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Generalized Ambrosetti–Rabinowitz condition for minimal period solutions of autonomous Hamiltonian systems

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Abstract. We show, under an iterative condition, generalizing that of Ambrosetti–Rabinowitz and using a variational method, the existence of a T-periodic solution for the autonomous superquadratic second order Hamiltonian system with even potential

$$\ddot{z} + V'(z) = 0, \qquad z \in \mathbb{R}^N, \qquad N \in \mathbb{N}^*$$

for any prescribed period $T > 0$. Moreover, under a certain symmetry condition, such a solution possesses T or $T/3$ as its minimal period.

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1. Introduction. In this paper, we consider the following second order Hamiltonian system

$$\ddot{z} + V'(z) = 0, \tag{1.1}$$

where $z : \mathbb{R} \to \mathbb{R}^N$ is a vector function, N is a positive integer, and V' is the gradient vector of the potential function V with respect to z.

In [24], Rabinowitz proved the following result

Theorem 1.1. *Suppose* $V \in C^1(\mathbb{R}^N, \mathbb{R})$ *and satisfies*
$(\overline{V_1})$ $\quad V(z) = o(|z|^2)$ *at* $z = 0$.
(AR) $\quad \exists \mu > 2, r > 0,$ *s.t.* $V \geq 0$ *and* $\forall |z| > r, \quad 0 < \mu V(z) \leq zV'(z)$.
Then, for every $T > 0$, *problem* (1.1) *has a non-constant* T*-periodic solution.*

He also conjectured that (1.1) possesses a non-trivial solution with any prescribed minimal period. Later, Ekeland, and Hoffer [9] confirmed this conjecture in the case of strictly convex Hamiltonian systems.

In the last years, many researchers have been interested in [24]. We can cite, for example, [1,2,4–6,18]. Most of these works are based on convex or weak convex assumptions to insure the existence of periodic solutions [7,8,12–14,19,20,26]. Only few of them treated the problem without relying on this condition. One can be referred, for example, to Girardi, Matzeu [10,11] and Long [16–18].

Recently, Souissi studied in [25] the problem (1.1) for the superquadratic second order Hamiltonian system with non-convex even potential, under the following new growth assumption

$$(ARS)\qquad \begin{aligned} &\text{There exist} \quad n \in \mathbb{N}^* \quad \text{and} \quad \alpha_n > 3-n, \quad \text{such that} \\ &\alpha_n z^{n-1} V^{(n-1)}(z) \leq z^n V^{(n)}(z), \quad \forall z \in \mathbb{R}\backslash\{0\}, \end{aligned}$$

where $V^{(n)}$ is the n^{th} derivative of V, for any integer $n \geq 1$. Under this condition, which is stronger than (AR), he proved that there exists at least one periodic solution with minimal period T/k for some integer $1 \leq k \leq 3$.

Throughout this work, we write $V^{(0)} = V$, $V^{(1)} = V'$, and $V^{(2)} = V''$. The main result of this paper reads as follows:

Theorem 1.2. *Assume that there exists an integer* $n \geq 2$ *such that* $V \in C^n(\mathbb{R}^N, \mathbb{R})$ *and satisfies*

$$(V_1) \quad V(0) < V(z) \quad \forall\, z \in \mathbb{R}^N\backslash\{0\}.$$

$$(V_2) \quad V'(z) = o(|z|) \quad at \quad z = 0.$$

$$(V_3) \quad V(-z) = V(z), \qquad \forall z \in \mathbb{R}^N.$$

(ARS_n) *There exist* $\alpha_n > 3-n$ *and* $r > 0$, *such that*

$$\alpha_n V^{(n-1)}(z).z^{n-1} \leq V^{(n)}(z).z^n, \qquad \forall\, |z| > r.$$

Then, for every $T > 0$, *problem* (1.1) *has at least one* T*-periodic solution* u *with* T *or* $T/3$ *as its minimal period. Moreover,* u *is even about* 0 *and* $T/2$ *and odd about* $T/4$ *and* $3T/4$.

Remark 1.3. (1) Take $N = 1$ and consider the potential V defined by

$$V(z) = z^2\, log(1 + z^2).$$

It is easily seen that V satisfies the conditions (V_1), (V_2), (V_3), and (ARS_5). However, it satisfies neither (AR) nor (ARS).

