Arch. Math. 110 (2018), 311–318

0003-889X/18/040311-8
published online January 6, 2018
https://doi.org/10.1007/s00013-017-1137-x

Archiv der Mathematik

Bounded Engel elements in groups satisfying an identity

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Abstract. We prove that a residually finite group G satisfying an identity $w \equiv 1$ and generated by a commutator closed set X of bounded left Engel elements is locally nilpotent. We also extend such a result to locally graded groups, provided that X is a normal set. As an immediate consequence, we obtain that a locally graded group satisfying an identity, all of whose elements are bounded left Engel, is locally nilpotent.

Mathematics Subject Classification. 20F45, 20E26, 20F40.

Keywords. Engel element, Residually finite group, Restricted Burnside problem.

1. Introduction. Let $w = w(x_1, \ldots, x_m)$ be a nonempty word in the free group generated by $x_1, \ldots, x_m$. A group G is said to satisfy the identity $w \equiv 1$ if $w(g_1, \ldots, g_m) = 1$ for all $g_1, \ldots, g_m \in G$. In the context of the Burnside problems, Zelmanov has recently proved that a residually finite p-group, for p a prime, which satisfies an identity is locally finite [23]. The result was already announced in [22] where the following conjecture was also given: "Apparently this theorem can be generalized from p-groups to periodic groups in the spirit of the theorem of P. Hall and G. Higman". It is still unclear how the Hall–Higman theorem [7] can be used to deal with the periodic case. However, as the Burnside problems are closely related to the theory of Engel groups (see, for instance, [19]), we formulate the above conjecture in terms of Engel groups.

R. Bastos was partially supported by FAPDF/Brazil. N. Mansuroğlu was supported by the Scientific and Technological Research Council of Turkey (TUBITAK) under the programme TUBITAK 2219-International Postdoctoral Research Fellowship and she would like to thank the Department of Mathematics at the University of Salerno for its excellent hospitality. A. Tortora and M. Tota are members of G.N.S.A.G.A. (INdAM).

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Conjecture 1.1. *Let G be a residually finite Engel group satisfying an identity. Then G is locally nilpotent.*

Given a group G, an element $g \in G$ is called a (left) Engel element if for any $x \in G$ there exists a positive integer $n = n(x, g)$ such that $[x,_n g] = 1$, where the commutator $[x,_n g]$ is defined inductively by the rules

$$[x,_1 g] = [x, g] = x^{-1}x^g \quad \text{and, for } n \geq 2, \quad [x,_n g] = [[x,_{n-1} g], g].$$

If n can be chosen independently of x, then g is called a (left) n-Engel element, or more generally a bounded (left) Engel element. The group G is an Engel group (resp. an n-Engel group) if all its elements are Engel (resp. n-Engel).

We say that a group G is a *nil group* if all elements of G are bounded Engel, i.e., for any $g \in G$ there is $n = n(g) \geq 1$ such that $[x,_n g] = 1$ for all $x \in G$. Of course, if G is n-Engel, then it satisfies the identity $[x,_n g] = 1$ and Conjecture 1.1 holds. In fact, by Zelmanov's solution of the restricted Burnside problem, it follows that a residually finite n-Engel group is locally nilpotent [19, Theorem 3.2] (see also [20]).

A subset X of a group is commutator closed if $[x, y] \in X$ for any $x, y \in X$. In this note we deal with groups generated by a commutator closed set of bounded Engel elements. Our main result is as follows.

Theorem A. *Let G be a residually finite group satisfying an identity $w \equiv 1$. Suppose that G is generated by a commutator closed set X of bounded Engel elements. Then G is locally nilpotent.*

Recall that a group is locally graded if every nontrivial finitely generated subgroup has a proper subgroup of finite index. The class of locally graded groups contains locally (soluble-by-finite) groups as well as residually finite groups.

We will extend Theorem A to locally graded groups, provided that X is a normal set. As an immediate consequence, we obtain that Conjecture 1.1 is true for locally graded nil groups.

Corollary B. *Let G be a locally graded nil group satisfying an identity. Then G is locally nilpotent.*

Notice that a nil group satisfying an identity might not be locally nilpotent, as announced by Juhasz and Rips in the case of an n-Engel group with $n \geq 40$ (see [11]). On the other hand, the following question remains open.

