Arch. Math. 111 (2018), 431–442

0003-889X/18/040431-12
published online April 24, 2018
https://doi.org/10.1007/s00013-018-1186-9

Archiv der Mathematik

Decay rates for the magneto-micropolar system in $L^2(\mathbb{R}^n)$

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Abstract. In this paper, the large time decay of the magneto-micropolar fluid equations on $\mathbb{R}^n$ $(n = 2, 3)$ is studied. We show, for Leray global solutions, that $\|(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{b})(\cdot, t)\|_{L^2(\mathbb{R}^n)} \to 0$ as $t \to \infty$ with arbitrary initial data in $L^2(\mathbb{R}^n)$. When the vortex viscosity is present, we obtain a *(faster)* decay for the micro-rotational field: $\|\boldsymbol{w}(\cdot, t)\|_{L^2(\mathbb{R}^n)} = o(t^{-1/2})$. Some related results are also included.

Mathematics Subject Classification. Primary 35Q35; Secondary 35B40, 76D05.

Keywords. Magneto-micropolar fluid equations, Long-time behavior, Decay rates in L^2.

1. Introduction. In this work, we derive a large time asymptotic decay estimate (see (1.3)) for global Leray solutions of the magneto-micropolar equations in $L^2(\mathbb{R}^n)$, where $n = 2, 3$. The 3D micropolar fluid model, firstly introduced by Eringen in [3], is a substantial generalization of the classical Navier–Stokes equations in the sense that the microstructure of the fluid particles is taken into account. When one considers also the effect of an induced magnetic field on the motion, one gets the more complete magnetic-micropolar fluids. The magneto-micropolar fluid flow in the whole 3D space is governed by the following equations,

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla \mathsf{p} = (\mu+\chi)\,\Delta\boldsymbol{u} + \chi\,\nabla\times\mathbf{w} + \boldsymbol{b}\cdot\nabla\boldsymbol{b}, \tag{1.1a}$$

$$\mathbf{w}_t + \boldsymbol{u}\cdot\nabla\mathbf{w} = \gamma\,\Delta\mathbf{w} + \nabla(\nabla\cdot\,\mathbf{w}) + \chi\,\nabla\times\,\boldsymbol{u} - 2\,\chi\,\mathbf{w}, \tag{1.1b}$$

$$\boldsymbol{b}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{b} = \nu\Delta\boldsymbol{b} + \boldsymbol{b}\cdot\nabla\boldsymbol{u}, \tag{1.1c}$$

$$\nabla\cdot\boldsymbol{u}(\cdot,t) = \nabla\cdot\boldsymbol{b}(\cdot,t) = 0, \tag{1.1d}$$

with initial data $(\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0) \in \boldsymbol{L}^2_\sigma(\mathbb{R}^3)\times\boldsymbol{L}^2(\mathbb{R}^3)\times\boldsymbol{L}^2_\sigma(\mathbb{R}^3)$ and

$$\|(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{b})(\cdot,t) - (\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0)\|_{L^2(\mathbb{R}^3)} \to 0, \quad \text{as } t \to 0.$$

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The two-dimensional magneto-micropolar fluid motion is a special case of the corresponding 3D motion (1.1), $\boldsymbol{u}(x,t) = (u_1(x_1,x_2,t), u_2(x_1,x_2,t), 0)$, $\boldsymbol{w}(x,t) = (0,0,w_3(x_1,x_2,t))$, and $\boldsymbol{b}(x,t) = (b_1(x_1,x_2,t), b_2(x_1,x_2,t), 0)$. Substituting $\boldsymbol{u}$, $\boldsymbol{w}$, and $\boldsymbol{b}$ of the above form into the system (1.1), one gets the following 2D governing equations (see [7]),

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla \mathsf{p} = (\mu+\chi)\,\Delta\boldsymbol{u} + \chi\,\nabla\times\mathbf{w} + \boldsymbol{b}\cdot\nabla\boldsymbol{b}, \tag{1.2a}$$

$$\mathbf{w}_t + \boldsymbol{u}\cdot\nabla\mathbf{w} = \gamma\,\Delta\mathbf{w} + \chi\,\nabla\times\,\boldsymbol{u} - 2\,\chi\,\mathbf{w}, \tag{1.2b}$$

$$\boldsymbol{b}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{b} = \nu\Delta\boldsymbol{b} + \boldsymbol{b}\cdot\nabla\boldsymbol{u}, \tag{1.2c}$$

$$\nabla\cdot\boldsymbol{u}(\cdot,t) = \nabla\cdot\boldsymbol{b}(\cdot,t) = 0, \tag{1.2d}$$

with initial data $(\boldsymbol{u}_0,\boldsymbol{w}_0,\boldsymbol{b}_0) \in \boldsymbol{L}^2_\sigma(\mathbb{R}^2)\times\boldsymbol{L}^2(\mathbb{R}^2)\times\boldsymbol{L}^2_\sigma(\mathbb{R}^2)$ and

$$\|(\boldsymbol{u},\boldsymbol{w},\boldsymbol{b})(\cdot,t) - (\boldsymbol{u}_0,\boldsymbol{w}_0,\boldsymbol{b}_0)\|_{L^2(\mathbb{R}^2)} \to 0,$$

as $t \to 0$.