(2) Condition (ARS_n) is just a local version of (ARS) in [25], which is not necessarily satisfied near zero. The condition (V_2) enables us to overcome the difficulty arising from this restriction.

2. Variational formulation. First of all, we remark that if V is such that $V(0) \neq 0$, then setting

$$\tilde{V}(z) = V(z) - V(0), \qquad \forall\, z \in \mathbb{R}^N,$$

we obtain a potential function $\tilde{V}$ vanishing at zero and satisfiying the conditions of Theorem 1.2. In addition, for any integer k such that $(2k+1 \leq n)$, we have by (V_3) that

$$\tilde{V}(0) = V^{(2k+1)}(0) = \tilde{V}^{(2k+1)}(0) = 0. \tag{2.1}$$

Moreover, the equation

$$\ddot{z}(t) + \tilde{V}'(z(t)) = 0, \qquad z : \mathbb{R} \to \mathbb{R}^N$$

has the same solutions as (1.1). It follows that we can replace, without loss of generality, in Theorem 1.2 the condition (V_1) by the following one

$$0 = V(0) < V(z), \qquad \forall\, z \in \mathbb{R}^N \backslash \{0\}. \tag{$\tilde{V_1}$}$$

Now, to look for solutions of (1.1), we denote by

$$S_T = \mathbb{R}/T\mathbb{Z},$$

and consider the Sobolev space

$$E_T = H^1(S_T, \mathbb{R}^N),$$

equipped with its usual norm

$$\|z\|_T^2 = \int_0^T \left(|z(t)|^2 + |\dot{z}(t)|^2\right) dt = \|z\|_2^2 + \|\dot{z}\|_2^2,$$

where $\|.\|_2$ is the usual norm in $L^2(\mathbb{R}, \mathbb{R}^N)$. Then, E_T is a Hilbert space with the corresponding inner product

$$<y, z>_T = \int_0^T y(t).z(t)dt + \int_0^T \dot{y}(t).\dot{z}(t)dt.$$

We define on E_T the functional

$$f(z) = \int_0^T \left(\frac{1}{2}|\dot{z}(t)|^2 - V(z)\right) dt. \tag{2.2}$$

It is easy to prove that $f \in C^n(E_T, \mathbb{R})$ and to see that the solutions of (1.1) are the critical points of the functional f.

3. Proof of Theorem 1.2. To look for non-trivial critical points of f, we start by defining the W-action for any T-periodic function $z : S_T \to \mathbb{R}^N$, where

$$W = \{\delta_1, \delta_2\}$$

with

$$\delta_1 z(t) = z(-t) \qquad \text{and} \qquad \delta_2 z(t) = -z(t + T/2), \qquad \text{a.e.}$$

Definition 3.1. Let $t_0 \in \mathbb{R}$ and $z : \mathbb{R} \to \mathbb{R}^N$, a vector function. Then z is

(1) t_0-even or even about t_0 if $z(t_0 - t) = z(t_0 + t)$, $\quad \forall\, t \in \mathbb{R}$.
(2) t_0-odd or odd about t_0 if $z(t_0 - t) = -z(t_0 + t)$, $\quad \forall\, t \in \mathbb{R}$.
(3) even if z is 0-even.
(4) odd if z is 0-odd.

Lemma 3.2. *Suppose that z is T-periodic on $\mathbb{R}$. Then, for all $t_0 \in \mathbb{R}$ and $k \in \mathbb{Z}$,*

(1) *if z is t_0-even (resp. t_0-odd), then z is also even (resp. odd) about $(t_0 + kT/2)$.*
(2) *if z is differentiable and t_0-even (resp. t_0-odd), then z' is t_0-odd (resp. t_0-even).*

Proof. (1) If z is t_0-even, then

$$\begin{aligned} z(t_0 + kT/2 + t) &= z(t_0 - kT/2 - t) \qquad &&\text{(by symmetry of } z \text{ about } t_0)\\ &= z(t_0 + kT/2 - t) \qquad &&\text{(by periodicity of } z). \end{aligned}$$

Then z is even about $t_0 + kT/2$.
A similar proof is given in the case where z is t_0-odd.
(2) is obvious. □

Corollary 3.3. *Suppose that z is T-periodic on $\mathbb{R}$. If z is even (resp. odd), then z is also even (resp. odd) about $T/2$.*

Now, we can define

Definition 3.4. A T-periodic vector function $z : S_T \to \mathbb{R}^N$ is said to have

