Arch. Math. 113 (2019), 63–72

0003-889X/19/010063-10
published online March 2, 2019
https://doi.org/10.1007/s00013-018-01297-z

Archiv der Mathematik

Plurisubharmonic geodesics and interpolating sets

Dario Cordero-Erausquin and Alexander Rashkovskii

Abstract. We apply a notion of geodesics of plurisubharmonic functions to interpolation of compact subsets of $\mathbb{C}^n$. Namely, two non-pluripolar, polynomially closed, compact subsets of $\mathbb{C}^n$ are interpolated as level sets $L_t = \{z : u_t(z) = -1\}$ for the geodesic u_t between their relative extremal functions with respect to any ambient bounded domain. The sets L_t are described in terms of certain holomorphic hulls. In the toric case, it is shown that the relative Monge–Ampère capacities of L_t satisfy a dual Brunn–Minkowski inequality.

Mathematics Subject Classification. Primary 32E30; Secondary 32U05, 32U15, 32U20, 52A20, 52A38, 52A40.

Keywords. Complex interpolation, Plurisubharmonic geodesic, Relative extremal function, Monge–Ampère capacity, Brunn–Minkowski inequality.

1. Introduction. In the classical complex interpolation theory of Banach spaces, originated by Calderón [6] (see [2] and, for more recent developments, [8] and references therein), a given family of Banach spaces X_ξ parameterized by boundary points ξ of a domain $C \subset \mathbb{C}^N$ gives rise to a family of Banach spaces X_ζ for all $\zeta \in C$. A basic setting is interpolation of two spaces, X_0 and X_1, for a partition $\{C_0, C_1\}$ of ∂C. More specifically, one can take C to be the strip $0 < \operatorname{Re}\zeta < 1$ in the complex plane and C_0, C_1 the corresponding boundary lines, then the interpolated norms depend only on $t = \operatorname{Im}\zeta$. In the finite dimensional case $X_j = (\mathbb{C}^n, \|\cdot\|_j)$, $j = 0, 1$, they are defined in terms of the family of mappings $C \to \mathbb{C}^n$, bounded and analytic in the strip, continuous up to the boundary and tending to zero as $\operatorname{Im}\zeta \to \infty$, see details in [2]. In this setting, the volume of the unit ball B_t of $(\mathbb{C}^n, \|\cdot\|_t)$, $0 < t < 1$, was proved in [7] to be a logarithmically concave function of t.

When the given norms $\|\cdot\|_j$ on $\mathbb{C}^n$ are toric, i.e., $\|(z_1,\ldots,z_n)\|_j = \|(|z_1|,\ldots,|z_n|)\|_j$, the interpolated norms are toric as well and the balls B_t are Reinhardt domains of $\mathbb{C}^n$ obtained as the multiplicative combinations (geometric means) of the balls B_0 and B_1. The logarithmic concavity implies that volumes of the multiplicative combinations

$$K_t^{\times} = K_0^{1-t}\, K_1^t \subset \mathbb{R}^n \tag{1.1}$$

of any two convex bounded neighbourhoods K_0 and K_1 of the origin in $\mathbb{R}^n$ satisfy the Brunn–Minkowski inequality

$$\mathrm{Vol}(K_t^{\times}) \geq \mathrm{Vol}(K_0)^{1-t}\mathrm{Vol}(K_1)^t, \quad 0 < t < 1. \tag{1.2}$$

Note also that in [20–22], the interpolated spaces were related to convex hulls and complex geodesics with convex fibers. In particular, it put the interpolation in the context of analytic multifunctions.

In this note, we develop a slightly different—albeit close—approach to the interpolation of compact, polynomially convex subsets of $\mathbb{C}^n$ by sets arising from a notion of plurisubharmonic geodesics. The technique originates from results on geodesics in the spaces of metrics on compact Kähler manifolds due to Mabuchi, Semmes, Donaldson, Berndtsson, and others (see [10] and the bibliography therein). Its local counterpart for plurisubharmonic functions from Cegrell classes on domains of $\mathbb{C}^n$ was introduced in [3,17]. We will need here a special case when the geodesics can be described as follows.