In (1.1) and (1.2), $\mu,\gamma > 0$ are the kinematic and spin viscosities, ν^{-1} is the magnetic Reynolds number, and $\chi \geq 0$ is the vortex viscosity and $\boldsymbol{u} = \boldsymbol{u}(x,t)$, $\boldsymbol{w} = \boldsymbol{w}(x,t)$, $\boldsymbol{b} = \boldsymbol{b}(x,t)$, and $p = p(x,t)$ are the flow velocity, microrotational velocity, the magnetic field, and the total pressure, respectively, for $t > 0$ and $x \in \mathbb{R}^n$. Here, $\boldsymbol{L}^2_\sigma(\mathbb{R}^n)$ denotes the space of solenoidal fields $\mathbf{v} = (v_1,v_2,\ldots,v_n) \in \boldsymbol{L}^2(\mathbb{R}^3) \equiv L^2(\mathbb{R}^n)^n$ with $\nabla\cdot\mathbf{v} = 0$ in the distributional sense. Our main result can be described as follows.

Theorem 1.1 (Main Theorem). *For a Leray solution* $(\boldsymbol{u},\boldsymbol{w},\boldsymbol{b})(\cdot,t)$ *of* (1.1) *and* (1.2), *one has*

$$\lim_{t\to\infty} \|(\boldsymbol{u},\boldsymbol{w},\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} = 0. \tag{1.3a}$$

Moreover, if $\chi > 0$, *then*

$$\lim_{t\to\infty} t^{1/2}\|\boldsymbol{w}(\cdot,t)\|_{L^2(\mathbb{R}^n)} = 0, \quad \textit{for } n = 2,3. \tag{1.3b}$$

For this purpose, it was necessary to establish some auxiliary results (Section 2), including the following gradient estimate

$$\lim_{t\to\infty} t^{1/2}\|(D\boldsymbol{u},D\boldsymbol{w},D\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} = 0,$$

$n = 2,3$. In the last section, we prove the main theorem.

A.C. Eringen proposed in his paper entitled *Theory of micropolar fluids* (see [3]) a study about the system (1.1) for the case of null magnetic field, i.e., $\boldsymbol{b} = 0$. In the literature, such fluids are called micropolar. Physically, micropolar fluids represent fluids consisting of rigid, randomly oriented (or spherical) particles suspended in a viscous medium, where the deformation of fluid particles is ignored. The non-Newtonian models of micropolar and magnetic-micropolar fluids have been used in modeling a variety of physical phenomena involving suspensions of rigid particles in fluids, such as human blood, polymeric suspensions, and so on, and therefore have found many applications in physiological and engineering problems. For more information on these type of fluids, see [8] and the references therein.

There are many results on the existence and uniqueness of solutions for problems related. The two-dimensional problem has been extensively studied and many interesting results involving the existence of solutions and asymptotic behavior have been established considering zero and partial viscosities, see, e.g., [1,2,7,10,11].

Notation. As usual, $\dot{\boldsymbol{H}}^1(\mathbb{R}^n) = \dot{H}^1(\mathbb{R}^n)^n$ where $\dot{H}^1(\mathbb{R}^n)$ denotes the homogeneous Sobolev space of order 1, $e^{\nu\Delta t}$ denotes the heat semigroup and $C_w(I, \boldsymbol{L}^2(\mathbb{R}^n))$ denotes the set of mappings from a given interval $I \subseteq \mathbb{R}$ to $\boldsymbol{L}^2(\mathbb{R}^n)$ that are L^2-weakly continuous at each $t \in I$, for $n = 2, 3$. As shown above, boldface letters are used for vector quantities, as in $\boldsymbol{u}(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$. Also, $\nabla p \equiv \nabla p(\cdot,t)$ denotes the spatial gradient of $p(\cdot,t)$, $D_j = \partial/\partial x_j$, $\nabla\cdot \boldsymbol{u} = D_1u_1 + D_2u_2 + D_3u_3$ is the (spatial) divergence of $\boldsymbol{u}(\cdot,t)$. $|\cdot|_2$ denotes the Euclidean norm in $\mathbb{R}^3$, and $\|\cdot\|_{L^q(\mathbb{R}^3)}$, $1 \leq q \leq \infty$, are the standard norms of the Lebesgue spaces $L^q(\mathbb{R}^3)$, with the vector counterparts

