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On the image, characterization, and automatic continuity of (σ, τ)-derivations

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Abstract. The first main theorem of this paper asserts that any (σ, τ)-derivation d, under certain conditions, either is a σ-derivation or is a scalar multiple of $(\sigma - \tau)$, i.e. $d = \lambda(\sigma - \tau)$ for some $\lambda \in \mathbb{C}\backslash\{0\}$. By using this characterization, we achieve a result concerning the automatic continuity of (σ, τ)-derivations on Banach algebras which reads as follows. Let $\mathcal{A}$ be a unital, commutative, semi-simple Banach algebra, and let $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ be two distinct endomorphisms such that $\varphi\sigma(\mathbf{e})$ and $\varphi\tau(\mathbf{e})$ are non-zero complex numbers for all $\varphi \in \Phi_{\mathcal{A}}$. If $d : \mathcal{A} \to \mathcal{A}$ is a (σ, τ)-derivation such that φd is a non-zero linear functional for every $\varphi \in \Phi_{\mathcal{A}}$, then d is automatically continuous. As another objective of this research, we prove that if $\mathfrak{M}$ is a commutative von Neumann algebra and $\sigma : \mathfrak{M} \to \mathfrak{M}$ is an endomorphism, then every Jordan σ-derivation $d : \mathfrak{M} \to \mathfrak{M}$ is identically zero.

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1. Introduction and preliminaries. Throughout the paper, $\mathcal{A}$ and $\mathfrak{M}$ will denote a Banach algebra and a von Neumann algebra, respectively. If $\mathcal{A}$ is unital, then $\mathbf{e}$ stands for its unit element. Before everything else, let us recall some basic definitions and set the notation which is used in the sequel. A non-zero linear functional φ on $\mathcal{A}$ is called a *character* if $\varphi(ab) = \varphi(a)\varphi(b)$ for every $a, b \in \mathcal{A}$. Throughout this article, $\Phi_{\mathcal{A}}$ denotes the set of all characters on $\mathcal{A}$. We know that, for an arbitrary element $\varphi \in \Phi_{\mathcal{A}}$, $\ker \varphi$, the kernel of φ, is a maximal ideal of $\mathcal{A}$ (see [4, Proposition 3.1.2]).

Let $\mathcal{H}$ be a Hilbert space. By $\mathcal{B}(\mathcal{H})$ we denote the set of all bounded linear mappings from $\mathcal{H}$ into itself. For each subset $\mathfrak{M}$ of $\mathcal{B}(\mathcal{H})$, let $\mathfrak{M}'$ denote the set of all bounded linear maps on $\mathcal{H}$ commuting with every linear mapping of $\mathfrak{M}$. Clearly, $\mathfrak{M}'$ is a Banach algebra containing the identity operator $\mathcal{I}$. If $\mathfrak{M}$ is invariant under the $*$-operation, that is, if $X \in \mathfrak{M}$ implies $X^* \in \mathfrak{M}$, then $\mathfrak{M}'$ is a C^*-algebra acting on the Hilbert space $\mathcal{H}$. A von Neumann algebra on the Hilbert space $\mathcal{H}$ is a $*$-subalgebra $\mathfrak{M}$ of $\mathcal{B}(\mathcal{H})$ such that $\mathfrak{M} = (\mathfrak{M}')' = \mathfrak{M}''$. For more details see [18]. By $\mathfrak{M}_{sa}$ we denote the set of all self-adjoint elements of $\mathfrak{M}$ (i.e. $\mathfrak{M}_{sa} = \{A \in \mathfrak{M} \mid A^* = A\}$) and the set of all projections in $\mathfrak{M}$ is denoted by $\mathcal{P}(\mathfrak{M})$ (i.e. $\mathcal{P}(\mathfrak{M}) = \{P \in \mathfrak{M} \mid P^2 = P,\ P^* = P\}$). Elements in $\mathfrak{M}$ which can be written as a finite real-linear combinations of mutually orthogonal projections in $\mathfrak{M}$ are usually called algebraic elements. It is known that the set of all algebraic elements of $\mathfrak{M}$ is norm dense in $\mathfrak{M}_{sa}$. For more details see [10,13,16].

