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A new inductive approach for counting dimension in large scale

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Abstract. We introduce the notion of *large scale dimensiongrad* as a large scale invariant of *asymptotic resemblance* spaces. Consequently it can be considered as a large scale invariant of metric spaces. The large scale dimensiongrad is a way of counting dimension in large scale but it is different from *asymptotic dimension* in general, as we show in the paper, too.

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1. Introduction. Large scale geometry as a tool for investigating properties of metric spaces is now well known and greatly in use. Simply, as a counterpart of small scale viewpoint, it tries to investigate metric spaces from afar. *Coarse equivalences* fill the place of *homeomorphisms* in large scale geometry. Large scale properties are those properties which are invariant under coarse equivalences. If one tries to make a list of already established properties, it will be relatively long. One of the most important large scale properties is *asymptotic dimension* which has been introduced by Gromov in [4]. There are also some inductive approaches for finding dimension in large scale, like *asymptotic inductive dimension*, *asymptotic dimensiongrad,* and *large scale inductive dimension* ([1,2,6]). In this paper we want to introduce a new large scale property which will be called the *large scale dimensiongrad.* It can be considered as a large scale way for counting dimension but we show it is not always equal to the asymptotic dimension. We present an example of a metric space with infinite asymptotic dimension but with the large scale dimensiongrad equal to zero (Example 3.15). We discuss the relation of the large scale dimensiongrad with the other inductive approaches for asymptotic dimension in Section 3. In Section 4 we prove that the class of all countable discrete groups with zero large scale dimensiongrad is equal to the class of all countable groups with zero asymptotic dimension. We introduce our notion of large scale dimensiongrad not only for metric spaces but also for all sets that are equipped with an *asymptotic resemblance relation*. Asymptotic resemblance relations have been introduced in [5]. We discuss asymptotic resemblance relations and inductive approaches for asymptotic dimension in Section 2 briefly.

2. Preliminaries.

2.1. Asymptotic resemblance. We introduce the notion of *asymptotic resemblance relation* (AS.R) as an equivalence relation λ on the family of all subsets of a set X with these two additional properties:

(i) If $A_1\lambda B_1$ and $A_2\lambda B_2$, then $(A_1\bigcup A_2)\lambda(B_1\bigcup B_2)$.
(ii) If $A_1, A_2 \neq \emptyset$ and $(A_1\bigcup A_2)\lambda B$, then there are nonempty subsets B_1 and B_2 of B such that $A_1\lambda B_1$ and $A_2\lambda B_2$. Let $A_1, A_2, B_1, B_2 \subseteq X$.

One can see [5] for the relation between this large scale structure on a set and the famous *coarse structures* introduced by Roe ([8]). We simply call the pair (X, λ) an *asymptotic resemblance space* (AS.R space). We call two subsets A and B of an AS.R space (X, λ) *asymptotically alike* if $A\lambda B$. A subset B of an AS.R space X is called *bounded* if for some point $x \in X$ we have $B\lambda x$. An AS.R space (X, λ) is said to be *asymptotically connected* if for each $a, b \in X$ we have $a\lambda b$. From now to the end of this paper all AS.R spaces are assumed to be connected. Let Y be a subset of an AS.R space (X, λ). For two subsets $A, B \subset Y$, define $A\lambda_Y B$ if $A\lambda B$. We call the AS.R space (Y, λ_Y) a *subspace AS.R* of X and λ_Y the induced AS.R on Y. On a topological space X the asymptotic resemblance λ is called to be compatible with the topology of X if for any subset A of X, $A\lambda\bar{A}$ and there exists an open subset $U \subseteq X$ such that $A\lambda U$. We call the open subset U the asymptotic neighbourhood of A.

Example 2.1. Suppose that (X, d) is a metric space. For two subsets A and B of X define $A\lambda_d B$ if $d_H(A, B) < \infty$, where d_H denotes the Hausdorff distance between A and B. It can be shown that λ_d is an AS.R on X and it is compatible with the topology of X ([5]). We call this AS.R the AS.R associated with the metric d.

