersolvable residual $S = G^{ss}$ is abelian. Then $\mathbf{F}(S)$ is abelian, and hence G is MANL by Theorem A. Also, if $H \subseteq G$ is an arbitrary subgroup, then $H/(H \cap S) \cong HS/S$ is supersolvable, and thus $H^{ss} \subseteq S$, so all Sylow subgroups of H^{ss} are abelian. It follows that every subgroup of G is MANL. In general, however, subgroups of solvable MANL groups need not be MANL, and in fact, there is nothing at all that can be said about such subgroups other than that they are solvable. In particular, there is no upper bound on the derived length, the Fitting length, or any similar measure of complexity of MANL groups. This is a consequence of the following.

Theorem B. *Given an arbitrary finite group G, there exists a finite MANL group W such that G is isomorphic to a subgroup of W and also to a factor group of W. In fact, we can take W to be the semidirect product of an elementary abelian p-group B acted on by G, where p is any prime not dividing the order of the Fitting subgroup $\mathbf{F}(G)$.*

In our proof of Theorem B, we shall see that the subgroup B of W is the unique maximal abelian normal subgroup of W, and that B is self-centralizing in W. It follows that W is an MANL group, as required. This argument seems somewhat unsatisfactory, however, because the maximal abelian normal subgroups of G play no role. If G is solvable, however, we can overcome this defect with a different embedding of G into an MANL group.

Theorem C. *Let G be a finite solvable group. Then G is a subgroup of some solvable MANL group Γ with the property that every maximal abelian normal subgroup of G has the form $A \cap G$, where A is a maximal abelian normal subgroup of Γ. In particular, Γ has at least as many maximal abelian normal subgroups as G.*

We observed previously that if G is solvable and all Sylow subgroups of $S = G^{ss}$ are abelian, then every subgroup $H \subseteq G$ is MANL. It is also true in this situation that every homomorphic image of G is MANL. (To see this, let $\theta : G \to H$ be a surjective homomorphism, and observe that $H^{ss} = \theta(S)$. Then each Sylow subgroup of H^{ss} is the image under θ of some Sylow subgroup of S, and hence it is abelian.)

It follows that if G is solvable and all Sylow subgroups of G^{ss} are abelian, then every section of G is MANL. (Recall that a **section** of a group G is any group of the form H/K, where $K \triangleleft H \subseteq G$.) The converse of this statement is false, however. For example, the normalizer of a Sylow 2-subgroup of $PSL(3,4)$ is solvable and all of its sections are MANL, but the supersolvable residual of this group is a nonabelian 2-group.

We close this introduction with a brief discussion of a connection between M-groups and MANL groups. First, recall that by definition, a group G is an **M-group** if every irreducible character χ of G is induced from a linear character of some (not necessarily proper) subgroup of G. Recall also that M-groups are necessarily solvable. Although not every solvable group is an M-group, it is a standard result that if G is solvable and all Sylow subgroups of G^{ss} are abelian, then G is an M-group. (See, for example, [3, Theorems 6.22 and 6.23]). In fact, something more general is true: if G is solvable and all sections of G are MANL, then G is an M-group.

To see this, suppose that all sections of G are MANL, and let $\chi \in \mathrm{Irr}(G)$. Let $H \subseteq G$ be minimal with the property that there exists a character $\psi \in \mathrm{Irr}(H)$ such that $\psi^G = \chi$, and observe that guarantees that ψ is primitive. Now let $K = \ker H$, and let A/K be a maximal abelian normal subgroup of H/K. Since ψ is a faithful primitive character of H/K, it follows that A/K is central in H/K. Also, H/K is MANL, so A/K is large in H/K, and thus $A/K = H/K$. We deduce that H/K is abelian, and hence ψ is linear. It follows that χ is monomial, as required.

Finally, we mention that it is not true that every solvable MANL group is an M-group, nor is it true that every M-group is MANL. The first of these two assertions holds since by Theorem B, every solvable group is a homomorphic image of a solvable MANL group, but only M-groups can occur as homomorphic images of M-groups. An example that demonstrates the second assertion is the normalizer of a Sylow 2-subgroup of the Suzuki group $Sz(8)$. This normalizer is an M-group but it is not MANL.

2. Theorem A.

In this section, we prove Theorem A.

Proof of Theorem A. To prove that G is MANL, we consider an arbitrary maximal abelian normal subgroup A of G, and we show that $A = \mathbf{C}_G(A)$. Otherwise, $A < \mathbf{C}_G(A)$, and we work to obtain a contradiction.