Let

$$A = \{\zeta \in \mathbb{C} :\ 0 < \log|\zeta| < 1\}$$

be the annulus bounded by the circles

$$A_j = \{\zeta :\ \log|\zeta| = j\}, \quad j = 0, 1.$$

Given two plurisubharmonic functions u_0 and u_1 in a bounded hyperconvex domain $\Omega \subset \mathbb{C}^n$, equal to zero on $\partial\Omega$, we consider the class $W(u_0, u_1)$ of all plurisubharmonic functions $u(z,\zeta)$ in the product domain $\Omega \times A$, such that $\limsup u(z,\zeta) \leq u_j(z)$ for all $z \in \Omega$ as $\zeta \to A_j$. The function

$$\widehat{u}(z,\zeta) = \sup\{u(z,\zeta) :\ u \in W(u_0,u_1)\} \tag{1.3}$$

belongs to the class and satisfies $\widehat{u}(z,\zeta) = \widehat{u}(z,|\zeta|)$, which gives rise to the functions $u_t(z) := \widehat{u}(z,e^t)$, $0 < t < 1$, the *geodesic* between u_0 and u_1. When the functions u_j are bounded, the geodesics u_t tend to u_j as $t \to j$, uniformly on Ω. One of the main properties of the geodesics is that they linearize the energy functional

$$\mathcal{E}(u) = \int_{\Omega} u(dd^c u)^n, \tag{1.4}$$

see [3,17] (where actually more general classes of plurisubharmonic functions are considered).

Given two non-pluripolar compact sets $K_0, K_1 \subset \mathbb{C}^n$, let u_j denote the relative extremal functions of K_j, $j = 0, 1$, with respect to a bounded hyperconvex neighbourhood Ω of $K_0 \cup K_1$, i.e.,

$$u_j(z) = \omega_{K_j,\Omega}(z) = \limsup_{y\to z} \sup\{u(y) :\ u \in \mathrm{PSH}_-(\Omega),\ u|_{K_j} \le -1\}, \qquad (1.5)$$

where $\mathrm{PSH}_-(\Omega)$ is the collection of all nonpositive plurisubharmonic functions in Ω. The functions u_j belong to $\mathrm{PSH}_-(\Omega)$ and satisfy $(dd^c u_j)^n = 0$ on $\Omega\backslash K_j$, see [12].

Assume, in addition, each K_j to be polynomially convex (in the sense that it coincides with its polynomial hull). This implies $\omega_{K_j,\Omega'} = -1$ on K_j for some (and thus any) bounded hyperconvex neighborhood Ω' of K_j and that $\omega_{K_j,\Omega'} \in C(\overline{\Omega'})$. In particular, the functions $u_j = -1$ on K_j and are continuous on $\overline{\Omega}$. The geodesics u_t converge to u_j uniformly as $t \to j$ [17] and so, by the Walsh theorem, the upper envelope $\widehat{u}(z,\zeta)$ (1.3) is continuous on $\Omega\times A$, which, in turn, implies $u_t \in C(\overline{\Omega}\times[0,1])$.

As was shown in [18], functions u_t in general are different from the relative extremal functions of any subsets of Ω. Consider nevertheless the sets where they attain their minimal value, -1:

$$L_t = \{z \in \Omega :\ u_t(z) = -1\}, \quad 0 < t < 1. \qquad (1.6)$$

By the continuity of the geodesic at the endpoints, the sets L_t converge (say, in the Hausdorff metric) to K_j when $t \to j \in \{0,1\}$ and so, they can be viewed as interpolations between K_0 and K_1.

The curve $t \mapsto L_t$ can be in a natural way identified with the multifunction $\zeta \mapsto K_\zeta := L_{\log|\zeta|}$. Note however that it is not an *analytic multifunction* (for the definition, see, e.g., [13,14,19]) because its graph $\{(z,\zeta) \in \Omega\times A : \widehat{u}(z,\zeta) = -1\}$ is not pseudoconcave.

In Section 2, we show that the interpolating sets L_t can be represented as sections $K_t = \{z :\ (z,e^t) \in \widehat{K}\}$ of the holomorphic hull $\widehat{K}$ of the set

$$K^A := (K_0\times A_0)\cup(K_1\times A_1) \subset \mathbb{C}^{n+1} \qquad (1.7)$$

with respect to functions holomorphic in $\mathbb{C}^n\times(\mathbb{C}\backslash\{0\})$.