$$\| \boldsymbol{u}(\cdot,t) \|_{L^q(\mathbb{R}^n)} = \left\{ \sum_{i=1}^{n} \int_{\mathbb{R}^n} | u_i(x,t) |^q \, dx \right\}^{1/q} \tag{1.4a}$$

$$\| D\boldsymbol{u}(\cdot,t) \|_{L^q(\mathbb{R}^n)} = \left\{ \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} | D_j u_i(x,t) |^q \, dx \right\}^{1/q} \tag{1.4b}$$

and, in general,

$$\| D^m\boldsymbol{u}(\cdot,t) \|_{L^q(\mathbb{R}^n)} = \left\{ \sum_{i,j_1,\ldots,j_m=1}^{n} \int_{\mathbb{R}^n} | D_{j_1}\cdots D_{j_m} u_i(x,t) |^q \, dx \right\}^{1/q} \tag{1.4c}$$

if $1 \leq q < \infty$.

When, $q = \infty$,

$$\|\boldsymbol{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} = \max\{\|u_i(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} : 1 \leq i \leq n\} \tag{1.4d}$$

and, for general $m \geq 1$:

$$\| D^m\boldsymbol{u}(\cdot,t) \|_{L^\infty(\mathbb{R}^n)} = \max\left\{ \| D_{j_1}\cdots D_{j_m} u_i(\cdot,t) \|_{L^\infty(\mathbb{R}^n)}: 1 \leq i, j_1, \ldots, j_m \leq n \right\}. \tag{1.4e}$$

Definitions (1.4) are convenient, but not essential. However, some choice for the vector norms has to be made to fix the values of constants. We also defined for simplicity the following norms for $(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{b})$ as usually made in the literature:

$$\|(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{b})\|^q_{L^q(\mathbb{R}^n)} := \|\boldsymbol{u}\|^q_{L^q(\mathbb{R}^n)} + \|\boldsymbol{w}\|^q_{L^q(\mathbb{R}^n)} + \|\boldsymbol{b}\|^q_{L^q(\mathbb{R}^n)} \tag{1.4f}$$

and more generally, for all integers $m \geq 1$

$$\|(D^m\boldsymbol{u}, D^m\boldsymbol{w}, D^m\boldsymbol{b})\|^q_{L^q(\mathbb{R}^n)} := \|D^m\boldsymbol{u}\|^q_{L^q(\mathbb{R}^n)} + \|D^m\boldsymbol{w}\|^q_{L^q(\mathbb{R}^n)} + \|D^m\boldsymbol{b}\|^q_{L^q(\mathbb{R}^n)} \tag{1.4g}$$

for all $1 \leq q < \infty$ and when $q = \infty$,

$$\|(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)} = \max\{\|\boldsymbol{u}\|_{L^\infty(\mathbb{R}^n)}, \|\boldsymbol{w}\|_{L^\infty(\mathbb{R}^n)}, \|\boldsymbol{b}\|_{L^\infty(\mathbb{R}^n)}\}. \tag{1.4h}$$

The constants will be represented by the letters C, c, or K. For economy, we will use typically the same symbol to denote constants with different numerical values.

2. Preliminaries. First, we will focus in the 3D case. Although it is not known that Leray solutions to the problem (1.1) are smooth for all $t > 0$, it is known that they do behave nicely for all $t > 0$ sufficiently large [8,9], say $t > t_*$, with

$$(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{b})(\cdot, t) \in C^\infty(\mathbb{R}^3 \times (t_*, \infty)) \tag{2.1a}$$

and, for each $m \in \mathbb{Z}_+$:

$$(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{b})(\cdot, t) \in C^0([t_*, \infty), \boldsymbol{H}^m(\mathbb{R}^3)) \tag{2.1b}$$

