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Inequalities for combinatorial sums

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Abstract. For $k, l \in \mathbf{N}$, let

$$P_{k,l} = \left(\frac{l}{k+l}\right)^{k+l} \sum_{\nu=0}^{k-1} \binom{k+l}{\nu} \left(\frac{k}{l}\right)^{\nu}$$

$$\text{and} \quad Q_{k,l} = \left(\frac{l}{k+l}\right)^{k+l} \sum_{\nu=0}^{k} \binom{k+l}{\nu} \left(\frac{k}{l}\right)^{\nu}.$$

We prove that the inequality

$$\frac{1}{4} \leq P_{k,l}$$

is valid for all natural numbers k and l. The sign of equality holds if and only if $k = l = 1$. This complements a result of Vietoris, who showed that

$$P_{k,l} < \frac{1}{2} \quad (k, l \in \mathbf{N}).$$

An immediate corollary is that

$$\frac{1}{4} \leq P_{k,l} < \frac{1}{2} < Q_{k,l} \leq \frac{3}{4} \quad (k, l \in \mathbf{N}).$$

The constant bounds are sharp.

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1. Introduction. In this paper, we are concerned with the combinatorial sums

$$P_{k,l} = \Big(\frac{l}{k+l}\Big)^{k+l} \sum_{\nu=0}^{k-1} \binom{k+l}{\nu} \Big(\frac{k}{l}\Big)^{\nu}$$

and

$$Q_{k,l} = \Big(\frac{l}{k+l}\Big)^{k+l} \sum_{\nu=0}^{k} \binom{k+l}{\nu} \Big(\frac{k}{l}\Big)^{\nu}$$

which are equal to the last k and $k+1$ terms (when arranged in descending order of powers of $k/(k+1)$), respectively, of the binomial expansion of $\big(k/(k+l)+l/(k+l)\big)^{k+l}$. Both sums can be written in terms of the beta and incomplete beta functions:

$$P_{k,l} = \frac{1}{B(k,l+1)} \int_{k/(k+l)}^{1} t^{k-1}(1-t)^l dt \tag{1.1}$$

and

$$Q_{k,l} = \frac{1}{B(k+1,l)} \int_{k/(k+l)}^{1} t^{k}(1-t)^{l-1} dt.$$

From the integral representations we conclude easily that $P_{k,l}$ and $Q_{l,k}$ are connected by the elegant identity

$$P_{k,l} + Q_{l,k} = 1 \quad (k,l \in \mathbf{N}). \tag{1.2}$$

Studies on mathematical statistics led Vietoris [2] in 1982 to the remarkable inequalities

$$P_{k,l} < \frac{1}{2} < Q_{k,l} \quad (k,l \in \mathbf{N}). \tag{1.3}$$

From (1.2), we see that the two inequalities are equivalent. Applying

$$P_{k,k} = \frac{1}{2} - \frac{1}{2^{2k+1}} \binom{2k}{k}$$

yields the limit relation

$$\lim_{k\to\infty} P_{k,k} = \frac{1}{2}. \tag{1.4}$$

This reveals that the upper bound 1/2 given in (1.3) cannot be replaced by a smaller constant. It is natural to ask whether there exists a positive constant lower bound for $P_{k,l}$. It is the aim of this note to give an affirmative answer to this question, namely, that the best possible constant lower bound for $P_{k,l}$ is 1/4. Our work is inspired by an interesting paper published by Raab [1] in 1984. He used the integral representation (1.1) to show that the inequality $P_{k,l} < 1/2$ holds for all positive real numbers k and l.

In the next section, we collect some lemmas. They play an important role in the proof of our main result which we present in Section 3.

2. Lemmas. The following product representation for $P_{k,l}$ was derived by Raab in [1]:

$$P_{k,l} = U_{k,l}\, V_{k,l}, \tag{2.1}$$

where

$$U_{k,l} = \exp \int_0^\infty g(t) h_{k,l}(t) dt\,, \tag{2.2}$$

with

$$g(t) = \frac{1}{t}\Big(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\Big), \tag{2.3}$$

$$h_{k,l}(t) = e^{-(k+l)t} - e^{-kt} - e^{-lt},$$

and

$$V_{k,l} = \frac{\sqrt{l}}{2\pi} \sum_{\nu=1}^\infty \frac{c_\nu(k/l)}{\sqrt{\nu}(\nu + l)}\,, \tag{2.4}$$

with

$$c_\nu(x) = \exp \int_0^\infty g(t)\Big(e^{-\nu(1+x)t} - e^{-\nu x t} - e^{-\nu t}\Big) dt \quad (\nu > 0; x > 0). \tag{2.5}$$

Moreover, let

$$b_\nu(l) = \frac{c_\nu(1/l)}{\sqrt{\nu}(\nu + l)} \quad (\nu > 0; l > 0). \tag{2.6}$$