Let K/A be a chief factor of G, where $K \subseteq \mathbf{C}_G(A)$, and note that K is not abelian by the maximality of A. The chief factor K/A is abelian because G is solvable, and since $A \subseteq \mathbf{Z}(K)$, we see that K is nilpotent, and thus $S \cap K$ is a normal nilpotent subgroup of S. Then $S \cap K \subseteq \mathbf{F}(S)$, and since we are assuming that $\mathbf{F}(S)$ is abelian, we deduce that $S \cap K$ is abelian.

Suppose $S \cap K \not\subseteq A$. Then $A < A(S \cap K) \subseteq K$, and since K/A is a chief factor of G and $A(S \cap K)$ is normal in G, we see that $A(S \cap K) = K$. Now $A \subseteq \mathbf{Z}(K)$ and $S \cap K$ is abelian, so $K = A(S \cap K)$ is abelian. This is a contradiction, and we conclude that $S \cap K \subseteq A$.

Now $K/(S \cap K)$ is isomorphic to KS/S, and under this isomorphism, the subgroup $A/(S \cap K)$ corresponds to AS/S. There is thus a natural isomorphism from K/A to KS/AS, and this isomorphism is compatible with the conjugation actions of G on these two groups. (In other words, K/A and KS/AS are isomorphic as G-operator groups.) Since K/A is a chief factor of G, it follows that KS/AS is also a chief factor of G, and hence it is a chief factor of the supersolvable group G/S. Then KS/AS is cyclic, so K/A is also cyclic, and we conclude that K is abelian because $A \subseteq \mathbf{Z}(G)$. This is our final contradiction. $\square$

3. Wreath products. We begin with a brief review of the definition and some basic properties of wreath products. Let H be a permutation group acting faithfully on some finite set Ω, and let U be an arbitrary group. We write $\tilde{U}$ to denote the set of all functions from Ω into U, and we observe that $\tilde{U}$ is a group with respect to pointwise multiplication.

Given $\alpha \in \Omega$, let U_α be the subgroup of $\tilde{U}$ consisting of the functions f such that $f(\beta) = 1$ whenever $\beta \neq \alpha$ for $\beta \in \Omega$. It is easy to see that $U_\alpha \cong U$ and that $\tilde{U}$ is the direct product of the subgroups U_α.

Observe that H acts on $\tilde{U}$ according to the formula

$$f^h(\alpha) = f(\alpha{\cdot}h^{-1})$$

for all $\alpha \in \Omega$, where $h \in H$, and $f \in \tilde{U}$. If U is nontrivial, the semidirect product $W = \tilde{U} \rtimes H$ is, by definition, the wreath product of U by H, and we write $W = U \wr H$. We view $\tilde{U}$ and H as subgroups of W, so $W = \tilde{U}H$, where $\tilde{U} \triangleleft W$ and $\tilde{U} \cap H = 1$. The normal subgroup $\tilde{U}$ of W is often referred to as the **base group** of the wreath product W.

It is easy to check that $(U_\alpha)^h = U_{\alpha\cdot h}$, so H acts to permute the direct factors U_α of $\tilde{U}$. Since by assumption, the action of H on Ω is faithful and U is nontrivial, we see that no nonidentity element of H can act trivially on $\tilde{U}$, and thus $\mathbf{C}_H(\tilde{U}) = 1$.

We mention for later use that if $R \subseteq U$ is an arbitrary subgroup, then $\tilde{R} \subseteq \tilde{U}$, and we observe that H normalizes $\tilde{R}$. If $R \triangleleft U$, it is easy to see that $\tilde{R} \triangleleft \tilde{U}$, and thus $\tilde{R} \triangleleft U \wr H$.

Proof of Theorem B. We are given an arbitrary finite group G, and we view G as a faithful permutation group on some set Ω. (For example, we can take $\Omega = G$, where G acts by right multiplication.) Choose a prime p not dividing $|\mathbf{F}(G)|$, and let U be a cyclic group of order p. Write $B = \tilde{U}$, so B is the group of all functions from Ω into U, and B is the base group of the wreath product $W = U \wr G$. Now W is the semidirect product of the elementary abelian p-group B acted on by G. In particular, G is both a subgroup and a homomorphic image of W, and it suffices to show that W is an MANL group.

