Arch. Math. 111 (2018), 129–133

0003-889X/18/020129-5
published online May 8, 2018
https://doi.org/10.1007/s00013-018-1187-8

Archiv der Mathematik

On classification of groups having Schur multiplier of maximum order II

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Abstract. We complete the classification of finite p-groups having Schur multiplier of maximum order.

Mathematics Subject Classification. 20J99, 20D15.

Keywords. Schur multiplier, Finite p-group.

1. Introduction. Let G be a group. The center and the commutator subgroup of G are denoted by $Z(G)$ and $\gamma_2(G)$, respectively. By $d(G)$ we denote the minimal number of generators of G. We write $\gamma_i(G)$ for the i-th term in the lower central series of G. Finally, the abelianization of the group G, i.e. $G/\gamma_2(G)$, is denoted by G^{ab}.

Let G be a non-abelian p-group of order p^n with $|\gamma_2(G)| = p^k$ and $M(G)$ be its Schur multiplier. Niroomand [4] proved that

$$|M(G)| \leq p^{\frac{1}{2}(n-k-1)(n+k-2)+1}. \tag{1.1}$$

He also classified p-groups such that $k = 1$ and bound (1.1) is attained. Note that if $k = 1$, then the group is of nilpotency class 2. The author ([6]) further classified the groups of nilpotency class 2 such that this bound is attained. Recently Hatui ([2]) proved that there are no p-groups, for $p \neq 3$, of nilpotency class 3 or more attaining the bound. She also gave an example of a 3-group of nilpotency class 3 such that the bound is attained.

In the following theorem we give a shorter proof of Hatui's result and classify 3-groups such that bound (1.1) is attained completing the classification of such groups.

Theorem 1.1. *Let G be a non-abelian finite p-group of order p^n with $|\gamma_2(G)| = p^k$. Then $|M(G)| = p^{\frac{1}{2}(n-k-1)(n+k-2)+1}$ if and only if G is one of the following groups.*

1. $G_1 = E_p \times C_p^{(n-3)}$, *where* E_p *is the extraspecial* p*-group of order* p^3 *and exponent* p *for an odd prime* p,
2. $G_2 = C_p^{(4)} \rtimes C_p$ *for an odd prime* p,
3. $$G_3 = \Big\langle \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \\ [\alpha_i, \beta_j] = 1, \alpha_i^p = \beta_i^p = 1 \ (p \text{ an odd prime}) \ (i, j = 1, 2, 3) \Big\rangle.$$
4. $$G_4 = \Big\langle \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3, \gamma \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \\ [\beta_i, \alpha_i] = \gamma, \quad [\alpha_i, \beta_j] = 1 (i \neq j), \alpha_i^3 = \beta_i^3 = 1 \ (i = 1, 2, 3, \quad j = 1, 2, 3) \Big\rangle.$$

2. Prerequisites. Let G be a finite p-group of nilpotency class c and $\overline{G}$ be the factor group $G/Z(G)$. Define the homomorphism

$$\Psi_2 : \overline{G}^{ab} \otimes \overline{G}^{ab} \otimes \overline{G}^{ab} \mapsto \frac{\gamma_2(G)}{\gamma_3(G)} \otimes \overline{G}^{ab}$$

by $\Psi(\overline{x_1} \otimes \overline{x_2} \otimes \overline{x_3}) = \overline{[x_1, x_2]} \otimes \overline{x_3} + \overline{[x_2, x_3]} \otimes \overline{x_1} + \overline{[x_3, x_1]} \otimes \overline{x_2}$.