In Section 3, we study the relative Monge–Ampère capacities $\mathrm{Cap}\,(L_t,\Omega)$ of the sets L_t; recall that for $K \Subset \Omega$,

$$\mathrm{Cap}\,(K,\Omega) = \sup\{(dd^c u)^n(K) :\ u \in \mathrm{PSH}_-(\Omega),\ u|_K \le -1\} = (dd^c\omega_{K,\Omega})^n(\Omega),$$

see [12]. It was shown in [17] that the function $t \mapsto \mathrm{Cap}\,(L_t,\Omega)$ is convex, which was achieved by using linearity of the energy functional (1.4) along the geodesics. In the case when Ω is the unit polydisk $\mathbb{D}^n$ and K_j are Reinhardt sets, the convexity of the Monge–Ampère capacities was rewritten in [18] as convexity of covolumes of certain unbounded convex subsets P_t of the positive orthant $\mathbb{R}^n_+$ (that is, volumes of their complements to $\mathbb{R}^n_+$). Here, we use a convex geometry technique to prove Theorem 3.2 stating that actually the covolumes of the sets P_t are *logarithmically convex*. Since in this case the sets L_t are exactly the geometric means $K_t^\times$ of K_0 and K_1, this implies the dual Brunn–Minkowski inequality for their Monge–Ampère capacities,

$$\mathrm{Cap}\,(K_t^\times,\mathbb{D}^n) \le \mathrm{Cap}\,(K_0,\mathbb{D}^n)^{1-t}\mathrm{Cap}\,(K_1,\mathbb{D}^n)^t, \quad 0 < t < 1. \qquad (1.8)$$

In addition, an equality here occurs for some $t \in (0,1)$ if and only if $K_0 = K_1$.

It is quite interesting that the volume of $K_t^\times$ satisfies the opposite Brunn–Minkowski inequality (1.2), i.e., it is logarithmically *concave*. Furthermore, so are the standard logarithmic capacity in the complex plane and the Newtonian capacity in $\mathbb{R}^n$ with respect to the Minkowski addition [4,5,16]. The difference here is that the relative Monge–Ampère capacity is, contrary to the logarithmic or Newton capacities, a local notion, which leads to the dual Brunn–Minkowski inequality (1.8), exactly like for the covolumes of coconvex bodies [11].

A natural question that remains open is to know whether the logarithmic convexity of the relative Monge–Ampère capacities is also true in the general, non-toric case. No non-trivial examples of (1.8) in this setting are known so far.

2. Level sets as holomorphic hulls. Let K_0, K_1 be two non-pluripolar compact subsets of a bounded hyperconvex domain $\Omega \subset \mathbb{C}^n$, and let $L_t = L_{t,\Omega}$ be the interpolating sets defined by (1.6) for the geodesic $u_t = u_{t,\Omega}$ with the endpoints $u_j = \omega_{K_j,\Omega}$. We start with an observation that if the sets K_j are polynomially convex, then the sets L_t are actually independent of the choice of the domain Ω containing $K_0 \cup K_1$.

Lemma 2.1. *If Ω' and Ω'' are bounded hyperconvex neighborhoods of non-pluripolar, polynomially convex, compact sets K_0 and K_1, then $L_{t,\Omega'} = L_{t,\Omega''}$.*

Proof. By the monotonicity of $\Omega \mapsto u_{t,\Omega}$, it suffices to show the equality for $\Omega' \Subset \Omega''$. Since $u_{t,\Omega''} \le u_{t,\Omega'}$, the inclusion $L_{t,\Omega'} \subset L_{t,\Omega''}$ is evident. Denote now

$$\delta = -\inf\{u_{j,\Omega''}(z) :\ z \in \partial\Omega',\ j = 0,1\} \in (0,1).$$

Recall that the geodesics $u_{t,\Omega}$ come from the maximal plurisubharmonic functions $\widehat{u}_\Omega$ in $\Omega \times A$ for the annulus A bounded by the circles A_j where $\log|\zeta| = j$. Then the function

$$\hat{v} := \frac{1}{1-\delta}(\widehat{u}_{\Omega''} + \delta) \in \mathrm{PSH}(\Omega' \times A) \cap C(\overline{\Omega' \times A})$$

satisfies $(dd^c\hat{v})^{n+1} = 0$ in $\Omega' \times A$ and

$$\lim \hat{v}(z,\zeta) = -1 \ \text{ as } (z,\zeta) \to K_j \times A_j. \tag{2.1}$$