We have $W = BG$ and $\mathbf{C}_G(B) = 1$. Writing $C = \mathbf{C}_W(B)$, we have $C \supseteq B$, and thus $C = B(G \cap C) = B$, so B is self-centralizing in W, and in particular, B is a maximal abelian normal subgroup of W. To prove that W is MANL, therefore, it suffices to show that B is the unique maximal abelian normal subgroup of W. Equivalently, we show that if $A \triangleleft W$, where A is abelian, then $A \subseteq B$.

Observe that $A/(A \cap B) \cong BA/B$, and this is an abelian normal subgroup of $W/B \cong G$. Then $|A : A \cap B|$ divides $|\mathbf{F}(G)|$, so p does not divide $|A : A \cap B|$, and thus $A \cap B$ is a Sylow p-subgroup of A. Since A is abelian, we can write $A = (A \cap B) \times Q$, where Q is the Hall p'-group of A. Now $Q \triangleleft W$ because by assumption, $A \triangleleft W$. Since B is a p-group, we have $Q \cap B = 1$, and thus $Q \subseteq \mathbf{C}_W(B) = B$. It follows that $Q = 1$, and we conclude that $A = A \cap B \subseteq B$, as required. □

4. Direct products and more wreath products. We begin work toward a proof of Theorem C with the following result.

Lemma 4.1. *Suppose G is the direct product of subgroups H_i for $1 \leq i \leq m$. Then the maximal abelian normal subgroups of G are exactly the subgroups of the form $\prod X_i$, where X_i is a maximal abelian normal subgroup of H_i for each subscript i. Furthermore, if all of the H_i are MANL groups, then G is also MANL.*

Proof. Working by induction on m, we see that it suffices to prove the result for $m = 2$. For notational simplicity, therefore, we suppose that $G = H \times K$, and we show that the maximal abelian normal subgroups of G are exactly the subgroups of the form XY, where X and Y are maximal abelian normal subgroups of H and K, respectively. Furthermore, we show that if H and K are MANL groups, then G is also MANL.

Given a maximal abelian normal subgroup A of G, let X and Y be the projections of A into H and K, respectively. Then $A \subseteq XY$, and since the projection maps from G to H and to K are surjective homomorphisms, it follows that X and Y are abelian normal subgroups of H and K.

We argue next that X and Y are maximal among the abelian normal subgroups of H and K. To see this, suppose $X \subseteq S \triangleleft H$ and $Y \subseteq T \triangleleft K$, where S and T are abelian. Then ST is an abelian normal subgroup of G and $A \subseteq XY \subseteq ST$. The maximality of A now guarantees that $A = ST$. It follows easily that $S = X$ and $T = Y$, and thus X and Y are maximal abelian normal subgroups of H and K, as wanted. Also, we have $A = XY$.

Conversely, suppose that X and Y are maximal abelian normal subgroups of H and K, respectively. Then XY is certainly an abelian normal subgroup of G, and we must show that XY is maximal with this property. To see this, suppose that A is a maximal abelian normal subgroup of G containing XY.

By the first part of the proof, $A = UV$, where U and V are the projections of A into H and K. Since $XY \subseteq A$, it follows that $X \subseteq U$ and $Y \subseteq V$, and thus $X = U$ and $Y = V$ by the assumed maximality of X and Y. Then $A = UV = XY$, and thus XY is a maximal abelian normal subgroup of G, as required.

Finally, suppose that H and K are MANL groups, and let A be a maximal abelian normal subgroup of G. By the first part of the proof, $A = XY$, where X and Y are maximal abelian normal subgroups of the MANL groups H and K, and thus X and Y are large in H and K, respectively. Now let $C = \mathbf{C}_G(A)$, and let U and V be the projections of C into H and K. Since the projection maps are homomorphisms, we see that U centralizes the projection of A into H, so $U \subseteq \mathbf{C}_H(X) \subseteq X$, and similarly, $V \subseteq Y$. Then $C \subseteq UV \subseteq XY = A$, and thus A is large in G, as required. □

Theorem 4.2. *Let U be a nontrivial MANL group, and let H be cyclic of prime order p, where H is viewed as a transitive permutation group via its regular representation. Then the wreath product $W = U \wr H$ is an MANL group.*

Proof. Let A be a maximal abelian normal subgroup of W, and let $C = \mathbf{C}_W(A)$, so our goal is to show that $C = A$.