Let $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ be linear maps. A linear mapping $d : \mathcal{A} \to \mathcal{A}$ is called a (σ, τ)-derivation (resp. Jordan (σ, τ)-derivation) if $d(ab) = d(a)\sigma(b) + \tau(a)d(b)$ (resp. $d(a^2) = d(a)\sigma(a) + \tau(a)d(a)$) holds for all $a, b \in \mathcal{A}$. In the case that $\sigma = \tau$, the linear mapping d is called a σ-derivation (resp. Jordan σ-derivation). Clearly, if $\sigma = \tau = \mathcal{I}$, the identity mapping on $\mathcal{A}$, then we reach to the usual notion of a derivation (resp. Jordan derivation) on the algebra $\mathcal{A}$. Note that every homomorphism θ is a $\frac{\theta}{2}$-derivation, since $\theta(ab) = \theta(a)\frac{\theta(b)}{2} + \frac{\theta(a)}{2}\theta(b)$. Hence, the theory of σ-derivations covers the theory of derivations and homomorphisms (see [7–9,11]). So far, many studies have been done about (σ, τ)-derivations. As can be seen, most of these articles have focused on the commutativity of rings, automatic continuity, amenability, stability, and so on (for instance, see [1,2,5,6,11,12]). It is noteworthy that the theory of automatic continuity of derivations has a fairly long history. Results on automatic continuity of linear operators defined on Banach algebras comprise a fruitful area of research intensively developed during the last sixty years. The references [3,4] review most of the main achievements obtained during the last sixty years. Furthermore, the problem of automatic continuity is also considered for σ-derivations (see [7,8,11]). In this study, by getting idea from [3, Proposition 1.8.10], we offer a characterization of (σ, τ)-derivations on Banach algebras. The first main theorem reads as follows.

Let $\mathcal{A}$ be a unital, commutative, semi-simple Banach algebra, and let $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ be two endomorphisms such that $\varphi\sigma(\mathbf{e})$ and $\varphi\tau(\mathbf{e})$ are non-zero complex numbers for all $\varphi \in \Phi_{\mathcal{A}}$. If $d : \mathcal{A} \to \mathcal{A}$ is a (σ, τ)-derivation such that φd is a non-zero linear functional for every $\varphi \in \Phi_{\mathcal{A}}$, then either

(1) $\sigma = \tau$ and d is a σ-derivation; or

(2) $\sigma \neq \tau$ and $d = \lambda(\sigma - \tau)$ for some $\lambda \in \mathbb{C}\backslash\{0\}$.

From the above-mentioned theorem, we achieve a corollary concerning the automatic continuity of (σ, τ)-derivations which reads as follows. Let $\mathcal{A}$ be a unital, commutative, semi-simple Banach algebra, and let $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ be two distinct endomorphisms such that $\varphi\sigma(\mathbf{e})$ and $\varphi\tau(\mathbf{e})$ are non-zero complex numbers for all $\varphi \in \Phi_{\mathcal{A}}$. If $d : \mathcal{A} \to \mathcal{A}$ is a (σ, τ)-derivation such that φd is a non-zero linear functional for every $\varphi \in \Phi_{\mathcal{A}}$, then d is automatically continuous. In the current study, we prove a Singer–Wermer type theorem for Jordan σ-derivations on von Neumann algebras. Now, we offer a short background in this issue. In 1955, Singer and Wermer [17] achieved a fundamental result which started a subsequent investigation about the image of derivations on Banach algebras. The result states that if $\mathcal{A}$ is a commutative Banach algebra and $d : \mathcal{A} \to \mathcal{A}$ is a bounded derivation, then $d(\mathcal{A}) \subseteq Rad(\mathcal{A})$, where $Rad(\mathcal{A})$ denotes the Jacobson radical of $\mathcal{A}$. It is evident that if $\mathcal{A}$ is semi-simple, i.e. $Rad(\mathcal{A}) = \{0\}$, then d is identically zero. In this paper, we show that if $\mathfrak{M}$ is a commutative von Neumann algebra and $\sigma : \mathfrak{M} \to \mathfrak{M}$ is an endomorphism, then every Jordan σ-derivation $d : \mathfrak{M} \to \mathfrak{M}$ is identically zero. From this theorem it is obtained that if $\mathfrak{M}$ is a commutative von Neumann algebra and $\sigma, \tau : \mathfrak{M} \to \mathfrak{M}$ are $*$-linear mappings such that $\frac{\sigma+\tau}{2}$ is an endomorphism, then every Jordan $* - (\sigma, \tau)$-derivation $d : \mathfrak{M} \to \mathfrak{M}$ is identically zero.