Moreover, since $\hat{v} \ge 0$ on $\partial\Omega' \times A$ and its restriction to each A_j satisfies $(dd^c\hat{v})^n = 0$ on $A_j \backslash K_j$, the boundary conditions (2.1) imply

$$\lim \hat{v}(z,\zeta) \ge u_{j,\Omega'} \text{ as } \zeta \to A_j.$$

Therefore, we have $\hat{v} \ge \hat{u}$ in the whole $\Omega' \times A$. If $z \in L_{t,\Omega''}$, this gives us $-1 \ge u_{t,\Omega'}(z)$ and so, $z \in L_{t,\Omega'}$, which completes the proof. ☐

Next step is comparing the sets L_t with other interpolating sets, K_t, defined as follows. Set

$$\widehat{K} = \widehat{K}(\Omega) = \{(z,\zeta) \in \Omega \times A :\ u(z,\zeta) \le M(u)\ \forall u \in \mathrm{PSH}_-(\Omega \times A)\}, \tag{2.2}$$

where $M(u) = \max_j M_j(u)$ and

$$M_j(u) = \limsup u(z,\zeta) \text{ as } (z,\zeta) \to K_j \times A_j, \quad j = 0,1.$$

Note that the set $\widehat{K}$ will not change if one replaces $\mathrm{PSH}_-(\Omega\times A)$ by the collection of all bounded from above (or just bounded) plurisubharmonic functions on $\Omega \times A$.

Denote by $\widehat{K}_\zeta$ the section of $\widehat{K}$ over $\zeta \in A$:

$$\widehat{K}_\zeta = \widehat{K}_\zeta(\Omega) = \{z \in \Omega : (z,\zeta) \in \widehat{K}\}, \quad \zeta \in A.$$

The set $\widehat{K}$ is invariant under rotation of the ζ-variable, so $\widehat{K}_\zeta$ depends only on $|\zeta|$. We set then

$$K_t = \widehat{K}_{e^t}, \quad 0 < t < 1.$$

Theorem 2.2. *If K_j are non-pluripolar, polynomially convex compact subsets of Ω, then $L_t = K_t$ for all $0 < t < 1$.*

Proof. First, we prove the inclusion

$$L_t \subset K_t, \tag{2.3}$$

that is,

$$u(z,t) \le M(u) \quad \forall z \in L_t \tag{2.4}$$

for all $u \in \mathrm{PSH}_-(\Omega\times A)$. By the scalings $u \mapsto c\,u$, we can assume $u - M(u) \le 1$ on $\Omega \times A$. Then the function

$$\phi(z,\zeta) = u(z,\zeta) - (1 - \log|\zeta|)M_0(u) - (\log|\zeta|)M_1(u) - 1$$

belongs to $\mathrm{PSH}_-(\Omega \times A)$ and

$$\limsup \phi(z,\zeta) \le -1 \text{ as } (z,\zeta) \to K_j \times A_j.$$

In other words, $\phi_t(z) := \phi(z, e^t)$ is a subgeodesic for u_0 and u_1, so $\phi_t \le u_t$. Therefore, $\phi_t \le -1$ on L_t, which gives us (2.4).

To get the reverse inclusion, assume $z \in K_t$. Then, by definition of $\widehat{K}$, we get $u_t(z) \le M(u_t) = -1$ and, since $u_t \ge -1$ everywhere, $u_t(z) = -1$. $\square$

The set $\widehat{K}$ can actually be represented as a holomorphic hull of the set

$$K^A = (K_0 \times A_0) \cup (K_1 \times A_1),$$

which is similar to what one gets in the classical interpolation theory. This can be concluded by standard arguments relating plurisubharmonic and holomorphic hulls (see, e.g., [15]).