(1) the W-symmetry if $\delta z = z, \quad \forall \delta \in W.$
(2) the W-anti-symmetry if $\delta z = -z, \quad \forall \delta \in W.$

Definition 3.5. A functional f, defined on E_T, is said to be W-invariant if

$$f(\delta z) = f(z), \qquad \forall \delta \in W, \quad \text{and} \quad \forall z \in E_T.$$

These definitions make us able to consider the following closed subspace of E_T,

$$SE_T = \{z \in E_T, \ \text{ s.t. } \ z \ \text{ has the } W\text{-symmetry}\}.$$

Lemma 3.6. *If $z \in SE_T$, then z is $(T/4)$-odd.*

Proof. If $z \in SE_T$, then

$$\begin{aligned} z(T/4 - t) &= z(T/2 - T/4 - t)\\ &= z(T/2 + T/4 + t) \text{ (by Corollary 3.3)}\\ &= \delta_2 z(T/4 + t)\\ &= -z(T/4 + t). \end{aligned}$$

Then z is $(T/4)$-odd.

Lemma 3.7. *Suppose that $V \in C^2(\mathbb{R}^N, \mathbb{R})$ and satisfies (ARS_2), then the functional f is W-invariant.*

Now, we can prove the following result.

Lemma 3.8. *Suppose that $V \in C^2(\mathbb{R}^N, \mathbb{R})$ and satisfies (ARS_2), then the following assertions are equivalent*:

(1) *z is a critical point of f on SE_T.*
(2) *z is a $C^2(S_T, \mathbb{R}^N)$-solution of (1.1) and it has the W-symmetry.*

Proof. (1) $\Longrightarrow$ (2): Suppose z is a critical point of f on SE_T. Then

$$f'(z).y = \int_0^T \Big(\dot{z}.\dot{y} - V'(z).y\Big)dt = 0, \qquad \forall y \in SE_T. \tag{3.1}$$

Since V is of class C^2, we have that $V'(z) \in E_T$, and so it is in $C(S_T, \mathbb{R}^N)$. This makes that the linear system

$$\begin{cases} \dot{p} = q, \\ \dot{q} = -V'(z), \\ p(T/2) = q(3T/4) = 0, \end{cases} \tag{3.2}$$

possesses a unique solution

$$(u, v) \in C^2(\mathbb{R}, \mathbb{R}^N) \times C^1(\mathbb{R}, \mathbb{R}^N)$$

Moreover, since z has the W-symmetry and by (V_3), it is the same for $V'(z)$. Therefore,

$$\int_0^{T/2} V'(z)dt = \int_0^T V'(z)dt = 0,$$

so that v is T-periodic and has the W-anti-symmetry. So, we have

$$\int_0^T v(t)dt = 0 \qquad \text{and} \qquad v(3T/4) = 0.$$

Thus u is T-periodic and has the W-symmetry. So $u \in SE_T$ and by (3.2),

$$\int_0^T (\dot{u}.\dot{y} - V'(z).y)\ dt = \int_0^T (\dot{u} - v).\dot{y}\ dt + v(t).y(t)|_0^T = 0, \qquad \forall y \in SE_T.$$

Using (3.1), this leads to

$$\int_0^T (\dot{u} - v)\dot{y}\ dt = 0, \qquad \forall y \in SE_T.$$

Particularly, for $y = z - u$ and by the fact that $z(3T/4) = v(3T/4) = 0$, we obtain

$$|z(t) - u(t)| \leq \int_{3T/4}^{t} |\dot{z} - \dot{u}|(s)\ ds\ \leq \sqrt{T}\|\dot{z} - \dot{u}\|_{L^2} = 0, \qquad \forall t \in [0, T].$$

It follows that $z \in C^2(S_T, \mathbb{R}^N)$ and is a solution of (1.1).

(2) $\Longrightarrow$ (1) is obvious and the proof is complete. □

Thanks to this lemma, we will look, in the following, for non-trivial critical points of the functional f on SE_T. For this, we begin by showing

Lemma 3.9. *Suppose that $V \in C^n(\mathbb{R}^N, \mathbb{R})$ and that (ARS_n) is satisfied, then (ARS_k) is also satisfied for any integer k such that $1 \leq k \leq n-1$.*

Proof. We suppose that $V \in C^n(\mathbb{R}^N, \mathbb{R})$ and that (ARS_n) is satisfied. We will prove that (ARS_{n-1}) is also satisfied. The proof will depend on the parity of n.

