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Irreducibility of the Hilbert scheme of smooth curves in $\mathbb{P}^3$ of degree g and genus g

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To the memory of our friend and collaborator Ryutaro Horiuchi

Abstract. We denote by $\mathcal{H}_{d,g,r}$ the Hilbert scheme of smooth curves, which is the union of components whose general point corresponds to a smooth irreducible and non-degenerate curve of degree d and genus g in $\mathbb{P}^r$. In this note, we show that any non-empty $\mathcal{H}_{g,g,3}$ is irreducible without any restriction on the genus g. This extends the result obtained earlier by Iliev (Proc Am Math Soc 134:2823–2832, 2006).

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1. An overview, preliminaries, and basic set-up. Given non-negative integers d, g and $r \geq 3$, let $\mathcal{H}_{d,g,r}$ be the Hilbert scheme of smooth curves parametrizing smooth irreducible and non-degenerate curves of degree d and genus g in $\mathbb{P}^r$.

After Severi asserted that $\mathcal{H}_{d,g,r}$ is irreducible for $d \geq g + r$ in [14, Anhang G, p. 368], the irreducibility of $\mathcal{H}_{d,g,r}$ has been widely studied by several authors. Ein proved Severi's claim for $r = 3$ and $r = 4$; cf. [6, Theorem 4] and [7, Theorem 7]. Shortly after, Keem and Kim gave a different proof in the same range $d \geq g+3$ while they also proved the irreducibility of $\mathcal{H}_{g+2,g,3}$ for $g \geq 5$ as well as $\mathcal{H}_{g+1,g,3}$ for $g \geq 11$; cf. [12, Theorem 1.5]. On the other hand, Severi's assertion turned out to be untrue for curves in higher dimensional projective spaces $\mathbb{P}^r$ with $r \geq 6$; cf. [11, Theorem 2.3]. Quite recently, Keem et al. proved that the irreducibility of $\mathcal{H}_{g+2,g,3}$ and $\mathcal{H}_{g+1,g,3}$ even holds for any genus g as long as they are non-empty; cf. [13, Proposition 2.1 & Proposition 3.2]. For the case $d = g$, Hristo Iliev proved that $\mathcal{H}_{g,g,3}$ is irreducible for $g \geq 13$; cf.

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[10, Theorem 3.1]. However, the irreducibility of $\mathcal{H}_{g,g,3}$ for lower genus $g \leq 12$ has been left open. In this article, we prove that every non-empty $\mathcal{H}_{g,g,3}$ is irreducible without any restriction on the genus g.

Before proceeding, we recall several related results which are rather well known; cf. [1] and [2]. Let $\mathcal{M}_g$ be the moduli space of smooth curves of genus g. For any given isomorphism class $[C] \in \mathcal{M}_g$ corresponding to a smooth irreducible curve C, there exist a neighborhood $U \subset \mathcal{M}_g$ of the class $[C]$ and a smooth connected variety $\mathcal{M}$ which is a finite ramified covering $h : \mathcal{M} \to U$, as well as varieties $\mathcal{C}$, $\mathcal{W}^r_d$, and $\mathcal{G}^r_d$ proper over $\mathcal{M}$ with the following properties:

(1) $\xi : \mathcal{C} \to \mathcal{M}$ is a universal curve, i.e. for every $p \in \mathcal{M}$, $\xi^{-1}(p)$ is a smooth curve of genus g whose isomorphism class is $h(p)$,
(2) $\mathcal{W}^r_d$ parametrizes the pairs (p, L) where L is a line bundle of degree d and $h^0(L) \geq r + 1$ on $\xi^{-1}(p)$,
(3) $\mathcal{G}^r_d$ parametrizes the couples $(p, \mathcal{D})$, where $\mathcal{D}$ is possibly an incomplete linear series of degree d and dimension r on $\xi^{-1}(p)$ - which is usually denoted by g^r_d.