Let (X, λ) and (Y, λ') be two AS.R spaces. We call a map $f : X \to Y$ proper if the inverse image of any bounded subset of Y is a bounded subset of X. A proper map $f : X \to Y$ is said to be an *AS.R mapping* if $A\lambda B$ implies $f(A)\lambda' f(B)$, where $A, B \subseteq X$. An AS.R mapping $f : X \to Y$ is said to be an asymptotic equivalence if there exists an AS.R mapping $g : Y \to X$ such that for each $A \subseteq X$ and $B \subseteq Y$ we have $g \circ f(A)\lambda A$ and $f \circ g(B)\lambda' B$. It can be shown that if the AS.R relations λ and λ' are associated with some metrics on X and Y, then this definition of AS.R mappings is equivalent to the definition of coarse maps and the definition of asymptotic equivalences is equivalent to the definition of coarse equivalences between metric spaces ([5]).
Let (X, d) be a proper metric space. A continuous and bounded map $f : X \to Y$ is called a *Higson function* if for each $R, \epsilon > 0$, there exists a compact

subset K of X such that if $(x, y) \in ((X \times X) \setminus (K \times K))$ and $d(x, y) < R$, then $| f(x) - f(y) | < \epsilon$. The family of all Higson function on a metric space X is denoted by $C_h(X)$. There exists a compactification hX of a proper metric space X such that $C_h(X)$ and the family of all continuous functions on hX are isomorphic C^*-algebras. The compactification hX is called the *Higson compactification* and the boundary $hX \setminus X$ is called the *Higson corona*. Let (X, λ) be an AS.R space. Two subsets of an AS.R space (X, λ) are called *asymptotically disjoint* if they do not have any asymptotically alike unbounded subsets. Clearly bounded subsets of X are asymptotically disjoint from all subsets of X. The AS.R space (X, λ) is said to be *asymptotically normal* if for each two asymptotically disjoint subsets A and B of X, we have $X = X_1 \bigcup X_2$ such that X_1 and X_2 are asymptotically disjoint from A and B, respectively. It can be shown that if (X, d) is a metric space, then (X, λ_d) is asymptotically normal ([5]). We say that an AS.R relation compatible with the topology of a topological space X is *proper* if every bounded subset of X has compact closure. Let X be a normal topological space and λ be a proper and asymptotically normal AS.R on X. (For example λ can be the AS.R associated to a proper metric on X.) In [5] we introduced a compactification for X called the *asymptotic compactification*. We denote this compactification here by $\mathcal{X}$. We call the boundary $\nu X = \mathcal{X} \setminus X$ the *asymptotic corona*. We are going to explain this compactification briefly, for more details see [5, Section 4]. Assume that $A, B \subseteq X$. We say $A\delta B$ if $\bar{A}$ and $\bar{B}$ are not disjoint subsets of X or are not asymptotically disjoint subsets of X (Notice that $\bar{A}$ and $\bar{B}$ denote the closures of A and B in the topological space X). The relation δ is a Hausdorff *proximity* on X. The *Smirnov* compactification associated to this proximity is the asymptotic compactification $\mathcal{X}$. For details on proximity spaces and their Smirnov compactification see [7]. We showed if λ is the AS.R associated to a proper metric d on X, then $\mathcal{X}$ is homeomorphic to the Higson compactification of X. From now on we will denote the closure of a subset A of X in the asymptotic compactification $\mathcal{X}$ by A' and the boundary $A' \bigcap \nu X$ by νA.

We need the following two propositions below.

Proposition 2.2. *Let (X, d) be a proper metric space. Then two subsets A and B of X are asymptotically disjoint if and only if $\nu A \bigcap \nu B = \emptyset$.*

Proof. It is an immediate consequence of [5, Corollary 4.23]. □

Proposition 2.3. *Let $f : X \to Y$ be an AS.R mapping between two AS.R spaces (X, λ) and (Y, λ').*

(*i*) *If f is an asymptotic equivalence, then the images of two asymptotically disjoint subsets of X are two asymptotically disjoint subsets of Y.*

(*ii*) *The inverse images of two asymptotically disjoint subsets of Y are two asymptotically disjoint subsets of X.*

Proof. See [6, Lemma 3.5]. □

2.2. Inductive approaches for asymptotic dimension. To keep abbreviated we do not mention the definition of asymptotic dimension here and refer the reader to [4]. We begin by recalling that in a topological space X a closed subset C of X is called a *separator* between two disjoint closed subsets A and B of X if $X \setminus C = U \bigcup V$ such that U and V are two disjoint open subsets of X and they contain A and B, respectively.