For $3 \leq i \leq c$ define homomorphisms

$$\Psi_i : \underbrace{\overline{G}^{ab} \otimes \overline{G}^{ab} \cdots \otimes \overline{G}^{ab}}_{i+1 \text{ times}} \mapsto \frac{\gamma_i(G)}{\gamma_{i+1}(G)} \otimes \overline{G}^{ab}$$

by

$$\begin{aligned} \Psi_i(\overline{x_1} \otimes \overline{x_2} \otimes \cdots \otimes \overline{x_{i+1}}) = {} & \overline{[x_1, x_2, \cdots, x_i]_l} \otimes \overline{x_{i+1}} + \overline{[x_{i+1}, [x_1, x_2, \cdots x_{i-1}]_l]} \otimes \overline{x_i} \\ & + \overline{[[x_i, x_{i+1}]_r, [x_1, \cdots, x_{i-2}]_l]} \otimes \overline{x_{i-1}} \\ & + \overline{[[x_{i-1}, x_i, x_{i+1}]_r, [x_1, x_2, \cdots, x_{i-3}]_l]} \otimes \overline{x_{i-2}} \\ & + \cdots + \overline{[x_2, \cdots, x_{i+1}]_r} \otimes \overline{x_1} \end{aligned}$$

where

$$[x_1, x_2, \cdots x_i]_r = [x_1, [\cdots [x_{i-2}, [x_{i-1}, x_i]] \ldots]$$

and

$$[x_1, x_2, \cdots x_i]_l = [\ldots [[x_1, x_2], x_3], \cdots, x_i].$$

The following proposition was given by Ellis and Wiegold [1, see Proposition 1 and the comments on page 192 following the proof of Theorem 2].

Proposition 2.1. *Let* G *be a finite* p*-group and* $\overline{G}$ *be the factor group* $G/Z(G)$. *Then*

$$\Big|M(G)\Big|\Big|\gamma_2(G)\Big| \prod_{i=2}^{c} \Big|\operatorname{Im} \Psi_i\Big| \leq \Big|M(G^{ab})\Big| \prod_{i=2}^{c} \Big|\frac{\gamma_i(G)}{\gamma_{i+1}(G)} \otimes \overline{G}^{ab}\Big|.$$

The following Lemma is from [5].

Lemma 2.2. [5, Lemma 2.1] *Let G be an abelian p-group of order p^n such that $G = C_{p^{\alpha_1}} \times C_{p^{\alpha_2}} \times \cdots \times C_{p^{\alpha_d}} (\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_d)$, then $|M(G)| \leq p^{\frac{1}{2}(d(G)-1)(n-(\alpha_1-\alpha_d))}$.*

3. Proof of Theorem 1.1. Let $|M(G)| = p^{\frac{1}{2}(n-k-1)(n+k-2)+1}$. In view of [6, Theorem 1.1] suppose that the nilpotency class c of G is at least 3. Let $|\gamma_c(G)| = p^r$. From the exact sequence [3, Corollary 3.2.4 (ii)]

$$1 \mapsto X \mapsto G/\gamma_2(G) \otimes \gamma_c(G) \mapsto M(G) \mapsto M(G/\gamma_c(G)) \mapsto \gamma_c(G) \mapsto 1,$$

it follows that

$$\frac{|M(G)|}{|G/\gamma_2(G) \otimes \gamma_c(G)|} \leq \frac{|M(G/\gamma_c(G))|}{|\gamma_c(G)|}.$$

Since $|G/\gamma_2(G) \otimes \gamma_c(G)| \leq p^{(n-k)r}$, we get that

$$|M(G/\gamma_c(G))| \geq p^{\frac{1}{2}(n-k-1)(n+k-2r-2)+1}.$$

This shows that the bound (1.1) is also attained for the group $G/\gamma_c(G)$. Applying induction, the bound (1.1) is attained for $G/\gamma_3(G)$. But $G/\gamma_3(G)$ is of nilpotency class 2, therefore by [6, Theorem 1.1], $p \neq 2$.

Let Ψ_i be as defined in Section 2, $d(G) = d$ and $d(G/Z(G)) = \delta$. Notice that $\left|\frac{\gamma_i(G)}{\gamma_{i+1}(G)} \otimes \overline{G}^{ab}\right| \leq \left|\frac{\gamma_i(G)}{\gamma_{i+1}(G)}\right|^{d(\overline{G}^{ab})}$. Therefore $\prod_{i=2}^{c} \left|\frac{\gamma_i(G)}{\gamma_{i+1}(G)} \otimes \overline{G}^{ab}\right| \leq |\gamma_2(G)|^{d(\overline{G}^{ab})} = p^{k\delta}$. From Proposition 2.1 we have

$$|M(G)||\gamma_2(G)| \prod_{i=2}^{c} |\operatorname{Im} \Psi_i| \leq |M(G^{ab})| p^{k\delta}.$$