2. Characterization of (σ, τ)-derivations on algebras. We begin with a characterization of (σ, τ)-derivations on unital, commutative, semi-simple Banach algebras.

Theorem 2.1. *Let $\mathcal{A}$ be a unital, commutative, semi-simple Banach algebra, and let $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ be two endomorphisms such that $\varphi\sigma(\mathbf{e})$ and $\varphi\tau(\mathbf{e})$ are non-zero complex numbers for all $\varphi \in \Phi_{\mathcal{A}}$. If $d : \mathcal{A} \to \mathcal{A}$ is a (σ, τ)-derivation such that φd is a non-zero linear functional for every $\varphi \in \Phi_{\mathcal{A}}$, then either*

(1) $\sigma = \tau$ and d is a σ-derivation; or

(2) $\sigma \neq \tau$ and $d = \lambda(\sigma - \tau)$ for some $\lambda \in \mathbb{C}\backslash\{0\}$.

Proof. Let φ be an arbitrary element of $\Phi_{\mathcal{A}}$. Putting $D = \varphi d$, $\Sigma = \varphi\sigma$, and $\Psi = \varphi\tau$, we have $D(ab) = D(a)\Sigma(b) + \Psi(a)D(b)$ for all $a, b \in \mathcal{A}$. It means that D is a non-zero (Σ, Ψ)-derivation from $\mathcal{A}$ into $\mathbb{C}$. Clearly, $\Sigma(\mathbf{e}) = \Psi(\mathbf{e}) = 1$ and so, $D(\mathbf{e}) = 0$. Therefore, ker (D) is a subalgebra of $\mathcal{A}$ containing the unit element $\mathbf{e}$. According to the main theorem of [15], D has one of the following forms:

(1) $D = \lambda\varphi_1$, where $\lambda \in \mathbb{C}\backslash\{0\}$ and $\varphi_1 \in \Phi_{\mathcal{A}}$,
(2) There is an element $\varphi_1 \in \Phi_{\mathcal{A}}$ such that $D(ab) = D(a)\varphi_1(b) + \varphi_1(a)D(b)$,
(3) $D = \lambda(\varphi' - \tau')$, where $\lambda \in \mathbb{C}\backslash\{0\}$ and φ', τ' are distinct elements of $\Phi_{\mathcal{A}}$.

In the following, we investigate the above-mentioned three cases.

Case 1: Suppose that D has the first form, i.e. $D = \lambda\varphi_1$. Then $0 = D(\mathbf{e}) = \lambda\varphi_1(\mathbf{e}) = \lambda$. This contradiction shows that case (1) above is impossible.

Case 2: Suppose that there exists an element φ_1 of $\Phi_{\mathcal{A}}$ satisfying

$$D(ab) = D(a)\varphi_1(b) + \varphi_1(a)D(b). \tag{2.1}$$

On the other side, we have

$$D(ab) = D(a)\Sigma(b) + \Psi(a)D(b). \tag{2.2}$$

Comparing (2.1) and (2.2), we obtain that

$$(\Sigma - \varphi_1)(b)D(a) + (\Psi - \varphi_1)(a)D(b) = 0 \quad \text{for all} \quad a, b \in \mathcal{A}. \tag{2.3}$$

If $D(a) = 0$, then $(\Psi - \varphi_1)(a) = 0$. By switching a and b in (2.3), we get that $(\Sigma - \varphi_1)(a) = 0$. Therefore, we have

$$(\Psi - \varphi_1)(a) = (\Sigma - \varphi_1)(a) = 0. \tag{2.4}$$

It follows from (2.4) that $(\Psi - \varphi_1)(a) + (\Sigma - \varphi_1)(a) = 0$, and it means that

$$(\Psi - 2\varphi_1 + \Sigma)(a) = 0. \tag{2.5}$$

If $D(a) \neq 0$, then by considering $a = b$ in (2.3) and factoring out $D(a)$, we find that

$$(\Sigma - 2\varphi_1 + \Psi)(a) = 0. \tag{2.6}$$

The Eqs. (2.5) and (2.6) imply that $(\Psi - 2\varphi_1 + \Sigma)(a) = 0$ for all $a \in \mathcal{A}$. Hence, we can write