Definition 2.4. Let (X, d) be a proper metric space. A subset C of X is called an *asymptotic separator* between asymptotically disjoint subsets A and B of X if νC is a separator between νA and νB in νX. We say that the *asymptotic inductive dimension* of X is equal to -1 if and only if X is bounded. For a nonnegative integer n, we say that the asymptotic inductive dimension of X is less than or equal to n if each two asymptotically disjoint subsets of X have an asymptotic separator with the asymptotic inductive dimension less than or equal to $n-1$. The asymptotic inductive dimension of the metric space (X, d) is denoted by $\operatorname{asInd} X$. For $n \in \mathbb{N} \bigcup \{0\}$ we say that $\operatorname{asInd} X = n$ if $\operatorname{asInd} X \leq n$ and $\operatorname{asInd} X \leq n-1$ is not true.

It has been shown that for a proper metric space X with finite asymptotic dimension, the asymptotic inductive dimension and the asymptotic dimension are equal ([2]).

Definition 2.5. Let (X, λ) be an AS.R space. A subset C of X is called a *large scale separator* between asymptotically disjoint subsets A and B of X if it is asymptotically disjoint from both A and B and we have $X = X_1 \bigcup X_2$ such that X_1 and X_2 are asymptotically disjoint from A and B, respectively, and if $L_1 \lambda L_2$ for two unbounded subsets $L_1 \subseteq X_1$ and $L_2 \subseteq X_2$, then there exists a subset L of C such that $L \lambda L_1$.

If in the Definition 2.4 we use the word large scale separator instead of the word asymptotic separator and instead of a proper metric space we assume to have an AS.R space (X, λ), then we have the definition of the *large scale inductive dimension*. We denote the large scale inductive dimension of an AS.R space (X, λ) by $\operatorname{lsInd}_\lambda X$. We showed that in a proper metric space (X, d) each large scale separator in X is an asymptotic separator ([6, Proposition 3.2]). Therefore an easy induction will lead to $\operatorname{asInd} X \leq \operatorname{lsInd}_{\lambda_d} X$, for every proper metric space (X, d). We showed in [6] that the large scale inductive dimension is a large scale property.

Definition 2.6. Let (X, d) be a metric space. For $r > 0$ an r-chain between two subsets A and B of X is a finite sequence $x_0, \ldots, x_n$ in X such that $x_0 \in A$ $x_n \in B$, and $d(x_{i-1}, x_i) \leq r$ for $i = 1, \ldots, n$. An *asymptotic cut* between two asymptotically disjoint subsets A and B of X is a subset C of X such that it is asymptotically disjoint from both A and B and for each $r > 0$ there exists some positive real number s such that each r-chain between A and B intersects the s neighbourhood of C.

It can be shown that in a metric space (X, d) each large scale separator (asymptotic separator) between asymptotically disjoint subsets of X is an asymptotic cut between them. If we replace the word asymptotic separator by asymptotic cut in the Definition 2.4, we have the definition of *asymptotic dimensiongrad*. We showed in [6] for a relatively big class of metric spaces that the asymptotic inductive dimension and asymptotic dimensiongrad are equal. Fortunately this class contains all geodesic metric spaces and all finitely generated groups.

3. Large scale dimensiongrad.

Definition 3.1. Let (X, λ) be an AS.R space. We call a subset $D \subseteq X$ a *large scale continuum* if $D = D_1 \bigcup D_2$ for two asymptotically disjoint subsets D_1 and D_2 of X implies that D_1 or D_2 is bounded.

Let us mention that a continuum in a topological space X is a compact connected subset of X.