Let $\widetilde{\mathcal{G}}$ be the union of components of $\mathcal{G}^r_d$ whose general element $(p, \mathcal{D})$ corresponds to a very ample linear series $\mathcal{D}$ on the curve $C = \xi^{-1}(p)$. Note that an open subset of $\mathcal{H}_{d,g,r}$ consisting of points corresponding to smooth irreducible and non-degenerate curves is a $\mathbb{P}\mathrm{GL}(r + 1)$-bundle over an open subset of $\widetilde{\mathcal{G}}$. Hence the irreducibility of $\widetilde{\mathcal{G}}$ guarantees the irreducibility of $\mathcal{H}_{d,g,r}$. We also make a note of the following basic and well-known facts regarding the scheme $\mathcal{G}^r_d$; cf. [2, Proposition 2.8] and [8, §2.a, p. 67].

Proposition 1.1. *For non-negative integers d, g, and r, let $\rho(d, g, r) := g-(r+1)(g - d + r)$ be the Brill-Noether number.*

(1) *The dimension of any component of $\mathcal{G}^r_d$ is at least $3g-3+\rho(d, g, r)$ which is denoted by $\lambda(d, g, r)$.*
(2) *Suppose $g > 1$ and let X be a component of $\mathcal{G}^2_d$ whose general element $(p, \mathcal{D})$ is such that $\mathcal{D}$ is a birationally very ample linear series on $\xi^{-1}(p)$. Then*

$$\dim X = 3g - 3 + \rho(d, g, 2) = 3d + g - 9.$$

We recall that the family of plane curves of degree d in $\mathbb{P}^2$ are naturally parametrised by the projective space $\mathbb{P}^N, N = \frac{d(d+3)}{2}$. Let $\Sigma_{d,g} \subset \mathbb{P}^N$ be the Severi variety of plane curves of degree d with geometric genus g. We also recall that a general point of $\Sigma_{d,g}$ corresponds to an irreducible plane curve of degree d having $\delta := \frac{(d-1)(d-2)}{2} - g$ nodes and no other singulariy. The following theorem of Harris is fundamental; cf. [4, Theorem 10.7 and 10.12, pp. 847-850] or [9].

Theorem 1.2. *$\Sigma_{d,g}$ is irreducible of dimension $3d + g - 1 = \lambda(d, g, 2) + \dim \mathbb{P}\mathrm{GL}(3)$.*

Denoting by $\mathcal{G}' \subset \mathcal{G}^2_d$ the union of components whose general element $(p, \mathcal{D}) \in \mathcal{G}'$ is such that $\mathcal{D}$ is birationally very ample on $C = \xi^{-1}(p)$, we remark that an open subset of the Severi variety $\Sigma_{d,g}$ is a $\mathbb{P}\mathrm{GL}(3)$-bundle over an open subset of $\mathcal{G}'$. Therefore, as an immediate consequence of Theorem 1.2, the irreducibility of $\Sigma_{d,g}$ implies the irreducibility of the locus $\mathcal{G}'$ and vice versa. We make a note of this simple observation as the following lemma.

Lemma 1.3. *Let $\mathcal{G}' \subset \mathcal{G}^2_d$ be the union of components whose general element $(p, \mathcal{D})$ is such that $\mathcal{D}$ is birationally very ample on $C = \xi^{-1}(p)$. Then $\mathcal{G}'$ is irreducible.*

We will utilize the following upper bound of the dimension of an irreducible component of $\mathcal{W}^r_d$, which was proved and used effectively in [10].

Proposition 1.4. ([10, Proposition 2.1]) *Let d, g and $r \geq 2$ be positive integers such that $d \leq g + r - 2$ and let $\mathcal{W}$ be an irreducible component of $\mathcal{W}^r_d$. For a general elment $(p, L) \in \mathcal{W}$, let b be the degree of the base locus of the line bundle $L = |D|$ on $C = \xi^{-1}(p)$. Assume further that for a general $(p, L) \in \mathcal{W}$ the curve $C = \xi^{-1}(p)$ is not hyperelliptic. If the moving part of $L = |D|$ is*

(a) *very ample and $r \geq 3$, then* $\dim \mathcal{W} \leq 3d + g + 1 - 5r - 2b$;
(b) *birationally very ample, then* $\dim \mathcal{W} \leq 3d + g - 1 - 4r - 2b$;
(c) *compounded, then* $\dim \mathcal{W} \leq 2g - 1 + d - 2r$.