$$\Psi - 2\varphi_1 + \Sigma \equiv 0. \tag{2.7}$$

We know that $\Phi_{\mathcal{A}}$ is a linearly independent subspace in $\mathcal{A}^{\times}$ (see [3, p. 38]). Since $\Psi, \Sigma, \varphi_1 \in \Phi_{\mathcal{A}}$ with $\varphi_1 = \frac{1}{2}(\Psi + \Sigma)$, we have $\Psi = \Sigma = \varphi_1$. Thus $\varphi(\tau(a) - \sigma(a)) = 0$ for all $a \in \mathcal{A}$. Since we are assuming that φ is arbitrary, $\tau(a) - \sigma(a) \in \bigcap_{\varphi \in \Phi_{\mathcal{A}}} \ker(\varphi)$. It is well known that in a unital, commutative, and semi-simple complex Banach algebra $\mathcal{A}$, $\bigcap_{\varphi \in \Phi_{\mathcal{A}}} \ker(\varphi) = Rad(\mathcal{A}) = \{0\}$. Hence, $\tau = \sigma$ and it means that d is a σ-derivation.

Case 3: Suppose that $D = \lambda(\varphi' - \tau')$, where $\lambda \in \mathbb{C}\backslash\{0\}$ and φ', τ' are distinct elements of $\Phi_{\mathcal{A}}$, i.e. $\varphi' \neq \tau'$. Hence, we can consider an element b_0 of $\mathcal{A}$ such that $\varphi'(b_0) = 1$ and $\tau'(b_0) = 0$. In this case, we have $D(b_0) = \lambda(\varphi'(b_0) - \tau'(b_0)) = \lambda$ and further,

$$\begin{aligned}
\lambda\varphi'(a) &= \lambda(\varphi'(ab_0) - \tau'(ab_0)) \\
&= D(ab_0) \\
&= D(a)\Sigma(b_0) + \Psi(a)D(b_0) \\
&= \lambda(\varphi'(a) - \tau'(a))\Sigma(b_0) + \lambda\Psi(a) \\
&= \lambda\left[(\varphi'(a) - \tau'(a))\Sigma(b_0) + \Psi(a)\right].
\end{aligned}$$

Since $\lambda \neq 0$ and a is an arbitrary element of $\mathcal{A}$, we have

$$\varphi' - \Sigma(b_0)(\varphi' - \tau') - \Psi \equiv 0. \tag{2.8}$$

If $\Sigma(b_0) = 0$, then we find that

$$\varphi' = \Psi. \tag{2.9}$$

Now, assume that $\Sigma(b_0) \neq 0$. It follows from (2.8) that

$$(1 - \Sigma(b_0))\varphi' + \Sigma(b_0)\tau' - \Psi \equiv 0. \tag{2.10}$$

Since $\tau', \varphi', \Psi \in \Phi_{\mathcal{A}}$ with $(1 - \Sigma(b_0))\varphi' = \Psi - \Sigma(b_0)\tau'$, we have $\Sigma(b_0) = 1$ and consequently,

$$\tau' = \Psi. \tag{2.11}$$

From (2.9) and (2.11) we obtain that

$$\varphi' = \Psi \ or \ \tau' = \Psi. \tag{2.12}$$

Similarly, let a_0 be an element of $\mathcal{A}$ such that $\varphi'(a_0) = 1$ and $\tau'(a_0) = 0$. So, $D(a_0) = \lambda(\varphi'(a_0) - \tau'(a_0)) = \lambda$ and further,

$$\begin{aligned}\lambda\varphi'(b) &= \lambda(\varphi'(a_0b) - \tau'(a_0b))\\ &= D(a_0b)\\ &= D(a_0)\Sigma(b) + \Psi(a_0)D(b)\\ &= \lambda(\varphi'(a_0) - \tau'(a_0))\Sigma(b) + \lambda\Psi(a_0)(\varphi'(b) - \tau'(b))\\ &= \lambda\Sigma(b) + \lambda\Psi(a_0)(\varphi'(b) - \tau'(b)).\end{aligned}$$

Since $\lambda \neq 0$ and b is an arbitrary element of $\mathcal{A}$, we get

$$\varphi' - \Sigma - \Psi(a_0)(\varphi' - \tau') \equiv 0. \tag{2.13}$$

If $\Psi(a_0) = 0$, then it follows from (2.13) that

$$\varphi' = \Sigma. \tag{2.14}$$

If $\Psi(a_0) \neq 0$, we have $(1 - \Psi(a_0))\varphi' - \Sigma + \Psi(a_0)\tau' \equiv 0$. By an argument similar to what was mentioned above, we obtain that

$$\varphi' = \Sigma \quad \text{or} \quad \tau' = \Sigma. \tag{2.15}$$

The above discussion can be summarized as follows.