and such that the *strong* energy inequality

$$\begin{aligned} &\|(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{b})(\cdot, t)\|^2_{L^2(\mathbb{R}^3)} + 2\mu \int_{t_0}^{t} \|D\boldsymbol{u}(\cdot, \tau)\|^2_{L^2(\mathbb{R}^3)} d\tau \\ &\quad + 2\gamma \int_{t_0}^{t} \|D\boldsymbol{w}(\cdot, \tau)\|^2_{L^2(\mathbb{R}^3)} d\tau + 2\nu \int_{t_0}^{t} \|D\boldsymbol{b}(\cdot, \tau)\|^2_{L^2(\mathbb{R}^3)} d\tau \\ &\quad + 2 \int_{t_0}^{t} \|\nabla \cdot \boldsymbol{w}(\cdot, \tau)\|^2_{L^2(\mathbb{R}^3)} d\tau + 2\chi \int_{t_0}^{t} \|\boldsymbol{w}(\cdot, \tau)\|^2_{L^2(\mathbb{R}^3)} d\tau \\ &\leq \|(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{b})(\cdot, t_0)\|^2_{L^2(\mathbb{R}^3)}, \quad \forall t > t_0 \end{aligned} \tag{2.2}$$

for a.e $t_0 \geq 0$, including $t_0 = 0$. Similarly, for the 2D case, one has

$$\begin{aligned} &\|(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{b})(\cdot, t)\|^2_{L^2(\mathbb{R}^2)} + 2\mu \int_{t_0}^{t} \|D\boldsymbol{u}(\cdot, \tau)\|^2_{L^2(\mathbb{R}^2)} d\tau \\ &+ 2\gamma \int_{t_0}^{t} \|D\boldsymbol{w}(\cdot, \tau)\|^2_{L^2(\mathbb{R}^2)} d\tau + 2\nu \int_{t_0}^{t} \|D\boldsymbol{b}(\cdot, \tau)\|^2_{L^2(\mathbb{R}^2)} d\tau \\ &+ 2\chi \int_{t_0}^{t} \|\boldsymbol{w}(\cdot, \tau)\|^2_{L^2(\mathbb{R}^2)} d\tau \leq \|(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{b})(\cdot, t_0)\|^2_{L^2(\mathbb{R}^2)}, \quad \forall t > t_0 \end{aligned} \tag{2.3}$$

and we actually have $t_* = 0$ in this case [8].

We will now establish some lemmas that are necessary to our analysis in the next section.

Lemma 2.1. *For* $(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{b})$ *Leray soltions of* (1.1) *and* (1.2), *one has*

$$\lim_{t \to \infty} t^{1/2} \|(D\boldsymbol{u}, D\boldsymbol{w}, D\boldsymbol{b})(\cdot, t)\|_{L^2(\mathbb{R}^n)} = 0, \quad n = 2, 3. \tag{2.4}$$

Proof. This next argument is adapted from [4]. Define, for simplicity, $z(\cdot, t) := (\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{b})(\cdot, t)$ and $D^m \boldsymbol{z} = (D^m \boldsymbol{u}, D^m \boldsymbol{w}, D^m \boldsymbol{w})$, for each $m \geq 0$ integer. In order to show (2.4), we use (1.1) and (2.1) to get, after a few computations,

$$\begin{aligned}
&\|D\boldsymbol{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\min\{\mu,\gamma,\nu\}\int_{t_0}^{t}\|D^2\boldsymbol{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau\\
&\quad + 2\int_{t_0}^{t}\|D\nabla\cdot\boldsymbol{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau + 2\chi\int_{t_0}^{t}\|D\boldsymbol{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau\\
&\quad\le \|D\boldsymbol{z}(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)}\\
&\quad + C\int_{t_0}^{t}\|\boldsymbol{z}(\cdot,\tau)\|^{1/2}_{L^2(\mathbb{R}^3)}\|D\boldsymbol{z}(\cdot,\tau)\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^2\boldsymbol{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau, \qquad (2.5)
\end{aligned}$$

where we have used a Sobolev–Nirenberg–Gagliardo (SNG) inequality (see (2.7)). By (2.2), we can choose $t_0 \ge t_*$ large enough such that

$$C^2\|(\boldsymbol{u}_0,\boldsymbol{w}_0,\boldsymbol{b}_0)\|_{L^2(\mathbb{R}^3)}\|(D\boldsymbol{u},D\boldsymbol{w},D\boldsymbol{b})(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} < (\min\{\mu,\gamma,\nu\})^2,$$

so that (2.5) gives $\|(D\boldsymbol{u},D\boldsymbol{w},D\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^3)} \le \|(D\boldsymbol{u},D\boldsymbol{w},D\boldsymbol{b})(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}$ for all t near t_0 by continuity. Actually, with this choice, it follows from [(2.5) again] that

$$C^2\|(\boldsymbol{u}_0,\boldsymbol{w}_0,\boldsymbol{b}_0)\|_{L^2(\mathbb{R}^3)}\|(D\boldsymbol{u},D\boldsymbol{w},D\boldsymbol{b})(\cdot,s)\|_{L^2(\mathbb{R}^3)} < (\min\{\mu,\gamma,\nu\})^2,\ \forall s \ge t_0.$$