Proposition 2.3. *Let K_0, K_1 be two non-pluripolar, polynomially convex compact subsets of a bounded hyperconvex domain $\Omega \subset \mathbb{C}^n$. Then the set $\widehat{K}$ defined by (2.2) is the holomorphic hull of the set K^A with respect to the collection of all functions holomorphic on $\Omega \times \mathbb{C}_*$ (here $\mathbb{C}_* = \mathbb{C}\setminus\{0\}$).*

Proof. The domain $\Omega \times \mathbb{C}_*$ is pseudoconvex, so it suffices to show that $\widehat{K}$ is the $\mathcal{F}$-hull $\widehat{K}_{\mathcal{F}}$ of the set (1.7) with respect to $\mathcal{F} = \mathrm{PSH}(\Omega \times \mathbb{C}_*)$.

Take any hyperconvex domain Ω' such that $K_0 \cup K_1 \subset \Omega' \Subset \Omega$. Since $\mathcal{F}$ forms a subset of the collection of all bounded from above psh functions on $\Omega' \times A$, we have $\widehat{K}' := \widehat{K}(\Omega') \subset \widehat{K}_{\mathcal{F}}$. Moreover, by Lemma 2.1 and Theorem 2.2, we have $\widehat{K}' = \widehat{K}$, which implies $\widehat{K} \subset \widehat{K}_{\mathcal{F}}$.

Let u_t be the geodesic of u_0, u_1 in Ω. Then its psh image $\hat{u}(z,\zeta)$ can be extended to $\Omega \times \mathbb{C}_*$ as $\hat{U}(z,\zeta) = u_0(z) - \log|\zeta|$ for $-\infty < \log|\zeta| \le 0$ and $\hat{U}(z,\zeta) = u_1(z) + \log|\zeta| - 1$ for $1 \le \log|\zeta| < \infty$. Indeed, the function

$$\hat{v}(z,\zeta) = \max\{u_0(z) - \log|\zeta|, u_1(z) + \log|\zeta| - 1\}$$

is psh on $\Omega \times A$, continuous on $\Omega \times \overline{A}$, and equal to u_j on $\Omega \times A_j$. Therefore, it coincides with $\hat{U}$ on $\Omega \times (\mathbb{C}_* \setminus A)$. Since $\hat{v} \le \hat{u}$ on $\Omega \times A$ and $\hat{v} = \hat{u}$ on $\Omega \times \partial A$, the claim is proved.

Let $(z^*, \zeta^*) \in \widehat{K}_{\mathcal{F}}$. By the definition of $\widehat{K}_{\mathcal{F}}$, since $\hat{U} \in \mathrm{PSH}(\Omega \times \mathbb{C}_*)$,

$$\hat{u}(z^*, \zeta^*) = \hat{U}(z^*, \zeta^*) \le \sup\{\hat{U}(z,\zeta) : (z,\zeta) \in K^A\} = -1,$$

so $z^* \in \widehat{K}_t$ with $t = \log|\zeta^*|$. □

Finally, since the sets L_t are independent of the choice of Ω, we get the following description of the interpolated sets K_t.

Corollary 2.4. *Let K_0, K_1 be two non-pluripolar, polynomially convex compact subsets of $\mathbb{C}^n$ and let Ω be a bounded hyperconvex domain containing $K_0 \cup K_1$. Denote by u_t the geodesic of the functions $u_j = \omega_{K_j,\Omega}$, $j = 0,1$. Then for any $\zeta \in A$,*

$$K_t = \{z \in \Omega : u_t(z) = -1\} = \{z \in \mathbb{C}^n : |f(z,\zeta)| \le \|f\|_{K^A} \ \forall f \in \mathcal{O}(\mathbb{C}^n \times \mathbb{C}_*)\}$$

with $t = \log|\zeta|$.

Remark 2.5. Note that the considered hulls are taken with respect to functions holomorphic in $\mathbb{C}^n \times \mathbb{C}_*$ and not in $\mathbb{C}^{n+1}$ (that is, not the polynomial hulls). This reflects the fact that in the definition of K^A, the circles A_0 and A_1 may be interchanged. Since for any polynomial $P(z,\zeta)$ and any ζ inside the disc $|\zeta| < e$, we have $|P(z,\zeta)| \le \max\{|P(z,\xi)| : |\xi| = e\}$, each section of the *polynomial* hull of K^A must contain K_1, so such a hull does not provide any interpolation between K_0 and K_1.