1) $\varphi' = \Psi$ or $\tau' = \Psi$.
2) $\Sigma = \varphi'$ or $\Sigma = \tau'$.

If $\varphi' = \Psi$ and $\varphi' = \Sigma$, then we have

$$\begin{aligned}D(ab) &= D(a)\Sigma(b) + \Psi(a)D(b)\\ &= D(a)\Sigma(b) + \Sigma(a)D(b).\end{aligned}$$

It means that

$$\lambda(\Sigma(ab) - \tau'(ab)) = \lambda(\Sigma(a) - \tau'(a))\Sigma(b) + \lambda\Sigma(a)(\Sigma(b) - \tau'(b)).$$

Since $\lambda \neq 0$ and Σ, τ' are homomorphisms, we find that

$$-\tau'(a)\,(\tau'(b) - \Sigma(b)) = \Sigma(a)\,(\Sigma(b) - \tau'(b)) \quad \text{for all} \quad a, b \in \mathcal{A}. \tag{2.16}$$

Now, we show that $\ker(\Sigma) \subsetneqq \ker(\tau')$. Suppose that $\ker(\Sigma) \subseteq \ker(\tau')$. Since $a - \Sigma(a)\mathbf{e} \in \ker(\Sigma)$ and $\ker(\Sigma) \subseteq \ker(\tau')$, $\tau'(a - \Sigma(a)\mathbf{e}) = 0$ for all $a \in \mathcal{A}$, and it implies that $\tau' = \Sigma$. This conclusion together with $\varphi' = \Sigma$ imply that $\varphi' = \tau'$, which is a contradiction. Therefore, $\ker(\Sigma)$ is not a subset of $\ker(\tau')$. So, there is an element $a_0 \in \ker(\Sigma)$ such that $a_0 \notin \ker(\tau')$, i.e. $\Sigma(a_0) = 0$ and $\tau'(a_0) \neq 0$. Replacing a with a_0 in (2.16), we obtain that $-\tau'(a_0)\,(\tau'(b) - \Sigma(b)) = 0$. Since $\tau'(a_0) \neq 0$ and b is an arbitrary element of $\mathcal{A}$, $\tau' = \Sigma$. Hence, we have $\tau' = \Sigma = \varphi'$, and it is a contradiction. Therefore, we have $\varphi' = \Psi$ and $\tau' = \Sigma$. Moreover, by a similar argument we will achieve that $\tau' = \Psi$ and $\Sigma = \varphi'$. Consequently, $D = \lambda(\varphi' - \tau') = \lambda(\Psi - \Sigma)$ or $D = \lambda(\varphi' - \tau') = \lambda(\Sigma - \Psi)$. So, we can write $D = \pm\lambda(\Sigma - \Psi)$. From this, we obtain that $\varphi\,(d(a) \pm \lambda(\sigma - \tau)(a)) = 0$ for all $a \in \mathcal{A}$, $\varphi \in \Phi_{\mathcal{A}}$. Since $\bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker\varphi = \{0\}$, $d = \pm\lambda(\sigma - \tau)$. This proves the theorem completely. $\square$

An immediate corollary about the automatic continuity of (σ,τ)-derivations reads as follows.

Corollary 2.2. *Let $\mathcal{A}$ be a unital, commutative, semi-simple Banach algebra, and let $\sigma,\tau:\mathcal{A}\to\mathcal{A}$ be two distinct endomorphisms such that $\varphi\sigma(\boldsymbol{e})$ and $\varphi\tau(\boldsymbol{e})$ are non-zero complex numbers for all $\varphi\in\Phi_{\mathcal{A}}$. If $d:\mathcal{A}\to\mathcal{A}$ is a (σ,τ)-derivation such that φd is a non-zero linear functional for every $\varphi\in\Phi_{\mathcal{A}}$, then d is automatically continuous.*

Proof. Since σ and τ are supposed to be two distinct endomorphisms, it follows from part (2) of Theorem 2.1 that $d=\lambda(\sigma-\tau)$ for some $\lambda\in\mathbb{C}\backslash\{0\}$. This proves the corollary. □

Theorem 2.3. *Let $\mathcal{A}$ be a C^*-algebra, and let $\sigma,\tau:\mathcal{A}\to\mathcal{A}$ be continuous endomorphisms. Then every Jordan (σ,τ)-derivation from $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule $\mathcal{M}$ is a continuous (σ,τ)-derivation.*