Recalling (2.5), this implies that

$$\|(D\boldsymbol{u},D\boldsymbol{w},D\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^3)} \le \|(D\boldsymbol{u},D\boldsymbol{w},D\boldsymbol{b})(\cdot,t_0)\|_{L^2(\mathbb{R}^3)},$$

for all $t \ge t_0$. For $n=2$, using the same argument, one has

$$\|(D\boldsymbol{u},D\boldsymbol{w},D\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} \le \|(D\boldsymbol{u},D\boldsymbol{w},D\boldsymbol{b})(\cdot,t_0)\|_{L^2(\mathbb{R}^n)},\quad t > t_0, \qquad (2.6)$$

for $n = 2, 3$.

Because a monotonic function $f \in C^0((a,\infty)) \cap L^1((a,\infty))$ has to satisfy $f(t) = o(1/t)$ as $t \to \infty$ (see, e.g., [4, p. 236]), we have

$$\lim_{t\to\infty} t\|(D\boldsymbol{u},D\boldsymbol{w},D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} = 0,\quad n = 2,3.$$

□

In $\mathbb{R}^n$, for $n = 2, 3$, we observe that pointwise values of functions can be estimated in terms of H^2 norms.

Lemma 2.2.

$$\begin{aligned}
&\|(\boldsymbol{u},\boldsymbol{w},\boldsymbol{b})\|_{L^\infty(\mathbb{R}^3)}\|(D\boldsymbol{u},D\boldsymbol{w},D\boldsymbol{b})\|_{L^2(\mathbb{R}^3)}\\
&\quad\le C\|(\boldsymbol{u},\boldsymbol{w},\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^3)}\|(D\boldsymbol{u},D\boldsymbol{w},D\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^3)}\|(D^2\boldsymbol{u},D^2\boldsymbol{w},D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^3)}. \qquad (2.7)
\end{aligned}$$

For $\mathbb{R}^2$ one has,

Lemma 2.3.

$$\begin{aligned}
&\|(\boldsymbol{u},\boldsymbol{w},\boldsymbol{b})\|_{L^\infty(\mathbb{R}^2)}\|(D\boldsymbol{u},D\boldsymbol{w},D\boldsymbol{b})\|_{L^2(\mathbb{R}^2)}\\
&\quad\le C\|(\boldsymbol{u},\boldsymbol{w},\boldsymbol{b})\|_{L^2(\mathbb{R}^2)}\|(D^2\boldsymbol{u},D^2\boldsymbol{w},D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^2)},
\end{aligned}$$

Finally, a well known heat kernel estimate is used,

Lemma 2.4. *Let $u \in L^r(\mathbb{R}^n)$ and $e^{\nu\Delta\tau}$ the heat kernel, then*

$$\| D^\alpha [e^{\nu\Delta\tau} u] \|_{L^2(\mathbb{R}^n)} \le K(n,m) \| u \|_{L^r(\mathbb{R}^n)} (\nu\tau)^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{|\alpha|}{2}} \tag{2.8}$$

for all $\tau > 0$ and α (multi-index), $1 \le r \le 2$, $n \ge 1$, and $m = |\alpha|$. (For a proof of (2.8), see, e.g., [5,6])

3. Proof of the main theorem. We start with $n = 3$. First, we will prove the result for the field $\boldsymbol{w}(\cdot,t)$. By (2.1), given $\epsilon > 0$, there exists $t_0 > 0$ such that

$$\|(D\boldsymbol{u}, D\boldsymbol{w}, D\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^3)} < \epsilon t^{-\frac{1}{2}}, \quad \forall t > t_0. \tag{3.1}$$

We begin with the inviscid vortex case, i.e., $\chi = 0$. By Duhamel's principle, we get

$$\begin{aligned}\|\boldsymbol{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)} &\le \underbrace{\|e^{\gamma\Delta(t-t_0)}\boldsymbol{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}}_{I} \\ &+ \underbrace{\int_{t_0}^{t} \|e^{\gamma\Delta(t-s)}(\boldsymbol{u}\cdot\nabla\boldsymbol{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)} ds}_{II} + \underbrace{\int_{t_0}^{t} \|e^{\gamma\Delta(t-s)}\nabla(\nabla\cdot\boldsymbol{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)} ds}_{III}.\end{aligned} \tag{3.2}$$

I is the solution of heat equation with initial condition $\boldsymbol{w}(\cdot,t_0) \in L^2(\mathbb{R}^3)$, and so,

$$\lim_{t\to\infty} \|e^{\gamma\Delta(t-t_0)}\boldsymbol{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} = 0. \tag{3.3}$$