3. Log-convexity of Monge–Ampère capacities. Let, as before, K_0 and K_1 be non-pluripolar, polynomially convex compact subsets of a bounded hyperconvex domain $\Omega \Subset \mathbb{C}^n$, u_t be the geodesic between $u_j = \omega_{K_j,\Omega}$, and let K_t be the corresponding interpolating sets as described in Section 2. As was mentioned, their relative Monge–Ampère capacities satisfy the inequality

$$\mathrm{Cap}\,(K_t, \Omega) \le (1-t)\,\mathrm{Cap}\,(K_0, \Omega) + t\,\mathrm{Cap}\,(K_1, \Omega).$$

Let now $\Omega = \mathbb{D}^n$ and assume the sets K_j to be Reinhardt. The polynomial convexity of K_j means then that their logarithmic images

$$Q_j = \mathrm{Log}\, K_j = \{s \in \mathbb{R}^n_- : (e^{s_1}, \ldots, e^{s_n}) \in K_j\}$$

are complete convex subsets of $\mathbb{R}^n_-$, i.e., $Q_j + \mathbb{R}^n_- \subset \mathbb{R}^n_-$; when this is the case, we will also say that K_j is complete logarithmically convex. In this situation, the sets K_t are, as in the classical interpolation theory, the geometric means of K_j. Note however that our approach extends the classical—convex—setting to a wider one.

Proposition 3.1. *The interpolating sets K_t of two non-pluripolar, complete logarithmically convex, compact Reinhardt sets $K_0, K_1 \subset \mathbb{D}^n$ coincide with*

$$K_t^\times := K_0^{1-t} K_1^t = \{z : \, |z_l| = |\eta_l|^{1-t} |\xi_l|^t, \; 1 \le l \le n, \; \eta \in K_0, \; \xi \in K_1\}.$$

Proof. We prove this by using the representation of the sets K_t as $L_t = \{z : u_t(z) = -1\}$ and a formula for the geodesics in terms of the Legendre transform [9,17]. By and large, this is Calderón's method.

As was noted in [17, Thm. 4.3], the inclusion $K_t^\times \subset L_t$ follows from convexity of the function $\check{u}_t(s) = u_t(e^{s_1}, \ldots, e^{s_n})$ in $(s,t) \in \mathbb{R}^n_- \times (0,1)$ since $s \in \log K_t^\times$ implies $\check{u}_t(s) \le -1$. To prove the reverse inclusion, we use a representation for $\check{u}_t$ given by [18, Thm. 6.1]:

$$\check{u}_t = \mathcal{L}[(1-t)\max\{h_{Q_0}+1, 0\} + t \max\{h_{Q_1}+1, 0\}], \quad 0 < t < 1,$$

where

$$\mathcal{L}[f](y) = \sup_{x \in \mathbb{R}^n} \{\langle x, y\rangle - f(x)\}$$

is the *Legendre transform* of f,

$$h_Q(a) = \sup_{s \in Q} \langle a, s\rangle, \quad a \in \mathbb{R}^n_+$$

is the support function of a convex set $Q \subset \mathbb{R}^n_-$, and $Q_j = \operatorname{Log} K_j$.

Let $z \notin K_t^\times$, then one can assume that none of its coordinates equals zero, so the corresponding point $\xi = (\log|z_1|, \ldots, \log|z_n|) \in \mathbb{R}^n_-$ does not belong to $Q_t = (1-t)Q_0 + tQ_1$. Therefore there exists $b \in \mathbb{R}^n_+$ such that

$$\langle b, \xi\rangle > h_{Q_t}(b) = (1-t)h_{Q_0}(b) + t\, h_{Q_1}(b);$$

by the homogeneity, one can assume $h_{Q_0}(b), h_{Q_1}(b) > -1$ as well. Then

$$\begin{aligned} \check{u}_t(\xi) &= \sup_{a \in \mathbb{R}^n_+} [\langle a, \xi\rangle - (1-t)\max\{h_{Q_0}(a)+1, 0\} - t\max\{h_{Q_1}(a)+1, 0\}] \\ &> (1-t)[h_{Q_0}(b) - (h_{Q_0}(b)+1)] + t[h_{Q_1}(b) - (h_{Q_1}(b)+1)] = 1, \end{aligned}$$

so ξ does not belong to $\operatorname{Log} L_t$ and consequently $z \notin L_t$. □