Proof. It is clear that $\mathcal{M}$ is a Banach $\mathcal{A}$-bimodule by the following module actions:

$$a\ltimes m=\tau(a)m,\quad m\rtimes a=m\sigma(a)\quad(a\in\mathcal{A},m\in\mathcal{M}).$$

We denote the above module by $\widehat{\mathcal{M}}$. We have $d(a^2)=d(a)\sigma(a)+\tau(a)d(a)=d(a)\rtimes a+a\ltimes d(a)$ for all $a\in\mathcal{A}$. It means that $d:\mathcal{A}\to\widehat{\mathcal{M}}$ is a Jordan derivation. In view of [14, Corollary 17] we get the required result. □

Below, we present a remark which is used in the proof of Theorem 2.5.

Remark 2.4. Let $\mathfrak{M}$ be a von Neumann algebra. Here, the spectrum of an arbitrary element $A\in\mathfrak{M}$ is denoted by $\mathfrak{S}(A)$ and recall that $\mathfrak{S}(A)=\{\lambda\in\mathbb{C}|\lambda\mathcal{I}-A$ is not invertible in $\mathfrak{M}\}$. Suppose that P is an idempotent of $\mathfrak{M}$, i.e. $P^2=P$. Clearly, if $P\neq 0,\mathcal{I}$, then $\{0,1\}\subseteq\mathfrak{S}(P)$. We show that $\mathfrak{S}(P)\subseteq\{0,1\}$ and it means that $\mathfrak{S}(P)=\{0,1\}$. For $\lambda\neq 0,1$ we have

$$\left(\frac{1}{1-\lambda}P-\frac{1}{\lambda}(\mathcal{I}-P)\right)(P-\lambda\mathcal{I})=(P-\lambda\mathcal{I})\left(\frac{1}{1-\lambda}P-\frac{1}{\lambda}(\mathcal{I}-P)\right)=\mathcal{I}.$$

It means that $P-\lambda\mathcal{I}$ is invertible and so, $\lambda\notin\mathfrak{S}(P)$. Therefore, $\mathfrak{S}(P)=\{0,1\}$.

The following theorem is a Singer-Wermer type theorem for Jordan σ-derivations.

Theorem 2.5. *Let $\mathfrak{M}$ be a commutative von Neumann algebra, and let $\sigma:\mathfrak{M}\to\mathfrak{M}$ be an endomorphism. Then every Jordan σ-derivation $d:\mathfrak{M}\to\mathfrak{M}$ is identically zero.*

Proof. We know that every von Neumann algebra is a C^*-algebra and every C^*-algebra is semi-simple (see [13]). Hence, [4, Proposition 5.1.1] implies that σ is continuous, i.e. $\|\sigma\|<\infty$. It follows from Theorem 2.3 that d is continuous. Suppose that φ is an arbitrary character on $\mathfrak{M}$, i.e. $\varphi\in\Phi_{\mathfrak{M}}$. It follows from [4, Proposition 3.1.2] that $\ker(\varphi)$ is a maximal ideal of $\mathfrak{M}$ of codimension 1, i.e. $\dim(\frac{\mathfrak{M}}{\ker\varphi})=1$ for each $\varphi\in\Phi_{\mathfrak{M}}$. Evidently, the algebra $\frac{\mathfrak{M}}{\ker(\varphi)}$ is commutative.

Since $\mathfrak{M}$ is a von Neumann algebra, the set of all algebraic elements of $\mathfrak{M}$ is norm dense in $\mathfrak{M}_{sa}$. Let Q be an arbitrary non-zero projection of $\mathfrak{M}$. We have

$$d(Q) + \ker\varphi = d(Q)\sigma(Q) + \sigma(Q)d(Q) + \ker\varphi = 2d(Q)\sigma(Q) + \ker\varphi.$$

So,

$$2d(Q)\left(\frac{1}{2}\mathcal{I} - \sigma(Q)\right) \in \ker\varphi. \tag{2.17}$$