To estimate the other terms, we use Lemma 2.4.

$$\begin{aligned} II &= \int_{t_0}^{t} \|e^{\gamma\Delta(t-s)}\boldsymbol{u}\cdot\nabla\boldsymbol{w}\|_{L^2(\mathbb{R}^3)} ds \le K\gamma^{-3/4} \int_{t_0}^{t} (t-s)^{-3/4} \|\boldsymbol{u}\cdot\nabla\boldsymbol{w}\|_{L^1(\mathbb{R}^3)} ds \\ &\le K\gamma^{-3/4} \int_{t_0}^{t} (t-s)^{-3/4} \|\boldsymbol{u}\|_{L^2(\mathbb{R}^3)} \|D\boldsymbol{w}\|_{L^2(\mathbb{R}^3)} ds \\ &\le K\epsilon\gamma^{-3/4} \|(\boldsymbol{u},\boldsymbol{w},\boldsymbol{b})(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} \int_{t_0}^{t} (t-s)^{-3/4} s^{-1/2} ds \\ &\le \int_{t_0}^{t} (t-s)^{-3/4} s^{-1/2} ds \le C t^{-1/4}. \end{aligned}$$

Therefore,

$$\lim_{t\to\infty} \int_{t_0}^{t} \|e^{\gamma\Delta(t-s)}(\boldsymbol{u}\cdot\nabla\boldsymbol{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)} ds = 0. \tag{3.4}$$

Similarly,

$$\lim_{t\to\infty}\int_{t_0}^{t}\|e^{\gamma\Delta(t-s)}\nabla(\nabla\cdot\boldsymbol{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)}ds=0. \tag{3.5}$$

That is, if $\chi=0$, then

$$\lim_{t\to\infty}\|\boldsymbol{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)}=0. \tag{3.6}$$

Now, suppose that $\chi>0$. Defining $\boldsymbol{z}=e^{2\chi t}\boldsymbol{w}$ and applying Duhamel's principle, we obtain that

$$\begin{aligned}\boldsymbol{w}(\cdot,t)&=e^{-2\chi(t-t_0)}e^{\gamma\Delta(t-t_0)}\boldsymbol{w}(\cdot,t_0)-\int_{t_0}^{t}e^{-2\chi(t-s)}e^{\gamma\Delta(t-s)}(\boldsymbol{u}\cdot\nabla\boldsymbol{w})(\cdot,s)ds\\&\quad+\int_{t_0}^{t}e^{-2\chi(t-s)}e^{\gamma\Delta(t-s)}\nabla(\nabla\cdot\boldsymbol{w})(\cdot,s)ds\\&\quad+\chi\int_{t_0}^{t}e^{-2\chi(t-s)}e^{\gamma\Delta(t-s)}(\nabla\times\boldsymbol{u})(\cdot,s)ds.\end{aligned}$$

Hence,

$$\begin{aligned}t^{\frac{1}{2}}\|\boldsymbol{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)}\le&\ \underbrace{t^{\frac{1}{2}}e^{-2\chi(t-t_0)}\|e^{\gamma\Delta(t-t_0)}\boldsymbol{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}}_{I}\\&+\underbrace{t^{\frac{1}{2}}\int_{t_0}^{t}e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}\boldsymbol{u}\cdot\nabla\boldsymbol{w}\|_{L^2(\mathbb{R}^3)}ds}_{II}\\&+\underbrace{t^{\frac{1}{2}}\int_{t_0}^{t}e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}\nabla(\nabla\cdot\boldsymbol{w})\|_{L^2(\mathbb{R}^3)}ds}_{III}\\&+\underbrace{\chi t^{\frac{1}{2}}\int_{t_0}^{t}e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}\nabla\times\boldsymbol{u}\|_{L^2(\mathbb{R}^3)}ds}_{IV}.\end{aligned}$$

By (3.3), we have that

$$\lim_{t\to\infty}t^{\frac{1}{2}}e^{-2\chi(t-t_0)}\|e^{\gamma\Delta(t-t_0)}\boldsymbol{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}=0.$$

To estimate the other terms, we use again Lemma 2.4.