First case: n is odd

Let us consider the function $\phi_{n,z}$ defined, for $t \in [0,1]$, by

$$\phi_{n,z}(t) = V^{(n-2)}(tz).z^{n-2} - V^{(n-1)}(0).z^{n-1}t.$$

We have immediately that $\phi_{n,z} \in C^n([0,1], \mathbb{R})$ and for all $t \in [0,1]$, we have

$$\phi'_{n,z}(t) = V^{(n-1)}(tz).z^{n-1} - V^{(n-1)}(0).z^{n-1},$$

$$\phi''_{n,z}(t) = V^{(n)}(tz).z^{n}.$$

It follows that

$$\phi_{n,z}(0) = \phi'_{n,z}(0) = \phi''_{n,z}(0) = 0,$$

so that we have

$$V^{(n-2)}(z).z^{n-2} = \phi_{n,z}(1) - \phi_{n,z}(0) + V^{(n-1)}(0).z^{n-1} = \int_0^1 V^{(n-1)}(tz).z^{n-1}dt.$$

Moreover, from (ARS_n), we deduce that

$$\alpha_n V^{(n-2)}(z).z^{n-2} \leq \int_0^1 tV^{(n)}(tz).z^n dt = V^{(n-1)}(z).z^{n-1} - V^{(n-2)}(z).z^{n-2}$$

which means that

$$(\alpha_n + 1)V^{(n-2)}(z).z^{n-2} \leq V^{(n-1)}(z).z^{n-1} \tag{3.3}$$

Second case: n is even

We consider the function $\psi_{n,z}$ defined, for $t \in [0,1]$, by

$$\psi_{n,z}(t) = V^{(n-2)}(tz).z^{n-2} - \frac{1}{2}V^{(n)}(0).z^n t^2.$$

Following the same procedure as in the first case and replacing $\phi_{n,z}$ by $\psi_{n,z}$, we also obtain (3.3).

Conclusion

To recapitulate, we have proved, independently on the parity of n, that taking $\alpha_{n-1} = \alpha_n + 1$, we obtain that (ARS_n) implies (ARS_{n-1}).

Obviously, iterating this procedure $(n-k)$ times, for some integer k such that $0 \leq k \leq n-1$, we obtain (ARS_k). □

We recall, in what follows, a fundamental condition, due to Palais and Smale, for the convergence of bounded sequences in Banach spaces.

Definition 3.10 (*Palais-Smale condition*). Let E be a Banach space and consider a functional $f \in C^1(E, \mathbb{R})$. We say that f satisfies the Palais-Smale condition (PS) on E if for all sequences $\{u_n\} \subset E$ such that $\{f(u_n)\}$ is bounded and $f'(u_n) \to 0$, there exists a convergent subsequence.

Lemma 3.11. *Assume that* $V \in C^1(\mathbb{R}^N, \mathbb{R})$ *and satisfies* $(\tilde{V_1}), (V_2)$, *and* (ARS_1), *then* f *satisfies* (PS) *on* E_T *and* SE_T.

Proof. If $V \in C^1(\mathbb{R}^N, \mathbb{R})$ and satisfies (ARS_1), then

$$0 < \alpha_1 V(z) \leq V'(z).z, \qquad \forall |z| > r. \tag{3.4}$$

Moreover, by $(\tilde{V_1})$ and (V_2), we obtain

$$V(z) = o(|z|^2), \qquad \text{at} \quad z = 0.$$

Then, proceeding as Rabinowitz [23,24], we deduce that the functional f satisfies (PS) on E_T. Or, SE_T is a closed subset of E_T, then f satisfies (PS) on SE_T. □

For a given $T > 0$ in order to find T-periodic solutions of (1.1), we use the Mountain-pass theorem due to Ambrosetti and Rabinowitz. For its proof, we refer the reader, for example, to [23].

Theorem 3.12. *Let* E *be a real Hilbert space and consider* $f \in C^2(E, \mathbb{R}^N)$. *Suppose that* f *satisfies the* (PS) *condition and the following*

(F_1) *There exist* ρ *and* $\alpha > 0$ *such that* $f(w) \geq \alpha, \quad \forall w \in \partial B_\rho(0)$.
(F_2) *There exist* $R > \rho$ *and* $e \in E$ *with* $\|e\| \geq R$ *such that* $f(e) \leq 0$.