Proposition 3.2. *Assume that (X, d) is a proper metric space. Then a subset $D \subseteq X$ is a large scale continuum in (X, λ_d) if and only if νD is a continuum in νX.*

Proof. Suppose that νD is a continuum in νX. Let $D = D_1 \bigcup D_2$ for two asymptotically disjoint subsets D_1 and D_2 of X. So by Proposition 2.2, $\nu D_1 \bigcap \nu D_2 = \emptyset$. Since νD is a continuum, $\nu D_1 = \emptyset$ or $\nu D_2 = \emptyset$ and it leads to D_1 is bounded or D_2 is bounded.
To prove the converse, assume that D is a large scale continuum and $\nu D = O_1 \bigcup O_2$ for two disjoint closed subsets O_1 and O_2 of νX. Since D' is a Hausdorff and compact topological space, it is a normal topological space and we can use Urysohn's lemma. Let $f : D' \to [0, 1]$ be a continuous function such that $f(O_1) = \{0\}$ and $f(O_2) = \{1\}$. Let $D_1 = f^{-1}([0, \frac{1}{2}]) \bigcap D$ and $D_2 = f^{-1}([\frac{1}{2}, 1]) \bigcap D$. Clearly $D = D_1 \bigcup D_2$. It is immediate that $f(\nu D_1) = \{0\}$ and $f(\nu D_2) = \{1\}$. So $\nu D_1 \bigcap \nu D_2 = \emptyset$. Thus by the Proposition 2.2 D_1 and D_2 are asymptotically disjoint and unbounded, a contradiction. □

Definition 3.3. Let (X, λ) be an AS.R space. We call $C \subseteq X$ a *large scale cut* between asymptotically disjoint subsets A and B of X if

(i) C is asymptotically disjoint from both A and B.

(ii) Each large scale continuum $D \subseteq X$ which is not asymptotically disjoint from A and B is not asymptotically disjoint from C too.

Proposition 3.4. *Let A and B be two asymptotically disjoint subsets of an AS.R. space (X, λ). Then each large scale separator between A and B is a large scale cut between them.*

Proof. Assume that C is a large scale separator between A and B. Let $X = X_1 \bigcup X_2$ such that X_1 and X_2 satisfy the properties of the Definition 2.5. Suppose that D is a large scale continuum in X and it is not asymptotically disjoint from A and B. Let $D_1 = D \bigcap X_1$ and $D_2 = D \bigcap X_2$. Since D is not asymptotically disjoint from A and B, D_1 and D_2 are unbounded. Since D is a large scale continuum and $D = D_1 \bigcup D_2$, so D_1 and D_2 are not asymptotically disjoint. Thus there are $L_1 \subseteq D_1$ and $L_2 \subseteq D_2$ such that $L_1 \lambda L_2$. By the Definition 2.5, L_1 and C are not asymptotically disjoint. Thus C and D are not asymptotically disjoint. It shows that C is a large scale cut. $\square$

The inverse of the above proposition is not true in general. Besides there is no any good relation between the notions asymptotic cut and large scale cut.

Example 3.5. Let $X = \bigcup_{n\in\mathbb{N}}(2^n \times [0, n])$. Assume that X has the subspace metric induced from $\mathbb{R}^2$. Let $A = \{(2^n, 0) \mid n \in \mathbb{N}\}$ and $B = \{(2^n, n) \mid n \in \mathbb{N}\}$. Clearly A and B are asymptotically disjoint subsets of X. Since there is no large scale continuum in X such that it is not asymptotically disjoint from both A and B, every bounded subset of X, like the set $C = \{(2, 0)\}$, can be a large scale cut between A and B. It is easy to see that C is not an asymptotic cut between A and B and therefore it is not an asymptotic separator (large scale separator) between them.

Even in finitely generated groups one can find a large scale cut which is not an asymptotic cut (asymptotic separator).

Example 3.6. Let $X = \mathbb{Z}^2$ and suppose that A and B are as the previous example. Let $C = \{\{2^{n-1}+2^n\} \times \mathbb{Z} \mid n \in \mathbb{N}\}$. It can be shown that each subset of $\mathbb{Z}^2$ which is asymptotically disjoint from C is not a large scale continuum, thus C is a large scale cut between A and B. Clearly C is not an asymptotic cut between A and B.