Applying Lemma 2.2 for G^{ab}, we get that

$$|M(G)| \prod_{i=2}^{c} |\operatorname{Im} \Psi_i| \leq p^{\frac{1}{2}(d-1)(n-k)+k(\delta-1)},$$

so that

$$|M(G)| \prod_{i=2}^{c} |\operatorname{Im} \Psi_i| \leq p^{\frac{1}{2}(d-1)(n+k)-k(d-\delta)}. \tag{3.1}$$

Following the proof of [1, Proposition 1], we have $|\operatorname{Im} \Psi_2| \geq p^{\delta-2}$. Therefore from Eq. (3.1) it follows that $d(G) = \delta$ and

$$|M(G)| \prod_{i=3}^{c} |\operatorname{Im} \Psi_i| \leq p^{\frac{1}{2}(d-1)(n+k-2)+1}. \tag{3.2}$$

Now using [4, Theorem 2.2] we see that $d(G) \geq 3$. Suppose $d(G) \geq 4$ and let Ψ_i be the maps as given in Section 2. By simplyfying notations

$$\Psi_3(\overline{x_1} \otimes \overline{x_2} \otimes \overline{x_3} \otimes \overline{x_4}) = \overline{[[x_1, x_2], x_3]} \otimes \overline{x_4} + \overline{[x_4, [x_1, x_2]]} \otimes \overline{x_3} + \overline{[[x_3, x_4], x_1]} \otimes \overline{x_2} + \overline{[x_2, [x_3, x_4]]} \otimes \overline{x_1}.$$

Since G is of nilpotency class at least 3 and $\delta \geq 4$, there exist $x_1, x_2, x_3, x_4 \in G$ such that $[[x_1, x_2], x_3] \notin \gamma_4(G)$ and $x_4\overline{G}^{ab} \notin \langle x_1\overline{G}^{ab}, x_2\overline{G}^{ab}, x_3\overline{G}^{ab}\rangle$. This shows that $\operatorname{Im}\Psi_3 \neq \{1\}$. Which is a contradiction in view of Eq. (3.2). Therefore $d(G) = 3$. Let G be generated by $\alpha_1, \alpha_2, \alpha_3$. Then for $i \neq j$

$$\Psi_3(\overline{\alpha_i} \otimes \overline{\alpha_j} \otimes \overline{\alpha_i} \otimes \overline{\alpha_j}) = 2(\overline{[\alpha_i, \alpha_j, \alpha_i]} \otimes \overline{\alpha_j}) + 2(\overline{[\alpha_j, [\alpha_i, \alpha_j]]} \otimes \overline{\alpha_i}).$$

This shows that $[\alpha_i, \alpha_j, \alpha_i] \in \gamma_4(G)$ because $p \neq 2$. Now for $i \neq j \neq k \neq i$, consider

$$\Psi_3(\overline{\alpha_i} \otimes \overline{\alpha_j} \otimes \overline{\alpha_k} \otimes \overline{\alpha_i}) = \overline{[\alpha_i, \alpha_j, \alpha_k]} \otimes \overline{\alpha_i} + \overline{[\alpha_j, [\alpha_k, \alpha_i]]} \otimes \overline{\alpha_i}.$$

Therefore

$$[\alpha_i, \alpha_j, \alpha_k]\gamma_4(G) = [\alpha_k, \alpha_i, \alpha_j]\gamma_4(G).$$

Putting $(i, j, k) = (1, 2, 3)$ and $(2, 3, 1)$ gives

$$[\alpha_1, \alpha_2, \alpha_3]\gamma_4(G) = [\alpha_3, \alpha_1, \alpha_2]\gamma_4(G)$$

and

$$[\alpha_2, \alpha_3, \alpha_1]\gamma_4(G) = [\alpha_1, \alpha_2, \alpha_3]\gamma_4(G),$$

respectively.