Then

(1) f *possesses a critical value* $c \geq \alpha$, *which is given by*

$$c = \inf_{h \in \Gamma} \max_{w \in h([0,1])} f(h(w)),$$

where $\Gamma = \{h \in C([0,1], E) /\ h(0) = 0,\ h(1) = e\}$.

(2) *There exists an element* $w_0 \in K_c = \{w \in E /\ f(w) = c$ *and* $f'(w) = 0\}$ *such that the negative Morse index* $i(w_0)$ *of* f *at* w_0 *satisfies*

$$i(w_0) \leq 1. \tag{3.5}$$

In order to apply this theorem, set

$$E = SE_T.$$

As it was shown in the proof of Lemma 3.11, the potential V satisfies $(\overline{V_1})$. It follows that for any $\varepsilon > 0$, there exists $\rho > 0$ such that

$$\|z\| \leq \rho \quad \Longrightarrow \quad 0 \leq V(z) \leq \varepsilon \|z\|^2.$$

Moreover, if $z \in E$, we have that $z(T/4) = z(3T/4) = 0$. This leads to

$$z(t) = \begin{cases} \displaystyle\int_{\frac{T}{4}}^{t} \dot{z}(s)\, ds, & \text{if } t \in [0, T/2], \\ \displaystyle\int_{\frac{3T}{4}}^{t} \dot{z}(s)\, ds, & \text{if } t \in]T/2, T], \end{cases}$$

so that

$$|z(t)| \leq \sqrt{\frac{T}{2}} \|\dot{z}\|_2,$$

and then,

$$\|z\|_2 \leq \frac{T}{\sqrt{2}} \|\dot{z}\|_2. \tag{3.6}$$

By Lemma 3.11, the functional f is of class C^2 and satisfies the (PS) condition on E. Then, for $\varepsilon > 0$, small enough, using (3.6), we obtain

$$f(z) \geq \frac{1}{2}(1 - \varepsilon T^2) \|\dot{z}\|_2^2,$$

which leads to condition (F_1) of Theorem 1.2.

Moreover, from (3.4), we deduce obviously the condition (F_2) of Theorem 1.2.

Now, following Rabinowitz [23], we get

$$\exists z \in SE_T \quad \text{such that} \quad f'(z) = 0 \quad \text{and} \quad f(z) > 0.$$

Next, for every non-constant T-periodic solution z of (1.1), we define the integer

$$O(z) = \sup\{k \geq 1, \text{ such that } z \text{ is } (T/k)\text{-periodic}\},$$

and we denote by $si_T(z)$ the negative Morse index of f at z. Then we can recall the following result, which is a simple corollary of Theorem 4.2 in [18].

Theorem 3.13. *Suppose that $V \in C^2(\mathbb{R}^N, \mathbb{R})$. Then, for $T > 0$ and for any non-constant $C^2(S_T, \mathbb{R}^N)$-solution z of (1.1), being even and $(T/4)$-odd, we have*

$$O(z) \leq 2(si_T(z)) + 1. \tag{3.7}$$

Lemma 3.14. *For $T > 0$, if $z \in SE_T \backslash \{0\}$, then z is not a $2mT$-periodic function for any $m \in \mathbb{N}^*$.*

Proof. Arguing by contradiction, we consider $z \in SE_T \backslash \{0\}$ and $m \in \mathbb{N}^*$. We suppose z to be a $2mT$-periodic function. Since z is even about 0 and $T/2$, then is also even about $mT/2$. Now, if z has the W-symmetry, it must be odd about $mT/2$. Therefore, $z \equiv 0$. □

Finally, we are interested in the minimal period of the solutions. Supposing that z has T/k as a minimal period, for some integer $k \geq 1$, we can deduce, combining (3.5) and (3.7), that

$$1 \leq k \leq 2si_T(z) + 1 \leq 3.$$

Now, suppose that $k = 2$, i.e. z is $(T/2)$-periodic. By Lemma 3.14, z can not be $[2m(T/2)]$-periodic, for any $m \in \mathbb{N}^*$. Particularly, for $m = 1$, we obtain that z can not be T-periodic. This contradicts the definition of z. It follows that

$$O(z) \in \{1, 3\}.$$

References

[1] A. Ambrosetti and V. Coti Zelati, Solutions with minimal period for Hamiltonian systems in a potential well, Ann. Inst. H. Poincaré Anal. Nonlinéaire **4** (1987), 275–296.