Example 3.7. Let $X = \bigcup_{n\in\mathbb{N}}(2^n \times \mathbb{R})$. Consider X with the induced metric from $\mathbb{R}^2$. Let $A = \{(2^n, 0) \mid n \in \mathbb{N}\}$ and $B = \{2\} \times \mathbb{R}$. Clearly A and B are asymptotically disjoint subsets of X. It is straightforward to show that $C = \{(2^n, 0)\}$ is an asymptotic cut between A and B. Since X is a large scale continuum that is not asymptotically disjoint from A and B and X is asymptotically disjoint from the bounded subset $C \subset X$, C is not a large scale cut between A and B.

Definition 3.8. Let (X, λ) be an AS.R space. If we use the word large scale cut instead of the word large scale separator in the definition of the large scale inductive dimension, we have the definition of the *large scale dimensiongrad* of the AS.R space (X, λ). We denote the large scale dimensiongrad of the AS.R space (X, λ) by $\operatorname{lsDg}_\lambda X$.

The following proposition can be proved by using Proposition 3.4 and applying an easy induction.

Proposition 3.9. *Let (X, λ) be an AS.R space. Then* $\operatorname{lsDg}_\lambda X \leq \operatorname{lsInd}_\lambda X$.

Let us recall that a closed subset D of a topological space X is called a cut between two closed disjoint subsets A and B of X if it is disjoint from both of them and each continuum that intersects both A and B intersects D too. It is easy to verify that in all topological spaces every separator between two disjoint subsets is a cut between them.

Proposition 3.10. *Assume that (X, d) is a proper metric space. Let A and B be two asymptotically disjoint subsets of X. If $C \subseteq X$ and νC is a cut between νA and νB in νX, then C is a large scale cut between A and B.*

Proof. Let $D \subseteq X$ be a large scale continuum such that it is not asymptotically disjoint from A and B. By Proposition 3.2 νD is a continuum in νX. Since D is not asymptotically disjoint from A and B, Proposition 2.2 leads to $\nu D \bigcap \nu A \neq \emptyset$ and $\nu D \bigcap \nu B \neq \emptyset$. Therefore $\nu D \bigcap \nu C \neq \emptyset$. It shows C and D are not asymptotically disjoint. □

Corollary 3.11. *Let (X, d) be a proper metric space. Then each asymptotic separator between asymptotically disjoint subsets of X is a large scale cut between them.*

Proof. Since each separator is also a cut, it is a straightforward consequence of Proposition 3.10. □

Corollary 3.12. *Let (X, d) be a proper metric space. Then* $\operatorname{lsDg}_{\lambda_d} X \leq \operatorname{asInd} X$.

Proof. By using Corollary 3.11 this can be proved inductively. □

Lemma 3.13. *Let $f : X \to Y$ be an asymptotic equivalence between AS.R spaces (X, λ) and (Y, λ'). If $D \subseteq X$ is a large scale continuum, then so is $f(D)$.*

Proof. Let $g : Y \to X$ be an AS.R mapping such that $g \circ f(A)\lambda A$ and $f \circ g(B)\lambda' B$ for all $A \subseteq X$ and $B \subseteq Y$. Contrary to our claim, suppose that $f(D) = L_1 \bigcup L_2$ such that L_1 and L_2 are asymptotically disjoint and unbounded subsets of Y. So $g(L_1)$ and $g(L_2)$ are asymptotically disjoint by part (i) of the Proposition 2.3. We have $g(f(D)) = g(L_1) \bigcup g(L_2)$. Since $g(f(D))\lambda D$, the property (ii) of AS.R relations shows that there are two subsets D_1 and D_2 of D such that $D = D_1 \bigcup D_2$ and they are asymptotically alike to $g(L_1)$ and $g(L_2)$, respectively. Thus D_1 and D_2 are unbounded and asymptotically disjoint, a contradiction. □

Theorem 3.14. *Assume that (X, λ) and (Y, λ') are two asymptotically equivalent AS.R spaces. Then* $\operatorname{asDg}_{\lambda} X = \operatorname{asDg}_{\lambda'} Y$.