Question 1.2. *Is any locally graded nil group necessarily locally nilpotent?*

The proof of Theorem A is based on Lie-theoretic techniques and uses a deep theorem of Zelmanov which generalizes the main result of his solution of the restricted Burnside problem (see [22] for an account). Other results in the same spirit were obtained in [3,4,14].

In the next section we will collect some definitions and results on Lie algebras satisfying an identity. In Section 3 we will prove Theorem A and the above mentioned version for locally graded groups. In Section 4 we will analyze the possible use of our results in order to show that bounded Engel elements of an arbitrary group do not form a subgroup.

2. Lie algebras with an identity. Let L be a Lie algebra over a field. We use the left normed convention for Lie brackets, that is,

$$[a_1, \dots, a_n] = [[\dots[[a_1, a_2], a_3], \dots], a_n]$$

for all $a_1, \dots, a_n \in L$. An element $a \in L$ is called ad-nilpotent if there exists a positive integer n such that

$$[x, \underbrace{a, \dots, a}_{n \ times}] = 0$$

for all $x \in L$. Following [23], we say that a subset X of L is a Lie set if $[a, b] \in X$ for any $a, b \in X$, and denote by $S\langle X\rangle$ the Lie set generated by X, namely the smallest Lie set containing X.

Let F be the free Lie algebra over the same field as L on the generators $x_1, \dots, x_m$. For a nonzero element $f = f(x_1, \dots, x_n)$ of F, the Lie algebra L is said to satisfy the polynomial identity $f \equiv 0$ if $f(a_1, \dots, a_n) = 0$ for all $a_1, \dots, a_n \in L$.

The main ingredient in the proof of Theorem A is the following powerful result, due to Zelmanov (see [23, Theorem 1.1]).

Theorem 2.1. *Let L be a Lie algebra satisfying a polynomial identity and generated by elements $a_1, \dots, a_m$. If every element $a \in S\langle a_1, \dots, a_m\rangle$ is ad-nilpotent, then L is nilpotent.*

Let p be a prime and G a group. A series of subgroups

$$G = G_1 \geq G_2 \geq \cdots \qquad (*)$$

is called an N-series if $[G_i, G_j] \leq G_{i+j}$ for all $i, j \geq 1$; in addition, the series is an N_p-series if $G_i^p \leq G_{pi}$ for all i. An important example of an N_p-series is the p-dimension central series $G = D_1 \geq D_2 \geq \cdots$, also known as Zassenhaus–Jennings–Lazard series, where

$$D_i = D_i(G) = \prod_{jp^k \geq i} \gamma_j(G)^{p^k}$$

(see [13, Proposition 2.10]).

Given an N-series (*), let $L^*(G)$ be the direct sum of the abelian groups G_i/G_{i+1}, written additively. Thus $L^*(G)$ has a Lie ring structure given by

$$[xG_{i+1}, yG_{j+1}] = [x, y]G_{i+j+1},$$

where $x \in G_i, y \in G_j$ and $[x, y]$ is the commutator in G. If all quotients G_i/G_{i+1} have prime exponent p, then $L^*(G)$ can be viewed as a Lie algebra over the field with p elements. This is always the case if the series is an N_p-series. We write $L_p(G)$ for the Lie algebra associated with the Zassenhaus–Jennings–Lazard series, and denote by $L(G)$ the subalgebra of $L_p(G)$ generated by G_1/G_2.

An important criterion for $L_p(G)$ to satisfy a polynomial identity follows from [21, Theorem 1].

Theorem 2.2. *Let G be a group satisfying an identity. Then, for any prime p, the Lie algebra $L_p(G)$ satisfies a polynomial identity.*

The following lemma is a well-known consequence of some remarkable results on pro-p groups (see [6] for relevant definitions and background).

Lemma 2.3. *Let G be a finitely generated pro-p group such that $L(G)$ is nilpotent. Then G has a faithful linear representation over the field of p-adic numbers.*

Proof. By a theorem of Lazard [8, 3.7, p. 206] the group G is p-adic analytic. Then, by [9, Theorem A], G is of finite rank and the claim follows from [6, Theorem 7.19]. □

For short we call a group G residually-p if it is residually a finite p-group, that is, if for every nontrivial element $x \in G$, there exists a normal subgroup N of G such that $x \notin N$ and G/N is a finite p-group.