Clearly, $\sigma(Q)$ is an idempotent and it follows from Remark 2.4 that $\frac{1}{2} \notin \mathfrak{S}(\sigma(Q))$. It means that $\frac{1}{2}\mathcal{I}-\sigma(Q)$ is an invertible element in $\mathfrak{M}$. This fact along with (2.17) imply that $d(Q) \in \ker(\varphi)$. Since φ is an arbitrary element of $\Phi_{\mathfrak{M}}$, $d(Q) \in \bigcap_{\varphi\in\Phi_{\mathfrak{M}}} \ker(\varphi)$. According to [4, Theorem 3.1.3] each member of $\Phi_{\mathfrak{M}}$ is continuous and so, $\ker(\varphi)$ $(\varphi \in \Phi_{\mathfrak{M}})$ is a closed subset and consequently, $\bigcap_{\varphi\in\Phi_{\mathfrak{M}}} \ker(\varphi)$ is also a closed subset of $\mathfrak{M}$. Our next task is to show that $\varphi(d(A)) = 0$ for every $A \in \mathfrak{M}$. Let X be an arbitrary algebraic element of $\mathfrak{M}$. Hence, $X = \sum_{i=1}^{m} r_i P_i$, where $P_1, P_2, \ldots, P_m$ are mutually orthogonal projections and $r_1, r_2, \ldots, r_m$ are real numbers. We have

$$\begin{aligned}\varphi(d(X)) &= \varphi\left(d\left(\sum_{i=1}^{m} r_i P_i\right)\right)\\ &= \sum_{i=1}^{m} r_i \varphi(d(P_i))\\ &= 0.\end{aligned}$$

Since the set of all algebraic elements is norm dense in $\mathfrak{M}_{sa}$ and d is a continuous σ-derivation, $\varphi(d(A)) = 0$ for every $A \in \mathfrak{M}_{sa}$. We know that each A in $\mathfrak{M}$ can be represented as $A = A_1 + iA_2$, where $A_1, A_2 \in \mathfrak{M}_{sa}$. Therefore, $\varphi(d(A)) = \varphi(d(A_1 + iA_2)) = 0$ for all $A \in \mathfrak{M}$ and $\varphi \in \Phi_{\mathfrak{M}}$. It means that $d(\mathfrak{M}) \subseteq \bigcap_{\varphi\in\Phi_{\mathfrak{M}}} \ker(\varphi)$. Since $\mathfrak{M}$ is commutative, it follows from [4, Proposition 3.2.1] that $\bigcap_{\varphi\in\Phi_{\mathfrak{M}}} \ker(\varphi) = Rad(\mathfrak{M})$. As we know, every von Neumann algebra is semi-simple and consequently, d is identically zero. □

Let $\mathcal{A}$ be a $*$-algebra and $T : \mathcal{A} \to \mathcal{A}$ be a linear mapping. We define a linear mapping T^* on $\mathcal{A}$ by $T^*(a) = (T(a^*))^*$ for all $a \in \mathcal{A}$. A linear mapping T is called a $*$-map if $T = T^*$. We are now ready to prove the above-mentioned theorem for Jordan $* - (\sigma, \tau)-$derivations.

Corollary 2.6. *Suppose that $\mathfrak{M}$ is a commutative von Neumann algebra and $\sigma, \tau : \mathfrak{M} \to \mathfrak{M}$ are $*$-linear mappings such that $\frac{\sigma+\tau}{2}$ is an endomorphism. Then every Jordan $* - (\sigma, \tau)$-derivation $d : \mathfrak{M} \to \mathfrak{M}$ is identically zero.*

Proof. According to the aforementioned assumptions, we have $d = d^*$, $\sigma = \sigma^*$, and $\tau = \tau^*$. By getting idea from [11], we have

$$\begin{aligned}d(A^2) = d^*(A^2) = (d(A^2)^*)^* &= (d(A^*)\sigma(A^*) + \tau(A^*)d(A^*))^*\\ &= \sigma(A)d(A) + d(A)\tau(A).\end{aligned}$$

Therefore,

$$\begin{aligned} d(A^2) &= \frac{1}{2}d(A^2) + \frac{1}{2}d(A^2) \\ &= d(A)\frac{\sigma(A)}{2} + \frac{\tau(A)}{2}d(A) + d(A)\frac{\tau(A)}{2} + \frac{\sigma(A)}{2}d(A) \\ &= d(A)\left(\frac{\sigma+\tau}{2}\right)(A) + \left(\frac{\sigma+\tau}{2}\right)(A)d(A). \end{aligned}$$

So d is a Jordan Σ-derivation, where $\Sigma = \frac{\sigma+\tau}{2}$. Now, Theorem 2.5 is exactly what we need to complete the proof. □

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