The crucial point for the Reinhardt case is a formula from [1, Thm. 7] (see also [18]) for the Monge–Ampère capacity of complete logarithmically convex compact sets $K \subset \mathbb{D}^n$:

$$\operatorname{Cap}(K, \mathbb{D}^n) = n! \operatorname{Covol}(Q^\circ) := n! \operatorname{Vol}(\mathbb{R}^n_+ \setminus Q^\circ),$$

where

$$Q^\circ = \{x \in \mathbb{R}^n_+ : \langle x, y\rangle \le -1 \; \forall y \in Q\}$$

is the *copolar* to the set $Q = \operatorname{Log} K$. In particular,

$$\operatorname{Cap}(K_t) = n! \operatorname{Covol}(Q_t^\circ) \tag{3.1}$$

for the copolar Q_t° of the set $Q_t = (1-t)Q_0 + t\, Q_1$.

Proposition 3.2. *We have*

$$\mathrm{Covol}(Q_t^\circ) \le \mathrm{Covol}(Q_0^\circ)^{1-t}\,\mathrm{Covol}(Q_1^\circ)^t, \quad 0 < t < 1. \tag{3.2}$$

If an equality here occurs for some $t \in (0,1)$, then $Q_0 = Q_1$.

Proof. Let, as before, h_Q be the restriction of the support function of a convex set $Q \subset \mathbb{R}_-^n$ to $\mathbb{R}_+^n$:

$$h_Q(x) = \sup\{\langle x, y\rangle : \, y \in Q\}, \ x \in \mathbb{R}_+^n.$$

We have then

$$\begin{aligned}
\int\limits_{\mathbb{R}_+^n} e^{h_{Q_t}(x)}\,dx &= \int\limits_{\mathbb{R}_+^n} dx \int\limits_{-h_{Q_t}(x)}^{\infty} e^{-s}\,ds = \int\limits_0^\infty e^{-s}\,ds \int\limits_{h_{Q_t}(x)\ge -s} dx \\
&= \int\limits_0^\infty e^{-s}\mathrm{Vol}(\{h_{Q_t}(x) \ge -s\})\,ds \\
&= \mathrm{Vol}(\{h_{Q_t}(x) \ge -1\}) \int\limits_0^\infty e^{-s}s^n\,ds \\
&= n!\,\mathrm{Covol}(Q_t^\circ).
\end{aligned}$$

Note that $h_{Q_t} = (1-t)h_{Q_0} + t\,h_{Q_1}$. Therefore, by Hölder's inequality with $p = (1-t)^{-1}$ and $q = t^{-1}$, we have

$$\int\limits_{\mathbb{R}_+^n} e^{h_{Q_t}(x)}\,dx \le \left(\int\limits_{\mathbb{R}_+^n} e^{h_{Q_0}(x)}dx\right)^{1-t} \left(\int\limits_{\mathbb{R}_+^n} e^{h_{Q_1}(x)}dx\right)^{t}, \tag{3.3}$$

which proves (3.2).

An equality in (3.2) implies the equality case in Hölder's inequality (3.3), which means the functions $e^{h_{Q_0}}$ and $e^{h_{Q_1}}$ are proportional, so $h_{Q_0}(x) = h_{Q_1}(x) + C$ for all $x \in \mathbb{R}_+^n$. Since both $h_{Q_0}(x)$ and $h_{Q_1}(x)$ converge to 0 as $x \to 0$ along $\mathbb{R}_+^n$, we get $C = 0$, which completes the proof. $\square$

Finally, by (3.1), we get

Theorem 3.3. *For polynomially convex, non-pluripolar compact Reinhardt sets $K_j \Subset \mathbb{D}^n$, the Monge–Ampère capacity* $\mathrm{Cap}\,(K_t, \mathbb{D}^n)$ *is a logarithmically convex function of t; in other words, the Brunn–Minkowski inequality* (1.8) *holds. An equality in* (1.8) *occurs for some $t \in (0,1)$ if and only if $K_0 = K_1$.*

Remark 3.4. The general situation of compact, polynomially convex Reinhardt sets reduces to the case $K_0, K_1 \subset \mathbb{D}^n$ because for K in the polydisk $\mathbb{D}_R^n$ of radius R, we have $\mathrm{Cap}\,(K, \mathbb{D}_R^n) = \mathrm{Cap}\,(\frac{1}{R}K, \mathbb{D}^n)$ and $(\frac{1}{R}K)_t = \frac{1}{R}K_t$.