For notations and conventions, we usually follow those in [3] and [4]; e.g., (d, r) is the maximal possible arithmetic genus of an irreducible and non-degenerate curve of degree d in $\mathbb{P}^r$. Throughout we work over the field of complex numbers.

2. Irreducibility of $\mathcal{H}_{g,g,3}$. The main result of this article is the following theorem.

Theorem 2.1. *Every non-empty $\mathcal{H}_{g,g,3}$ is irreducible.*

We make a note of the following well-known facts - which are also very easy to prove - when the genus of curves under consideration is relatively low.

Proposition 2.2. (1) *$\mathcal{H}_{g,g,3} = \emptyset$ for $1 \leq g \leq 7$*
(2) *$\mathcal{H}_{8,8,3}$ and $\mathcal{H}_{9,9,3}$ is irreducible of dimension 32, 36, respectively.*

Proof. (1) If $1 \leq g \leq 7$, one has $g \leq \pi(d, 3) < g$ by the Castelnuovo genus bound, a contradiction.
(2) We refer [5, Theorem 5.2.1] for a detailed treatment.

□

Therefore we shall assume that $g \geq 10$ for the rest of this section. The following lemma is a crucial step toward the proof of the irreducibility of $\mathcal{H}_{g,g,3}$.

Lemma 2.3. *Let $\mathcal{G} \subset \mathcal{G}^3_g$ be an irreducible component whose general element $(p, \mathcal{D})$ is a very ample linear series $\mathcal{D}$ on the curve $C = \xi^{-1}(p)$ and assume $g \geq 10$. Then*

(1) *$\mathcal{D}$ is complete and* $\dim \mathcal{G} = 4g - 15$.

(2) *a general element of the component* $\mathcal{W}^\vee \subset \mathcal{W}^2_{g-2}$ *consisting of the residual series (with respect to the canonical series on the corresponding curve) of those elements in* $\mathcal{G}$ *is a base-point-free, complete, and birationally very ample net.*

Proof. By Proposition 1.1, we have

$$\lambda(g,g,3) = 3g-3+\rho(g,g,3) = 4g-15 \le \dim \mathcal{G}.$$

We set $r := h^0(C,|\mathcal{D}|)-1$ for a general $(p,\mathcal{D}) \in \mathcal{G}$.

Let $\mathcal{W} \subset \mathcal{W}^r_g$ be the component containing the image of the natural rational map $\mathcal{G} \overset{\iota}{\dashrightarrow} \mathcal{W}^r_g$ with $\iota(\mathcal{D}) = |\mathcal{D}|$. Since $\dim \mathcal{G} \le \dim \mathcal{W} + \dim \mathbb{G}(3,r)$, it follows by Proposition 1.4(a) that

$$\lambda(g,g,3) = 4g-15 \le \dim \mathcal{G} \le (4g+1-5r)+4(r-3) = 4g-r-11$$

and hence $r \le 4$.

Let $\mathcal{W}^\vee \subset \mathcal{W}^{r-1}_{g-2}$ be the locus consisting of the residual series (with respect to the canonical series on the corresponding curve) of those elements in $\mathcal{W}$, i.e. $\mathcal{W}^\vee = \{(p,\omega_C \otimes L^{-1}) : (p,L) \in \mathcal{W}\}$.

(a) If a general element of $\mathcal{W}^\vee$ is compounded, then by Proposition 1.4(c),

$$\begin{aligned} 4g-15 &\le \dim \mathcal{G} \le \dim \mathcal{W} + \dim \mathbb{G}(3,r) = \dim \mathcal{W}^\vee + 4(r-3) \\ &\le (2g-1+(g-2)-2(r-1)) + 4(r-3) \\ &= 3g+2r-13 \end{aligned}$$

implying $10 \le g \le 2r+2 \le 10$ and hence $(g,r) = (10,4)$. If $(g,r) = (10,4)$, by the Castelnuovo genus bound one has $g = 10 \le \pi(10,4) = 9$, a contradiction. Therefore it follows that a general element of $\mathcal{W}^\vee$ is not compunded.