Suppose first that A is contained in the base group B of W. We have seen that

$$B = U_1 \times U_2 \times \cdots \times U_p,$$

where $U_i \cong U$ for all i, and H acts by conjugation on the set $\{U_i \mid 1 \leq i \leq p\}$. The projection maps from B to the factors U_i are surjective group homomorphisms, so the projection A_i of A into U_i is abelian and normal in U_i. It follows that $\prod A_i$ is an abelian normal subgroup of B that contains A. Also, since A is normalized by H and H permutes the subgroups U_i, we see that H permutes the projections A_i, and thus $\prod A_i$ is normalized by H. We have $A \subseteq \prod A_i \lhd W$, and so the maximality of A guarantees that $A = \prod A_i$.

Next, we claim that for each subscript i, the subgroup A_i is maximal among the abelian normal subgroups of U_i. It suffices by symmetry to show that A_1 is a maximal abelian normal subgroup of U_1, so suppose that $A_1 \subseteq T_1 \lhd U_1$, where T_1 is abelian. Let T_i be the unique conjugate of T_1 contained in U_i, and observe that $T = \prod T_i$ is an abelian normal subgroup of W that contains $\prod A_i = A$. By assumption, A is a maximal abelian normal subgroup of W, so $T = A$, and hence $T_1 = A_1$. It follows that A_1 is a maximal abelian subgroup of U_1, and thus for each subscript i, the subgroup A_i is a maximal abelian normal subgroup of U_i, as claimed.

Since $A = \prod A_i$, it follows by Lemma 4.1 that A is a maximal abelian normal subgroup of B. Also, since each of the direct factors U_i of B is isomorphic to the MANL group U, Lemma 4.1 guarantees that B is an MANL group, so $\mathbf{C}_B(A) = A$. Then $C \cap B = A$, and to complete the proof in this case, it suffices to show that $C \subseteq B$.

Given a subscript i, we have seen that A_i is a maximal abelian normal subgroup of the MANL group U_i, and thus A_i is nontrivial. It follows that U_i is the only one of the direct factors U_j that contains A_i, and since C normalizes A_i, it follows that C normalizes U_i. Thus C acts trivially on the set of direct factors U_i of B, and since B is the kernel of the conjugation action on this set, we deduce that $C \subseteq B$, as wanted.

We can now assume that $A \not\subseteq B$, and we show that B is abelian in this case. Since $|W : B|$ is prime, it follows that $BA = W$, and thus $C = (C \cap B)A$. Now let $D = B \cap A$, so D is abelian and it is normal in W. Also, $[B, A] \subseteq D$, so A acts trivially on B/D, and thus A normalizes each of the subgroups U_iD.

Since $BA = W$, it follows that A acts transitively on the set of subgroups U_i, and thus A acts transitively on the set of subgroups U_iD. However A stabilizes each of these subgroups, so it follows that the U_iD are all equal. Then U_1D contains U_i for all i, and thus $U_1D = B$. Now $B/U_1 = U_1D/U_1 \cong D/(D \cap U_1)$, and since D is abelian, we deduce that B/U_1 is abelian, and thus

$B' \subseteq U_1$. Similarly, $B' \subseteq U_2$, so $B' \subseteq U_1 \cap U_2 = 1$, and thus B is abelian, as claimed.

Now $BA = W$ and $A \subseteq C$, and thus $C = (C \cap B)A$. Since B is abelian, we see that $C \cap B$ and A are abelian subgroups that centralize each other, and it follows that C is abelian. The maximality of A thus guarantees that $C = A$, as required. □

Now let G be an arbitrary finite group, and suppose that $1 < N \lhd G$. It is fairly well known that G can be isomorphically embedded in $N \wr (G/N)$, where to construct the wreath product, G/N is viewed as a regular permutation group (acting by right multiplication on itself). This result appears as [2, Satz I.15.9], and also as [1, Theorem A.18.9]. For our proof of Theorem C, we need the following slightly more precise version of this embedding theorem.

To state our result, we recall that the left-regular action of a group H is the action of H on itself given by $x{\cdot}h = h^{-1}x$, where $h, x \in H$. (It is easy to see that the left-regular action of H is permutation-isomorphic to the right-regular action given by $x{\cdot}h = xh$, but we shall not need this fact).