Lemma 2.4. *Let G be a residually-p group such that the Lie algebra $L_p(G)$ satisfies a polynomial identity. Let X be a commutator closed subset of G consisting of bounded Engel elements, and assume that $G = \langle x_1, \ldots, x_m \rangle$ for some $x_1, \ldots, x_m \in X$. Then G is linear.*

Proof. Of course $L(G)$ satisfies the same polynomial identity as $L_p(G)$. For any x_i, denote by a_i the element $x_iG_2 \in L(G)$. Then $L(G)$ is generated by $a_1, \ldots, a_m$. Take any $a \in S\langle a_1, \ldots, a_m \rangle$ and let x be the group-commutator in x_i having the same system of brackets as a. We have $x \in X$ and so x is a bounded Engel element. This implies that a is ad-nilpotent. Thus, by Theorem 2.1, $L(G)$ is nilpotent.

Let $\hat{G}$ be the pro-p completion of G, that is, the inverse limit of all quotients of G which are finite p-groups. Notice that $\hat{G}$ is finitely generated, as G is finitely generated. Furthermore, $L(\hat{G})$ can be identified with $L(G)$ (see [10, Proposition 3.2.2] for more details). Hence $L(\hat{G})$ is nilpotent and so, by Lemma 2.3, the pro-p group $\hat{G}$ is linear. On the other hand, since G is residually-p, G embeds in $\hat{G}$ (see [6, pp. 18–19]) and therefore G is also linear. □

The next result is the analogous of [23, Theorem 2.1] for nil groups. It follows from Lemma 2.4, together with the fact that linear Engel groups are locally nilpotent (see, for instance, [19, Theorem 2.6]).

Theorem 2.5. *Let G be a residually-p nil group such that the Lie algebra $L_p(G)$ satisfies a polynomial identity. Then G is locally nilpotent.*

3. The main results. Before proving Theorem A, we quote a straightforward corollary of [20, Lemma 2.1] (see [13, Lemma 3.5] for the proof).

Lemma 3.1. *Let G be a finitely generated residually finite-nilpotent group. For p a prime, denote by R_p the intersection of all normal subgroups of G of finite p-power index. If G/R_p is nilpotent for all p, then G is nilpotent.*

We restate Theorem A for the reader's convenience: *let G be a residually finite group satisfying an identity $w \equiv 1$. Suppose that G is generated by a commutator closed set X of bounded Engel elements. Then G is locally nilpotent.*

Proof of Theorem A. Let H be a finitely generated subgroup of G. Consider a subgroup K of G containing H which is generated by finitely many elements of X. Clearly, K is also a residually finite group satisfying the same identity as G. Set $X_K = X \cap K$. Then X_K is a commutator closed set of bounded Engel elements of K. Since soluble-by-finite groups generated by Engel elements are nilpotent (see [12, 12.3.7, 12.3.3]), K is residually finite-nilpotent. Therefore, by Lemma 3.1, we may assume that K is residually-p for some prime p.

Let $L_p(K)$ be the Lie algebra associated with the Zassenhaus–Jennings–Lazard series of K. Thus, by Theorem 2.2, $L_p(K)$ satisfies a polynomial identity. It follows from Lemma 2.4 that K is linear. Notice that K cannot contain a nonabelian free subgroup, because it satisfies an identity. Hence, by Tits' alternative [18], K is soluble-by-finite and therefore nilpotent, as already explained above. In particular, H is nilpotent. This proves that G is locally nilpotent. □

Remark 3.2. In the case when G is a *periodic* residually finite Engel group satisfying an identity, Conjecture 1.1 can be easily derived from Zelmanov's results. In fact, arguing as in the first part of the above proof, the subgroup H is residually-p and therefore finite by [23, Theorem 2.1]. Consequently, H is nilpotent (see [12, 12.3.4]) and G is locally nilpotent.

The following lemma is a particular case of [4, Corollary 5].

Lemma 3.3. *Let X be a normal commutator closed subset of a group G. Suppose that G is generated by finitely many elements of X. If x is Engel for all $x \in X$, then each term of the derived series of G is finitely generated.*

Next we extend Theorem A to locally graded groups, assuming that the set of bounded Engel elements is normal. Related results were obtained in [4,16] (see also [15,17]).

In what follows, as usual, the finite residual of a group G is the intersection of all (normal) subgroups of finite index of G. This is a characteristic subgroup of G.