[2] A. Ambrosetti and G. Manchini, Solutions of minimal period for a class of convex Hamiltonian systems, Math. Ann. **255** (1981), 405–421.

[3] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Functional Analysis **14** (1973), 349–381.

[4] A. Bahri and P. L. Lions, Morse index of min-max critical points I. Applications to multiplicity results, Comm. Pure Appl. Math. **8** (1988), 1027–1037.

[5] V. Benci and P. H. Rabinowitz, Critical point theorems for indefinite functionals, Invent. Math. **52** (1979), 241–273.

[6] L. Čaklović, S.J. Li, and M. Willem, A note on Palais-Smale condition and coercivity, Differential Integral Equations **3** (1990), 799–800.

[7] Y. Dong and Y. Long, Closed characteristics on partially symmetric compact convex hyper surfaces in $\mathbb{R}^{2n}$, J. Differential Equations **196** (2004), 226–248.

[8] I. Ekeland, Convexity Methods in Hamiltonian Mechanics, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 19, Springer Verlag, Berlin, 1990.

[9] I. Ekeland and H. Hoffer, Periodic solutions with prescribed period for autonomous Hamiltonian systems, Invent. Math. **81** (1985), 155–188.

[10] M. Girardi and M. Matzeu, Some results on solutions of minimal period to superquadratic Hamiltonian systems, Nonlinear Anal. **7** (1983), 475–482.

[11] M. Girardi and M. Matzeu, Solutions of minimal period for a class of nonconvex Hamiltonian systems and applications to the fixed energy problem, Nonlinear Anal. **10** (1986), 371–383.

[12] C. Liu, Minimal period estimates for brake orbits of nonlinear symmetric Hamiltonian systems, arXiv 07/2009, doi:10.3934/dcds.2010.27.337.

[13] C. Liu, Y. Long, and C. Zhu, Multiplicity of closed characteristics on symmetric convex hypersurfaces in $\mathbb{R}^{2n}$, Math. Ann. **323** (2002), 201–215.

[14] C. Liu and D. Zhang, Multiple brake orbits on compact convex symmetric reversible hypersurfaces in $\mathbb{R}^{2n}$, arXiv:1111.0722v1 [math.DS] (2011).

[15] Y. Long, Index Theory for Symplectic Paths with Applications, Progress in Mathematics, 207, Birkhäuser Verlag, Basel, 2002.

[16] Y. Long, On the minimal period for periodic solution problem of nonlinear Hamiltonian systems, Chinese Ann. Math. Ser. B **18** (1997), 481–484.

[17] Y. Long, The minimal period problem of periodic solutions for autonomous second order Hamiltonian systems, J. Differential Equations **111** (1994), 147–171.

[18] Y. Long, The minimal period problem of classical Hamiltonian systems with even potentials, Ann. Inst. H. Poincaré Anal. Non Linéaire **10** (1993), 605–626.

[19] Y. Long, D. Zhang, and C. Zhu, Multiple brake orbits in bounded convex symmetric domains, Adv. Math. **203** (2002), 568–635.

[20] Y. Long and C. Zhu, Closed characteristics on compact convex hypersurfaces in $\mathbb{R}^{2n}$, Ann. of Math. (2) **155** (2002), 317–368.

[21] J. Mawhin and M. Willem, Critical Point Theory and Hamiltonian Systems, Applied Mathematical Sciences, 74, Springer-Verlag, New York, 1989.

[22] R. Palais and S. Smale, A generalized Morse theory, Bull. Amer. Math. Soc. **70** (1964), 165–171.

[23] P.H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, CBMS Regional Conference Series in Mathematics, 65, Published for the Conference Board of the Mathematical Sciences, Washington, DC, by the American Mathematical Society, Providence, RI, 1986.

[24] P. H. Rabinowitz, Periodic solutions of Hamiltonian systems, Comm. Pure Appl. Math. **31** (1978), 157–184.

[25] C. Souissi, Orbits with minimal period for a class of autonomous second order one-dimensional Hamiltonian systems, to appear in Georgian Mathematical Journal.

[26] D. Z. Zhang, Symmetric period solutions with prescribed minimal period for even autonomous semipositive Hamiltonian systems, Sci. China Math. **57** (2014), 81–96.

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