Theorem 4.3. *Let $1 < N \lhd G$, and let $W = N \wr (G/N)$, where we view G/N as a permutation group acting on itself via the left-regular action. Then there exists an injective homomorphism $\theta : G \to W$. Also, if $R \lhd G$, where $R \subseteq N$, then $\theta(R)$ is contained in the subgroup $\tilde{R}$ of the base group $\tilde{N}$ of W.*

Proof. Fix a set of representatives for the cosets of N in G. For an arbitrary element $t \in G$, write $[t]$ to denote the chosen representative for the coset tN, so for arbitrary elements $t, x \in G$, we have $[t] \in tN$, and $x[t] \in x(tN) = [xt]N$. We see, therefore, that $[xt]^{-1}x[t]$ lies in N.

Write $\overline{G} = G/N$ and use the standard "bar convention", so the coset tN is denoted by $\bar{t}$. Now for $x \in G$, we define the function $f_x : \overline{G} \to N$, by writing $f_x(\bar{t}) = [xt]^{-1}x[t]$, and observe that f_x is an element of the base group $\tilde{N}$ of the wreath product $W = N \wr \overline{G}$. (Note that in this situation, the set Ω permuted by $\overline{G}$ is $\overline{G}$ itself, so $\tilde{N}$ is the group of functions from $\overline{G}$ into N).

Given a function $f \in \tilde{N}$, we have

$$f^{\overline{y}}(\bar{t}) = f(\bar{t}{\cdot}\overline{y}^{-1}) = f(\overline{y}\bar{t}),$$

and in particular

$$(f_x)^{\overline{y}}(\bar{t}) = f_x(\overline{y}\bar{t}) = [xyt]^{-1}x[yt].$$

We argue that the map $\theta : G \to W$ defined by $\theta(x) = \overline{x}f_x$ is a homomorphism. We have

$$\theta(x)\theta(y) = \overline{x}f_x\overline{y}f_y = \overline{xy}(f_x)^{\overline{y}}f_y,$$

so to prove that θ is a homomorphism, it suffices to show for all $\bar{t} \in \overline{G}$ that

$$(f_x)^{\overline{y}}(\bar{t})f_y(\bar{t}) = f_{xy}(\bar{t}).$$

We have

$$(f_x)^{\overline{y}}(\bar{t})f_y(\bar{t}) = ([xyt]^{-1}x[yt])([yt]^{-1}y[t]) = [xyt]^{-1}xy[t] = f_{xy}(\bar{t}),$$

as wanted, and thus θ is a homomorphism.

To show that θ is injective, suppose $\theta(x) = 1$, so $f_x\overline{x} = 1$, and in particular, $\overline{x} = 1$, and thus $x \in N$. Also, f_x is the constant function with value 1, so $[xt]^{-1}x[t] = 1$ for all t. Since $x \in N$, we see that $[xt] = [t]$, and it follows that $x = 1$, and thus θ is injective, as required.

Now suppose that $R \lhd G$, with $R \subseteq N$. If $x \in R$, then $x \in N$, and thus $\overline{x} = 1$, so $\theta(x) = f_x$. Now $f_x(\bar{t}) = [xt]^{-1}x[t] = [t]^{-1}x[t] \in R$, because by assumption, $R \lhd G$. Then f_x lies in $\tilde{R}$, as required. □

Proof of Theorem C. Given a solvable group G, we work to embed G as a subgroup of a solvable MANL group Γ, and we want to do this so that each maximal abelian normal subgroup R of G is contained in some maximal abelian normal subgroup A of Γ. Once we accomplish this, we shall have $R \subseteq A \cap G$, and since $A \cap G$ is abelian and normal in G, the maximality of R will guarantee that $R = A \cap G$, as required.

If G is an MANL group, we can take $\Gamma = G$, and there is nothing further to prove. We assume, therefore, that G is not MANL, and in particular, G is not nilpotent, and we proceed by induction on $|G|$.

Since G is solvable but not nilpotent, there exists a normal subgroup N of G such that $\mathbf{F}(G) \subseteq N$ and $|G : N|$ is prime. By the inductive hypothesis, we can assume that N is a subgroup of some solvable MANL subgroup Δ, and in addition, we can assume that each abelian normal subgroup of N is contained in some abelian normal subgroup of Δ.

Now let C be a cyclic group of order p, where $p = |G : N|$. View C as a regular permutation group, and note that $G/N \cong C$, so by Theorem 4.3, there is an isomorphism θ from G into $N \wr C$.

Since $N \subseteq \Delta$, we see that $\tilde{N} \subseteq \tilde{\Delta}$, where $\tilde{N}$ and $\tilde{\Delta}$ are the groups of functions from C into N and Δ, respectively. It follows that there is a natural embedding of $N \wr C = \tilde{N}C$ into $\Delta \wr C = \tilde{\Delta}C$. Also, since Δ is solvable, we see that $\Delta \wr C$ is solvable, and furthermore, $\Delta \wr C$ is an MANL group by Theorem 4.2.