$$
\begin{aligned}
II &= t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}\boldsymbol{u}\cdot\nabla\boldsymbol{w}\|_{L^2(\mathbb{R}^3)}ds\\
&\le K\gamma^{-3/4}t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-s)}(t-s)^{-3/4}\|\boldsymbol{u}\cdot\nabla\boldsymbol{w}\|_{L^1(\mathbb{R}^3)}ds\\
&\le K\gamma^{-3/4}t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-s)}(t-s)^{-3/4}\|\boldsymbol{u}\|_{L^2(\mathbb{R}^3)}\|D\boldsymbol{w}\|_{L^2(\mathbb{R}^3)}ds\\
&\le K\gamma^{-3/4}\|(\boldsymbol{u},\boldsymbol{w},\boldsymbol{b})(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-s)}(t-s)^{-3/4}\|D\boldsymbol{w}\|_{L^2(\mathbb{R}^3)}ds\\
&\le K\epsilon\gamma^{-3/4}\|(\boldsymbol{u},\boldsymbol{w},\boldsymbol{b})(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-s)}(t-s)^{-3/4}s^{-1/2}ds\\
&\le K\epsilon\gamma^{-3/4}\|(\boldsymbol{u},\boldsymbol{w},\boldsymbol{b})(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}\left(e^{-\chi t}t^{1/4}+(2\chi)^{-1/4}\Gamma\left(\frac{1}{4}\right)\right),
\end{aligned}
$$

where Γ is the so-called gamma function. Therefore

$$\lim_{t\to\infty} t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}\boldsymbol{u}\cdot\nabla\boldsymbol{w}\|_{L^2(\mathbb{R}^3)}ds = 0.$$

Similarly,

$$\lim_{t\to\infty} t^{\frac{1}{2}}\int_{t_0}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}\nabla(\nabla\cdot\boldsymbol{w})\|_{L^2(\mathbb{R}^3)}ds = 0.$$

And, using again the same idea

$$\lim_{t\to\infty} \chi t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}(\nabla\times\boldsymbol{u})(\cdot,s)\|_{L^2(\mathbb{R}^3)}ds = 0. \tag{3.7}$$

That is, if $\chi > 0$, then

$$\lim_{t\to\infty} t^{1/2}\|\boldsymbol{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)} = 0. \tag{3.8}$$

Now, we show that $\|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \to 0$ whether $\chi = 0$ or $\chi > 0$. Rewrite the equation (1.1a) as

$$\boldsymbol{u}_t = (\mu+\chi)\Delta\boldsymbol{u} + \mathbf{F}(\cdot,\tau),$$

where $\mathbf{F}(\cdot,\tau) = \chi\nabla\times\boldsymbol{w} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla p + \boldsymbol{b}\cdot\nabla\boldsymbol{b}$. We can write $\mathbf{F}(\cdot,\tau)$ as

$$\mathbf{F}(\cdot,\tau) = \mathbb{P}_h[\chi\nabla\times\boldsymbol{w} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \boldsymbol{b}\cdot\nabla\boldsymbol{b}],$$

where $\mathbb{P}_h$ denotes the Helmholtz–Leray projector [6,8,9]. By Duhamel's principle (again)

$$\boldsymbol{u}(\cdot,t)=e^{(\mu+\chi)\Delta(t-t_0)}\boldsymbol{u}(\cdot,t_0)+\int_{t_0}^{t}e^{(\mu+\chi)\Delta(t-s)}\mathbf{F}(\cdot,\tau)(\cdot,s)ds.$$

Since the heat kernel commutes with the Helmholtz projector, we get the following inequality,

$$\begin{aligned}\|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^3)}\le&\underbrace{\|e^{(\mu+\chi)\Delta(t-t_0)}\boldsymbol{u}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}}_{I}\\&+\underbrace{\int_{t_0}^{t}\|e^{(\mu+\chi)\Delta(t-s)}(\boldsymbol{u}\cdot\nabla\boldsymbol{u})(\cdot,s)\|_{L^2(\mathbb{R}^3)}ds}_{II}\\&+\underbrace{\chi\int_{t_0}^{t}\|e^{(\mu+\chi)\Delta(t-s)}(\nabla\times\boldsymbol{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)}ds\,.}_{III}\\&+\underbrace{\int_{t_0}^{t}\|e^{(\mu+\chi)\Delta(t-s)}(\boldsymbol{b}\cdot\nabla\boldsymbol{b})(\cdot,s)\|_{L^2(\mathbb{R}^3)}ds\,.}_{IV}\end{aligned}$$

As in (3.3), I is the solution of the heat equation with initial condition $\boldsymbol{u}(\cdot,t_0)$. Hence,

$$\lim_{t\to\infty}\|e^{\gamma\Delta(t-t_0)}\boldsymbol{u}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}=0.$$

To estimate the other terms, we use again Lemma 2.4.