(b) Suppose that the moving part of a general element of $\mathcal{W}^\vee$ is very ample and let b be the degree of the base locus of a general element of $\mathcal{W}^\vee$. By Proposition 1.4(a), we have

$$\begin{aligned} 4g-15 &\le \dim \mathcal{G} \le \dim \mathcal{W} + \dim \mathbb{G}(3,r) = \dim \mathcal{W}^\vee + 4(r-3) \\ &\le (3(g-2)+g+1-5(r-1)-2b) + 4(r-3) \\ &= 4g-r-2b-12, \end{aligned}$$

implying $r=3$ and $b=0$. Therefore a general element of $\mathcal{W}^\vee \subset \mathcal{W}^{r-1}_{g-2}$ is a smooth plane curve C of degree $g-2$ equipped with a base-point-free and very ample g^2_{g-2}. However the equality $g = p_a(C) = \frac{(g-3)(g-4)}{2}$ does not have an integer solution.

(c) Thus the moving part of a general element $(p,\mathcal{E}) \in \mathcal{W}^\vee$ is birationally very ample on $C = \xi^{-1}(p)$. Applying Proposition 1.4(b), we get

$$\begin{aligned} 4g-15 &\le \dim \mathcal{G} \le \dim \mathcal{W} + \dim \mathbb{G}(3,r) = \dim \mathcal{W}^\vee + 4(r-3) \\ &\le (3(g-2)+g-1-4(r-1)-2b) + 4(r-3) \\ &= 4g-15-2b, \end{aligned}$$

implying $b=0$, i.e. $\mathcal{E}$ is base-point-free, birationally very ample, and

$$\dim \mathcal{W} = \dim \mathcal{W}^\vee = 4g-4r-3. \qquad (2.1)$$

We finally claim that $r = 3$. Suppose that $r = 4$. By (2.1) we have $\dim \mathcal{W}^\vee = 4g - 19$. We consider the following diagram.

$$\begin{array}{ccc} \mathcal{W}^2_{g-4} \underset{\mathcal{M}}{\times} \mathcal{W}_2 & \stackrel{q}{\longrightarrow} & \mathcal{W}^2_{g-2} \\ \big\downarrow \pi & & \\ \mathcal{W}^2_{g-4} & & \end{array}$$

where $q(\mathcal{E}', \mathcal{O}_C(R+S)) = \mathcal{E}' \otimes \mathcal{O}_C(R+S)$ and $\pi(\mathcal{E}', \mathcal{O}_C(R+S)) = \mathcal{E}'$. Since a general element $(p, \mathcal{E}) \in \mathcal{W}^\vee \subset \mathcal{W}^3_{g-2} \subset \mathcal{W}^2_{g-2}$ is birationally very ample and base-point-free (which can never be very ample by semi-continuity), $q^{-1}(\mathcal{E}) \neq \emptyset$ for a general $(p, \mathcal{E}) \in \mathcal{W}^\vee$. Let Σ be a component of $q^{-1}(\mathcal{W}^\vee)$ such that $q(\Sigma) = \mathcal{W}^\vee$. By the birationality of a general $(p, \mathcal{E}) \in \mathcal{W}^\vee$, we see that $\dim q^{-1}(\mathcal{E}) = 0$ and hence

$$\dim \Sigma = \dim \mathcal{W}^\vee = 4g - 19.$$

Setting $\mathcal{Z} := \pi(\Sigma) \subset \mathcal{W}^2_{g-4}$, we have the following induced diagram:

$$\begin{array}{ccc} \mathcal{W}^2_{g-4} \underset{\mathcal{M}}{\times} \mathcal{W}_2 \supset \Sigma & \stackrel{q}{\longrightarrow} & \mathcal{W}^\vee \subset \mathcal{W}^3_{g-2} \subset \mathcal{W}^2_{g-2} \\ \big\downarrow \pi & & \\ \mathcal{W}^2_{g-4} \supset \mathcal{Z} & & \end{array}$$