Acknowledgements. Part of the work was done while the second named author was visiting Université Pierre et Marie Curie in March 2017; he thanks Institut de Mathématiques de Jussieu for the hospitality. The authors are grateful to the anonymous referee for careful reading of the text.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

[1] Aytuna, A., Rashkovskii, A., Zahariuta, V.: Widths asymptotics for a pair of Reinhardt domains. Ann. Polon. Math. **78**, 31–38 (2002)

[2] Bergh, J., Löfström, J.: Interpolation Spaces. An Introduction. Springer, Berlin (1976)

[3] Berman, R.J., Berndtsson, B.: Moser–Trudinger type inequalities for complex Monge–Ampère operators and Aubin's "hypothèse fondamentale". arXiv:1109.1263

[4] Borell, C.: Capacitary inequalities of the Brunn–Minkowski type. Math. Ann. **263**(2), 179–184 (1983)

[5] Borell, C.: Hitting probabilities of killed Brownian motion: a study on geometric regularity. Ann. Sci. École Norm. Sup. (4) **17**(3), 451–467 (1984)

[6] Calderón, A.-P.: Intermediate spaces and interpolation, the complex method. Studia Math. **24**, 113–190 (1964)

[7] Cordero-Erausquin, D.: Santaló's inequality on $\mathbb{C}^n$ by complex interpolation. C. R. Acad. Sci. Paris Ser. I **334**, 767–772 (2002)

[8] Cordero-Erausquin, D., Klartag, B.: Interpolations, convexity and geometric inequalities. In: Klartag, B., Mendelson, S., Milman, V.D. (eds.) Geometric Aspects of Functional Analysis. Lecture Notes in Math., vol. 2050, pp. 151–168. Springer, Berlin (2012)

[9] Guan, D.: On modified Mabuchi functional and Mabuchi moduli space of Kähler metrics on toric bundles. Math. Res. Lett. **6**(5–6), 547–555 (1999)

[10] Guedj, V. (ed.): Complex Monge–Ampère Equations and Geodesics in the Space of Kähler Metrics. Lecture Notes in Math., vol. 2038. Springer, Berlin (2012)

[11] Khovanski, A., Timorin, V.: On the theory of coconvex bodies. Discrete Comput. Geom. **52**(4), 806–823 (2014)

[12] Klimek, M.: Pluripotential Theory. Oxford University Press, London (1991)

[13] Oka, K.: Note sur les familles de finctions analytiques multiformes etc. J. Sci. Hiroshima Univ. Ser. A **4**, 93–98 (1943)

[14] Poletsky, E.A.: Jensen measures and analytic multifunctions. Ark. Mat. **42**(2), 335–352 (2004)

[15] Range, M.: Holomorphic Functions and Integral Representations in Several Complex Variables. Springer, Berlin (1986)

[16] Ransford, T.: Computation of logarithmic capacity. Comput. Methods Funct. Theory **10**(2), 555–578 (2010)

[17] Rashkovskii, A.: Local geodesics for plurisubharmonic functions. Math. Z. **287**, 73–83 (2017)

[18] Rashkovskii, A.: Copolar convexity. Ann. Polon. Math. **120**(1), 83–95 (2017)

[19] Słodkowski, Z.: Analytic set-valued functions and spectra. Math. Ann. **256**(3), 363–386 (1981)

[20] Słodkowski, Z.: Polynomial hulls with convex sections and interpolating spaces. Proc. Am. Math. Soc. **96**(2), 255–260 (1986)

[21] Słodkowski, Z.: Complex interpolation of normed and quasinormed spaces in several dimensions. I. Trans. Am. Math. Soc. **308**(2), 685–711 (1988)

[22] Słodkowski, Z.: Polynomial hulls with convex fibers and complex geodesics. J. Funct. Anal. **94**(1), 156–176 (1990)

Dario Cordero-Erausquin
Institut de Mathématiques de Jussieu
Sorbonne Université
4, Place Jussieu
75252 Paris Cedex 05
France
e-mail: dario.cordero@imj-prg.fr

Alexander Rashkovskii
Tek/Nat
University of Stavanger
4036 Stavanger
Norway
e-mail: alexander.rashkovskii@uis.no

Received: 4 October 2018