$$\begin{aligned}&\int_{t_0}^{t}\|e^{(\mu+\chi)\Delta(t-s)}\boldsymbol{u}\cdot\nabla\boldsymbol{u}\|_{L^2(\mathbb{R}^3)}ds\le K(\mu+\chi)^{-\frac{3}{4}}\int_{t_0}^{t}(t-s)^{-\frac{3}{4}}\|\boldsymbol{u}\cdot\nabla\boldsymbol{u}\|_{L^1(\mathbb{R}^3)}ds\\&\le K(\mu+\chi)^{-\frac{3}{4}}\int_{t_0}^{t}(t-s)^{-\frac{3}{4}}\|\boldsymbol{u}(\cdot,s)\|_{L^2(\mathbb{R}^3)}\|D\boldsymbol{u}(\cdot,s)\|_{L^2(\mathbb{R}^3)}ds\\&\le K(\mu+\chi)^{-\frac{3}{4}}\|(\boldsymbol{u},\boldsymbol{w},\boldsymbol{b})(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}\int_{t_0}^{t}(t-s)^{-\frac{3}{4}}\|D\boldsymbol{u}(\cdot,s)\|_{L^2(\mathbb{R}^3)}ds\\&<K\epsilon\int_{t_0}^{t}(t-s)^{-\frac{3}{4}}s^{-\frac{1}{2}}ds\le K\epsilon t^{-\frac{1}{4}}.\end{aligned}\tag{3.9}$$

Therefore

$$\lim_{t\to\infty}\int_{t_0}^{t}\|e^{(\mu+\chi)\Delta(t-s)}(\boldsymbol{u}\cdot\nabla\boldsymbol{u})(\cdot,s)\|_{L^2(\mathbb{R}^3)}ds=0.$$

We only need to worry about III when $\chi>0$, but in this case we may assume that t_0 is large enough [see (3.8)] such that

$$\|\boldsymbol{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)}<\epsilon t^{-\frac{1}{2}},\qquad \forall t>t_0$$

and so, we have

$$\begin{aligned}\chi\int_{t_0}^{t}\|e^{(\mu+\chi)\Delta(t-s)}\nabla\times\boldsymbol{w}\|_{L^2(\mathbb{R}^3)}ds\\ &\le K\chi(\mu+\chi)^{-1/2}\int_{t_0}^{t}(t-s)^{-1/2}\|\boldsymbol{w}\|_{L^2(\mathbb{R}^3)}ds\\ &\le K\chi\epsilon(\mu+\chi)^{-1/2}\int_{t_0}^{t}(t-s)^{-1/2}s^{-1/2}ds\le K\chi\epsilon(\mu+\chi)^{-1/2}.\end{aligned}$$

Therefore

$$\lim_{t\to\infty}\chi\int_{t_0}^{t}\|e^{(\mu+\chi)\Delta(t-s)}(\nabla\times\boldsymbol{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)}ds=0 \tag{3.10}$$

and the last term (IV) can be estimated following (3.9). That is,

$$\lim_{t\to\infty}\|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^3)}=0. \tag{3.11}$$

Similarly, in order to show that $\|\boldsymbol{b}(\cdot,t)\|_{L^2(\mathbb{R}^3)}\to 0$ as $t\to\infty$, we can apply the same previous idea to prove (3.11). More specifically,

$$\begin{aligned}\|\boldsymbol{b}(\cdot,t)\|_{L^2(\mathbb{R}^3)}\le \underbrace{\|e^{(\mu+\chi)\Delta(t-t_0)}\boldsymbol{b}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}}_{I}\\ +\underbrace{\int_{t_0}^{t}\|e^{(\nu)\Delta(t-s)}(\boldsymbol{b}\cdot\nabla\boldsymbol{u})(\cdot,s)\|_{L^2(\mathbb{R}^3)}ds}_{II}+\underbrace{\int_{t_0}^{t}\|e^{(\nu)\Delta(t-s)}(\boldsymbol{u}\cdot\nabla\boldsymbol{b})(\cdot,s)\|_{L^2(\mathbb{R}^3)}ds}_{III}\end{aligned}$$

and, as before, (I), (II), and (III) above can be estimated without problems [see (3.9)]. That is,

$$\lim_{t\to 0}\|\boldsymbol{b}(\cdot,t)\|_{L^2(\mathbb{R}^3)}=0. \tag{3.12}$$

For $n = 2$, we can apply the previous argument for the system (1.2) and use the Lemma (2.3) to show (3.6), (3.8), (3.11), and (3.12) in the two-dimensional case. Hence the proof of the main theorem is complete.

Acknowledgements. We would like to express our gratitude to Prof. P. Zingano for his interest in this work and various helpful suggestions. We also thank the Brazilian agencies CAPES and CNPq for their financial support to this work (Under CNPq Grant 140445/2017-9 and CAPES Grants 1510610, 1442127.)

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Received: 9 January 2018