Theorem 3.4. *Let G be a locally graded group satisfying an identity $w \equiv 1$. Suppose that G is generated by a normal commutator closed set X of bounded Engel elements. Then G is locally nilpotent.*

Proof. Let H be a finitely generated subgroup of G, and take $x_1, \ldots, x_m \in X$ such that $H \le K = \langle x_1, \ldots, x_m \rangle$. Obviously, every subgroup and quotient of K satisfies the identity $w \equiv 1$. Let R be the finite residual of K.

First suppose $R \neq 1$. Since K/R is a residually finite group, and the set $\{xR \mid x \in X \cap K\}$ is a commutator closed subset of K/R consisting of bounded Engel elements, Theorem A implies that K/R is nilpotent. Then, for some $d \geq 1$, the dth term $K^{(d)}$ of the derived series of K is a subgroup of R. On the other hand, $K/K^{(d)}$ is nilpotent, because it is a soluble group generated by Engel elements [12, 12.3.3]. Hence, $R/K^{(d)}$ is a subgroup of the finitely generated nilpotent group $K/K^{(d)}$, so that $R/K^{(d)}$ is also finitely generated. Moreover, by Lemma 3.3, $K^{(d)}$ is finitely generated and therefore R is finitely generated, as well. By hypothesis G is a locally graded group, so there exists a proper subgroup of R with finite index. This implies that the finite residual S of R is a proper subgroup of R. Now R/S is a residually finite group, and the set $\{xS \mid x \in X \cap R\}$ is a commutator closed subset of R/S of bounded Engel elements. Then, by Theorem A, R/S is nilpotent. It follows that K/S is soluble group. Applying again Theorem A, we obtain that K/S is a finitely generated nilpotent group, whence K/S is residually finite. This gives $R = S$, which is a contradiction.

Finally, if $R = 1$, then K is residually finite and as above, by Theorem A, we conclude that K is nilpotent. Thus, H is nilpotent and G is locally nilpotent. □

Notice that Theorem 3.4 applies in particular to nil groups, proving Corollary B. Also, if G is a *periodic* locally graded group satisfying an identity, then, using Remark 3.2 instead of Theorem A in the proof of Theorem 3.4, G is locally nilpotent under the weaker hypothesis that it is an Engel group.

4. Concluding remarks. Given a group G, let $E(G)$ be the set of all bounded Engel elements of G. It is clear that $E(G)$ is invariant under automorphisms of G, but it is still unknown whether it is a subgroup (see, for instance, [1]). Assume that the group G is locally graded and satisfies an identity. Denote by $HP(G)$ its Hirsch–Plotkin radical, i.e., the unique maximal normal locally nilpotent subgroup containing all normal locally nilpotent subgroups of G (see [12, 12.1.3]). Then, according to Theorem 3.4, if $E(G)$ were a subgroup, then $E(G)$ would be locally nilpotent and therefore contained in $HP(G)$. This means that, in order to show that $E(G)$ is not a subgroup, one could try to solve the following problem.

Problem 4.1. *Find a locally graded group G satisfying an identity such that $E(G) \neq 1$ and $HP(G) = 1$.*

In [5] the authors give some examples of residually finite groups satisfying an identity. One of these is the group $G = \mathbb{Z} \ltimes B(\mathbb{Z}, 4)$, where $B(\mathbb{Z}, 4)$ is the free group of exponent 4 on the generators x_m, with $m \in \mathbb{Z}$. By [5, Theorem 8], G is a residually finite group such that $[x, y]^4 = 1$, for all $x, y \in G$. Moreover, any involution of G is a 3-Engel element. In fact, for any involution g of an arbitrary group, we have

$$[x,_n g] = [x, g]^{(-2)^{n-1}}$$

for any $n \geq 1$ and all elements x of the group [1, Proposition 3.3]. However, in our case, $HP(G) = B(\mathbb{Z}, 4)$.

For completeness, we point out that there exists a group, based on the (first) Grigorchuk group, in which the set of Engel elements is not a subgroup. This is an (unpublished) example of Bludov which has been refined by Bartholdi, who showed that the Grigorchuk group is not Engel even if it is generated by Engel elements [2, Theorem 1].

Acknowledgements. The authors wish to thank Professor P. Shumyatsky for interesting discussions and the anonymous referee for useful comments.

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Received: 20 June 2017