Proof. Let $f : X \to Y$ and $g : Y \to X$ be two AS.R mappings such that $g \circ f(A)\lambda A$ for all $A \subseteq X$ and $f \circ g(B)\lambda' B$ for all $B \subseteq Y$. We proceed by induction on $\operatorname{asDg}_{\lambda'} Y$. If $\operatorname{asDg}_{\lambda'} Y = -1$ then Y is bounded, so X is bounded and the theorem holds. Assume that the theorem is true for $n - 1$. Let $\operatorname{asDg}_{\lambda'} Y = n$. Let A and B be two asymptotically disjoint subsets of X. Thus $f(A)$ and $f(B)$ are asymptotically disjoint subsets of Y by part (i) of Proposition 2.3. There exists a large scale cut $C \subseteq Y$ between $f(A)$ and $f(B)$ such that $\operatorname{asDg}_{\lambda'_C} C \leq n - 1$. Assume that $O \subseteq X$ is an asymptotic continuum such that it is not asymptotically disjoint from A and B. Thus $f(O)$ is an asymptotic continuum in Y by Lemma 3.13 and it is not asymptotically disjoint from $f(A)$ and $f(B)$ by Proposition 2.3 part (ii). So $f(O)$ and C are not asymptotically disjoint and it shows that $g(C)$ is not asymptotically disjoint from $g \circ f(O)$. Since $g(f(O))\lambda O$ an λ is an equivalence relation so $g(C)$ and O are not asymptotically disjoint. It shows that $g(C)$ is a large scale cut between A and B. By [5, Lemma 2.20] $g(C)$ and C are asymptotic equivalent and the assumption of our induction shows that $\mathrm{lsDg}_{\lambda_{g(C)}} g(C) = n-1$. Therefore $\mathrm{lsDg}_\lambda X \leq n = \mathrm{lsDg}_{\lambda'} Y$. Similarly one shows that $\mathrm{lsDg}_{\lambda'} Y \leq \mathrm{lsDg}_\lambda X$, and it will prove this theorem. $\square$

The previous theorem showed that the large scale dimensiongrad is a large scale invariant. In the following example we show that the notions asymptotic dimension and large scale dimensiongrad are different large scale properties.

Example 3.15. Let $Y = \bigoplus_{\mathbb{N}} \mathbb{R}$, i.e. the set of all finitely supported functions form $\mathbb{N}$ to $\mathbb{R}$. For $f, g \in Y$, define $d(f,g) = \sqrt{\Sigma_{i\in\mathbb{N}}(f(i)-g(i))^2}$ and assume Y with this metric. For each $n \in \mathbb{N}$ suppose that A_n denotes the set of all $f \in Y$ such that $f(1) \in [3^n - n, 3^n + n]$ and $f(i) \in [-n, +n]$ for all $1 < i \leq n$ and $f(i) = 0$ for all $i > n$. Let $X = \bigcup_{n\in\mathbb{N}} A_n$. Assume that $D \subseteq X$ is an unbounded subset. Let $J = \{n \in \mathbb{N} \mid A_n \bigcap D \neq \emptyset\}$. Since for each $n \in \mathbb{N}$, A_n is bounded and D is unbounded, J is an infinite countable set. Consider J as a sequence like $n_1, n_2, \ldots$. Let $D_1 = \bigcup_{\text{i is even}} A_{n_i}$ and $D_2 = \bigcup_{\text{i is odd}} A_{n_i}$. We have

$$dist(A_n, A_m) \geq \mid 3^m - 3^n - m - n \mid$$

for each $m, n \in \mathbb{N}$. So clearly D_1 and D_2 are two asymptotically disjoint subset of X. It shows that X does not contain any unbounded large scale continuum. Thus each bounded subset of X can be a large scale cut between two asymptotically disjoint subsets of X. Therefore $\mathrm{asDg}_{\lambda_d} X = 0$. Now we are going to show that X has infinite asymptotic dimension. For $n \in \mathbb{N}$ suppose that X_n denotes the set of all $f \in X$ such that for $i > n$, $f(i) = 0$. So X_n can be considered as a subset of $\mathbb{R}^n$. Let $\sigma \in X_n$ be such that $\sigma(1) = 3^n$ and for all $i > 1$, $\sigma(i) = 0$. Suppose that $r > 0$ is given. Choose $m \in \mathbb{N}$ such that $m > n$ and $m > r$. One can easily see that X_n contains a ball of radius r around σ. Thus X_n is a subset of $\mathbb{R}^n$ which contains arbitrarily big balls. So the asymptotic dimension of X_n equals n ([3, Proposition 3.5]). Since X_n is a subspace of X, the asymptotic dimension of X should be bigger than n. It shows that the asymptotic dimension of X is infinite.