Lemma 1. *With $g(t)$ as defined in (2.3), if $t > 0$, then $0 < tg(t) < 1/2$.*

Proof. Let $t > 0$ and $w(t) = tg(t)$. Since

$$w'(t) = \frac{1}{t^2} - \frac{e^t}{(e^t - 1)^2} = \frac{2e^t}{t^2(e^t - 1)^2} \sum_{j=2}^\infty \frac{t^{2j}}{(2j)!} > 0,$$

we conclude that w is strictly increasing on $(0, \infty)$. Moreover, we have

$$\lim_{t\to 0} w(t) = \frac{1}{2} + \lim_{t\to 0} \frac{1 + t - e^t}{te^t - t} = 0 \quad \text{and} \quad \lim_{t\to\infty} w(t) = \frac{1}{2}.$$

This implies that $0 < w(t) < 1/2$. □

Lemma 2. *If $l > 0$, then $\nu \mapsto b_\nu(l)$ is strictly decreasing on $(0, \infty)$.*

Proof. Let $\nu > 0$. We have

$$\log b_\nu(l) = \log c_\nu(1/l) - \frac{1}{2}\log \nu - \log(\nu + l)$$

and

$$\frac{\partial}{\partial \nu} \log b_\nu(l) = \int_0^\infty tg(t)\phi_{\nu,l}(t) dt - \frac{1}{2\nu} - \frac{1}{\nu + l} \tag{2.7}$$

with

$$\phi_{\nu,l}(t) = e^{-\nu t}\Big(1 - e^{-\nu t/l}\Big) + \frac{e^{-\nu t/l}}{l}\Big(1 - e^{-\nu t}\Big).$$

Since $\phi_{\nu,l}$ is positive on $(0,\infty)$, we conclude from (2.7) and Lemma 1 that

$$\frac{\partial}{\partial\nu}\log b_\nu(l) < \frac{1}{2}\int_0^\infty \phi_{\nu,l}(t)dt - \frac{1}{2\nu} - \frac{1}{\nu+l} = -\frac{1}{\nu+l} < 0.$$

This implies that $b_\nu(l)$ is strictly decreasing with respect to ν. □

Lemma 3. *If $l \geq 1$ and $\nu > 0$, then*

$$b_\nu(1) \leq l^{3/2} b_{l\nu}(l). \tag{2.8}$$

Proof. From

$$\log c_{l\nu}(1/l) - \log c_\nu(1) = \int_0^\infty g(t)e^{-\nu t}\Big(1-e^{-\nu(l-1)t}\Big)\Big(1-e^{-\nu t}\Big)dt \geq 0,$$

we conclude that

$$c_{l\nu}(1/l) \geq c_\nu(1).$$

Thus,

$$\frac{l^{3/2}b_{l\nu}(l)}{b_\nu(1)} = \frac{c_{l\nu}(1/l)}{c_\nu(1)} \geq 1.$$

This leads to (2.8). □

3. Main result. We are now in a position to present the best possible constant lower bound for $P_{k,l}$.

Theorem. *For all natural numbers k and l, we have*

$$\frac{1}{4} \leq P_{k,l}.$$

The sign of equality holds if and only if $k = l = 1$.

Proof. We make use of Raab's product representation (2.1) for $P_{k,l}$.

First, we prove that

$$U_{k,l} \geq U_{1,1} \tag{3.1}$$

with equality only if $k = l = 1$. Let k and l be real numbers with $k \geq l \geq 1$. Then, for $t > 0$,

$$\frac{\partial}{\partial k}h_{k,l}(t) = te^{-(k+1)t}\Big(e^{lt}-1\Big) > 0.$$

This yields

$$h_{k,l}(t) \geq h_{l,l}(t) = e^{-2lt} - 2e^{-lt}. \tag{3.2}$$

Since

$$\frac{\partial}{\partial l}h_{l,l}(t) = 2te^{-2lt}\Big(e^{lt}-1\Big) > 0,$$

we obtain

$$h_{l,l}(t) \geq h_{1,1}(t). \tag{3.3}$$

Applying (2.2), (3.2), (3.3), and Lemma 1 leads to

$$\log U_{k,l} \geq \int_0^\infty g(t)h_{1,1}(t)dt = \log U_{1,1}$$

which implies (3.1). Moreover, if $U_{k,l} = U_{1,1}$, then $h_{k,l}(t) = h_{l,l}(t)$ and $h_{l,l}(t) = h_{1,1}(t)$. This gives $k = l$ and $l = 1$.

Next, we estimate $V_{k,l}$. Let $\nu \geq 1$ and $t > 0$. Since

$$\frac{\partial}{\partial x}\Big(e^{-\nu(1+x)t} - e^{-\nu xt} - e^{-\nu t}\Big) = \nu t e^{-\nu(1+x)t}\Big(e^{\nu t} - 1\Big) > 0,$$

we conclude from (2.5) and Lemma 1 that $c_\nu(x)$ is strictly increasing with respect to x. Using (2.4), we obtain

$$V_{k,l} \geq V_{1,l}. \tag{3.4}$$

Finally, we show that

$$V_{1,l} \geq V_{1,1}. \tag{3.5}$$

Let $1 \leq \mu \leq l$ and $\nu \geq 1$. Applying Lemma 2 gives

$$b_{\nu l+\mu}(l) \geq b_{\nu l+l}(l).$$

This yields

$$\sum_{\nu=1}^\infty b_\nu(l) = \sum_{\nu=0}^\infty \sum_{\mu=1}^l b_{\nu l+\mu}(l) \geq \sum_{\nu=0}^\infty l\, b_{(\nu+1)l}(l).$$