Identifying G with $\theta(G)$ in Theorem 4.3, we can view G as a subgroup of $N \wr C$, so G is also a subgroup of $\Delta \wr C$. To complete the proof, therefore, it suffices to show that each abelian normal subgroup of G is contained in some abelian normal subgroup of $\Delta \wr C$.

Let $R \lhd G$, where R is abelian. Then $R \subseteq \mathbf{F}(G) \subseteq N$, and we observe that by Theorem 4.3, the image $\theta(R)$ of R in $N \wr C$ is contained in the subgroup $\tilde{R}$ of $\tilde{N}$. Since we are identifying G with $\theta(G)$, therefore, we can write $R \subseteq \tilde{R}$.

By the inductive hypothesis, there is an abelian normal subgroup S of Δ such that $R \subseteq S$. Then $\tilde{R} \subseteq \tilde{S}$, and we have $R \subseteq \tilde{R} \subseteq \tilde{S}$.

Since S is abelian, we see that $\tilde{S}$ is abelian. Also $S \lhd \Delta$, and it follows that $\tilde{S} \lhd \tilde{\Delta}$. Since C normalizes $\tilde{S}$, we see that $\tilde{S}$ is a normal abelian subgroup of $\tilde{\Delta}C = \Delta \wr C$. Now R is contained in the abelian normal subgroup $\tilde{S}$ of Δ, and thus $\tilde{S}$ is contained in some maximal abelian subgroup A of $\Delta \wr C$, as required. □

5. Further remarks. Suppose we relax the MANL condition and instead we fix a positive integer r, and we ask if it is true in a group G that every nilpotent normal subgroup maximal with the property that its nilpotence class is at most r is large. (Of course, if $r = 1$, we are asking if G is MANL.) If $r > 1$, the following result shows that the answer is "yes" for all solvable groups G.

Theorem 5.1. *Let $N \triangleleft G$, where G is solvable, and fix an integer $r > 1$. Let N be maximal among nilpotent normal subgroups of G having nilpotence class at most r. Then N is large in G.*

Proof. Let $C = \mathbf{C}_G(N)$, and write $Z = \mathbf{Z}(N)$. Then $C \cap N = Z$, and we must show that $Z = C$. Otherwise, $Z < C$, and we can choose a chief factor K/Z of G with $K \subseteq C$. Then $N \cap K = Z$, so $NK/Z \cong (N/Z) \times (K/Z)$. Now N/Z is nilpotent with class at most $r - 1$, and K/Z is abelian because it is a chief factor of the solvable group G. It follows that NK/Z is nilpotent with class at most $r - 1$. Also, Z is central in NK, and hence that NK is nilpotent with class at most r. The maximality of N thus yields $N = NK \supseteq K$, so $K \subseteq N \cap C = Z$. This is a contradiction because $K/Z > 1$. $\square$

Finally, we mention a consequence of the fact that nilpotent groups are MANL. This pretty result is not new, but perhaps it is not as well known as it should be, and so we have decided to include it here. (See [4, Problem 1D.11].)

Theorem 5.2. *Let G be an arbitrary finite group, and let m be a positive integer. If $|A| \leq m$ for every abelian subgroup A of G, then $|G|$ divides $m!$.*

Proof. It suffices to show for every prime p that the order of a Sylow p-subgroup P of G divides $m!$. To see this, let A be a maximal abelian normal subgroup of P. Since P is nilpotent, it is MANL, and thus $A = \mathbf{C}_P(A)$. It follows that P/A is isomorphic to a subgroup of $\mathrm{Aut}(A)$.

Now $\mathrm{Aut}(A)$ acts faithfully on the set of nonidentity elements of A, so $\mathrm{Aut}(A)$ is isomorphic to a subgroup of the symmetric group of degree $|A| - 1$, and thus $|P/A|$ divides $(|A|-1)!$. We see, therefore, that $|P| = |A||P/A|$ divides $|A|(|A| - 1)! = |A|!$. By hypothesis, $|A| \leq m$, so $|A|!$ divides $m!$, and thus $|P|$ divides $m!$, as required. $\square$

The requirement that G is finite is essential in Theorem 5.2 because there exist infinite groups for which the orders of the abelian subgroups are bounded. For example, a "Tarski monster" has this property. (See [5].)

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