4. Groups with zero large scale dimensiongrad. Let us recall that every discrete countable group can be equipped with a proper left invariant metric which is unique up to coarse equivalences. So speaking of the large scale properties of countable groups is relevant.

Lemma 4.1. *Let G be a countable group and $g \in G$. If the set $X = \{g^n \mid n \in \mathbb{N}\}$ is unbounded, then it is an asymptotic continuum.*

Proof. Let d be a left invariant proper metric on G. Suppose that for two unbounded subsets A and B of X, we have $X = A \bigcup B$. Let $O = \{n \in \mathbb{N} \mid g^n \in A, g^{n+1} \in B\}$. We claim that O is infinite. Assume that, contrary to our claim, O is finite. So there exists $N \in \mathbb{N}$ such that for each $n \geq N$ if $g^n \in A$, then so is g^{n+1}. Suppose that there exists some $m \geq N$ such that $g^m \in A$.

An easy induction shows that for each $n \geq m$, $g^n \in A$. Thus B is finite and it contradicts our assumption of unboundedness of B. So we should suppose that for each $n \geq N$, $g^n \notin A$. Clearly this leads to the finiteness of A, and this is a contradiction. Now let $L_1 = \{g^n \mid n \in O\}$ and $L_2 = \{g^{n+1} \mid n \in O\}$. So $L_1 \subseteq A$ and $L_2 \subseteq B$, and they are infinite (finiteness of L_1 and L_2 will show g has finite order and it will lead to finiteness of X, a contradiction). Since d is proper, L_1 and L_2 are unbounded. Choose $r > 0$ such that $d(g, e) < r$, where e denotes the neutral element of G. Since d is left invariant, we have $d(g^n, g^{n+1}) = d(g, e) < r$, for each $n \in \mathbb{N}$. Thus $d_H(L_1, L_2) \leq r$, and it shows that A and B are not asymptotically disjoint. □

Lemma 4.2. *Let X be an unbounded metric space. Then there exist two unbounded subsets A and B of X such that they are asymptotically disjoint.*

Proof. Let $x_1 \in X$. Choose $y_1 \in X$ such that $d(x_1, y_1) > 1$. Suppose that x_n and y_n have been chosen. Choose x_{n+1} and y_{n+1} such that $d(x_{n+1}, y_i) > n+1$ and $d(y_{n+1}, x_i) > n + 1$, for $i \in \{1, \ldots, n\}$. Let $A = \{x_n \mid n \in \mathbb{N}\}$ and $B = \{y_n \mid n \in \mathbb{N}\}$. It is straightforward to show that A and B are two asymptotically disjoint and unbounded subsets of X. □

It is known that a countable discrete group G has asymptotic dimension zero if and only if each element of G has finite order [9, Corollary 2].

Proposition 4.3. *Let G be an infinite countable group. Then the large scale dimensiongrad of G is zero if and only if the asymptotic dimension of G is zero.*

Proof. Let d be a proper left invariant metric on G. Assume that the asymptotic dimension of G is zero. Since the asymptotic dimension and asymptotic inductive dimension of proper metric spaces are equal, $\operatorname{asInd} X = 0$. By Proposition 3.9, $\operatorname{lsDg}_{\lambda_d} X \leq 0$. Since G is unbounded, $\operatorname{lsDg}_{\lambda_d} X = 0$. To prove the converse, assume that $\operatorname{lsDg}_{\lambda_d} X = 0$. Suppose that $g \in G$. Let $X = \{g^n \mid n \in \mathbb{N}\}$. We want to show that X is finite. Assume that, contrary to our claim, X is infinite. Since d is proper, X is unbounded. Let A and B be two unbounded asymptotically disjoint subsets of X (Lemma 4.2). By Lemma 4.1, X is a large scale continuum. In addition X is not asymptotically disjoint from both A and B. Since X is asymptotically disjoint from each bounded subset of G, no bounded subset of G can be a large scale cut between A and B. Thus the large scale dimensiongrad of G is bigger than zero, a contradiction. Therefore g has finite order. □

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