From Lemma 3 we obtain

$$l b_{(\nu+1)l}(l) \geq \frac{1}{\sqrt{l}}\, b_{\nu+1}(1).$$

Hence

$$\sum_{\nu=1}^\infty b_\nu(l) \geq \frac{1}{\sqrt{l}} \sum_{\nu=1}^\infty b_\nu(1). \tag{3.6}$$

Using (2.6), (2.4), and (3.6) gives

$$2\pi(V_{1,l} - V_{1,1}) = \sqrt{l}\sum_{\nu=1}^\infty b_\nu(l) - \sum_{\nu=1}^\infty b_\nu(1) \geq 0.$$

This settles (3.5).

Combining (3.1), (3.4), and (3.5) leads to

$$P_{k,l} = U_{k,l}V_{k,l} \geq U_{1,1}V_{1,1} = P_{1,1} = \frac{1}{4}.$$

If $P_{k,l} = P_{1,1}$, then $U_{k,l} = U_{1,1}$ which implies $k = l = 1$. This completes the proof of the Theorem. □

Using (1.2), (1.3), (1.4), and our theorem yields the following chain of inequalities which complements (1.3).

Corollary 1. *For all natural numbers k and l, we have*

$$\frac{1}{4} \le P_{k,l} < \frac{1}{2} < Q_{k,l} \le \frac{3}{4}. \tag{3.7}$$

All constant bounds are sharp.

When we apply (3.7) to specific values of k, then we obtain a sequence of elementary but non-trivial inequalities, the first three are listed below.

Corollary 2. *For all integers $l > 1$, we have*

$$\tfrac{1}{4} < \left(\tfrac{l}{l+l}\right)^{l+1} < \tfrac{1}{2} < \tfrac{2l+1}{l+1}\left(\tfrac{l}{l+1}\right)^{l} < \tfrac{3}{4},$$

$$\tfrac{1}{4} < \tfrac{3l+4}{l+2}\left(\tfrac{l}{l+2}\right)^{l+1} < \tfrac{1}{2} < \tfrac{5l^2+10l+4}{(l+2)^2}\left(\tfrac{l}{l+2}\right)^{l} < \tfrac{3}{4},$$

$$\tfrac{1}{4} < \tfrac{17l^2+63l+54}{2(l+3)^2}\left(\tfrac{l}{l+3}\right)^{l+1} < \tfrac{1}{2} < \tfrac{26l^3+117l^2+153l+54}{2(l+3)^3}\left(\tfrac{l}{l+3}\right)^{l} < \tfrac{3}{4}.$$

Applications of (1.2), (1.3), (1.4), and the theorem lead to sharp upper and lower bounds for the ratio of two combinatorial sums. We define

$$R_{k,l} = \sum_{\nu=0}^{k-1} \binom{k+l}{\nu}\Big(\frac{k}{l}\Big)^{\nu} \Big/ \sum_{\nu=k}^{k+l} \binom{k+l}{\nu}\Big(\frac{k}{l}\Big)^{\nu}$$

and

$$S_{k,l} = \sum_{\nu=0}^{k} \binom{k+l}{\nu}\Big(\frac{k}{l}\Big)^{\nu} \Big/ \sum_{\nu=k+1}^{k+l} \binom{k+l}{\nu}\Big(\frac{k}{l}\Big)^{\nu}.$$

Corollary 3. *For all natural numbers k and l, we have*

$$\frac{1}{3} \le R_{k,l} < 1 \tag{3.8}$$

and

$$1 < S_{k,l} \le 3. \tag{3.9}$$

All bounds are best possible. Equality holds if and only if $k = l = 1$.

Proof. The proofs for (3.8) and (3.9) are similar. Therefore, we only establish (3.8). Raab [1] pointed out that $P_{k,l} < 1/2$ is equivalent to the right-hand side of (3.8). Indeed, we have

$$0 < \Big(\frac{k+l}{l}\Big)^{k+l}(1 - 2P_{k,l}) = \Big(\sum_{\nu=k}^{k+l} - \sum_{\nu=0}^{k-1}\Big)\binom{k+l}{\nu}\Big(\frac{k}{l}\Big)^{\nu}.$$

This leads to the second inequality in (3.8). Moreover, using (1.4) gives

$$\lim_{k\to\infty} R_{k,k} = \lim_{k\to\infty} \frac{P_{k,k}}{1 - P_{k,k}} = \frac{1/2}{1 - 1/2} = 1.$$

It follows that the upper bound 1 is sharp. From

$$0 \le \Big(\frac{k+l}{l}\Big)^{k+l}(4P_{k,l} - 1) = \Big(3\sum_{\nu=0}^{k-1} - \sum_{\nu=k}^{k+l}\Big)\binom{k+l}{\nu}\Big(\frac{k}{l}\Big)^{\nu},$$

we conclude that the left-hand side of (3.8) is valid with equality if and only if $k = l = 1$. $\square$

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