Applying Hall-Witt identity we see that $[\alpha_2, \alpha_3, \alpha_1]^3 \in \gamma_4(G)$. Since $[\alpha_i, \alpha_j, \alpha_i] \in \gamma_4(G)$ and $\gamma_3(G)/\gamma_4(G)$ is non-trivial, we have $[\alpha_1, \alpha_2, \alpha_3] \notin \gamma_4(G)$. It follows that $p = 3$. Let G be a group of nilpotency class at least 4. Consider the map Ψ_4. By simplyfying notations

$$\begin{aligned}\Psi_4(\overline{x_1} \otimes \overline{x_2} \otimes \overline{x_3} \otimes \overline{x_4} \otimes \overline{x_5}) = {} & \overline{[x_1, x_2, x_3, x_4]} \otimes \overline{x_5} + \overline{[x_5, [x_1, x_2, x_3]]} \otimes \overline{x_4} \\ & + \overline{[[x_4, x_5], [x_1, x_2]]} \otimes \overline{x_3} + \overline{[[x_3, [x_4, x_5]], x_1]} \otimes \overline{x_2}. \\ & + \overline{[x_2, [x_3, [x_4, x_5]]]} \otimes \overline{x_1}.\end{aligned}$$

Since $\gamma_4(G)/\gamma_5(G)$ is non-trivial, one of the elements $[\alpha_1, \alpha_2, \alpha_3, \alpha_i]$, $i = 1, 2, 3$ does not belong to $\gamma_5(G)$. Suppose $[\alpha_1, \alpha_2, \alpha_3, \alpha_1] \notin \gamma_5(G)$. Then $\Psi_4(\overline{\alpha_1} \otimes \overline{\alpha_2} \otimes \overline{\alpha_1} \otimes \overline{\alpha_2} \otimes \overline{\alpha_3})$ is non-identity so that $\operatorname{Im}\Psi_4$ is non-trivial. Similarly supposing $[\alpha_1, \alpha_2, \alpha_3, \alpha_2] \notin \gamma_5(G)$, the element $\Psi_4(\overline{\alpha_1} \otimes \overline{\alpha_2} \otimes \overline{\alpha_2} \otimes \overline{\alpha_1} \otimes \overline{\alpha_3})$, while supposing $[\alpha_1, \alpha_2, \alpha_3, \alpha_3] \notin \gamma_5(G)$, the element $\Psi_4(\overline{\alpha_1} \otimes \overline{\alpha_2} \otimes \overline{\alpha_3} \otimes \overline{\alpha_3} \otimes \overline{\alpha_1})$ give that $\operatorname{Im}\Psi_4$ is non-trivial. This, in view of Eq. (3.2), gives a contradiction. Therefore G is a 3-group of nilpotency class 3. Hence we have

$$[\alpha_1, \alpha_2, \alpha_3] = [\alpha_3, \alpha_1, \alpha_2] = [\alpha_2, \alpha_3, \alpha_1].$$

Since $[\alpha_1, \alpha_2, \alpha_3] \neq 1$, we get that $[\alpha_i, \alpha_j] \notin \gamma_3(G)$ for $i, j = 1, 2, 3$ and $i \neq j$. Also, since $[\alpha_i, \alpha_j, \alpha_i] = 1$, it follows that $[\alpha_i, \alpha_j]\gamma_3(G)$ cannot be generated by $\{[\alpha_k, \alpha_l]\gamma_3(G) \mid (k, l) \neq (i, j) \text{ or } (j, i)\}$. This shows that $\gamma_2(G)/\gamma_3(G)$ is generated by 3 elements. Using Eq. (3.2) $G/\gamma_2(G)$ is elementary abelian, so that $\gamma_2(G)/\gamma_3(G)$ is elementary abelian. Hence $|\gamma_2(G)/\gamma_3(G)| = 3^3$. Therefore $|G| = 3^7$. Now it can be checked using GAP ([7]) that the bound is attained if and only if $G = G_4$. This completes the proof.

Acknowledgements. I am very grateful to my post-doctoral supervisor Prof. Boris Kunyavskiĭ for his encouragement and support. I am also thankful to the referee for his/her valuable comments and suggestions. This research was supported by Israel Council for Higher Education's fellowship program and by ISF grant 1623/16.

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Received: 22 January 2018