We now show that $\dim \pi^{-1}(\mathcal{E}') = 0$ for a general $(p, \mathcal{E}') \in \mathcal{Z}$. We choose $(p, \mathcal{E}) \in \mathcal{W}^\vee$ and fix $(p, \mathcal{E}') \in \mathcal{Z}$ such that $(\mathcal{E}', \mathcal{O}_C(R+S)) \in q^{-1}(\mathcal{E})$ for some $R, S \in C = \xi^{-1}(p)$, i.e. $\mathcal{E} \cong \mathcal{E}' \otimes \mathcal{O}_C(R+S)$. Recall that by our initial setting, $\omega_C \otimes \mathcal{E}^{-1} = \mathcal{D} \in \mathcal{W} \subset \mathcal{W}^4_g$ is a very ample line bundle for a general $\mathcal{E} \in \mathcal{W}^\vee \subset \mathcal{W}^3_{g-2}$. We also note that the very ample, base-point-free and complete linear system $\mathcal{D} = \omega_C \otimes \mathcal{E}^{-1} = \omega_C \otimes \mathcal{E}'^{-1} \otimes \mathcal{O}_C(-R-S)$ is a subsystem of $\omega_C \otimes \mathcal{E}'^{-1}$. Hence $\omega_C \otimes \mathcal{E}'^{-1}$ is birationally very ample; otherwise the isomorphism induced by the very ample $\mathcal{D}$ on $C = \xi^{-1}(p)$ factors non-trivially through the morphism induced by $\omega_C \otimes \mathcal{E}'^{-1}$, which is an absurdity. Therefore by noting that $\omega_C \otimes \mathcal{E}'^{-1} = g^5_{g+2}$, there are only finitely many choices of $\mathcal{O}_C(-\widetilde{R} - \widetilde{S})$'s such that $\omega_C \otimes \mathcal{E}'^{-1} \otimes \mathcal{O}_C(-\widetilde{R} - \widetilde{S}) = g^4_g \in \mathcal{W} \subset \mathcal{W}^4_g$, i.e. $(\mathcal{E}', \mathcal{O}_C(\widetilde{R} + \widetilde{S})) \in \Sigma$ or equivalently $\mathcal{E}' \otimes \mathcal{O}_C(\widetilde{R} + \widetilde{S}) \in \mathcal{W}^\vee$, which implies $\dim \pi^{-1}(\mathcal{E}') = 0$. By semi-continuity, we have $\dim \pi^{-1}(\mathcal{E}') = 0$ for a general $(p, \mathcal{E}') \in \mathcal{Z}$ and hence

$$\dim \mathcal{Z} = \dim \Sigma = 4g - 19.$$

The following three possibilities may occur.

(i) A general element of $\mathcal{Z} \subset \mathcal{W}^2_{g-4}$ is compounded; by Proposition 1.4(c),

$$4g-19 = \dim \mathcal{Z} \leq (2g-1+(g-4)-2\cdot 2) = 3g-9$$

which is impossible unless $g = 10$. However, a curve of genus 10 cannot have a very ample $g^4_{10} \in \mathcal{W} \subset \mathcal{W}^4_g$ by exactly the same reason when we were eliminating the possibility $(g,r) = (10,4)$ in (a).

(ii) An (general) element of $\mathcal{Z} \subset \mathcal{W}^2_{g-4}$ is very ample; in this case, one has $p_a(C) = \frac{(g-5)(g-6)}{2} = g \geq 10$ and hence $g = 10$. On the other hand, a smooth plane curve of degree $g-4 = 6$ cannot have a very ample $g^4_{10} \in \mathcal{W} \subset \mathcal{W}^4_g$ by the same reason as in (i) or (a).

(iii) A general element of $\mathcal{Z} \subset \mathcal{W}^2_{g-4}$ is birationally very ample; by Proposition 1.4(b),

$$4g-19 = \dim \mathcal{Z} \leq (3(g-4)+g-1-4\cdot 2) = 4g-21$$

which is a contradiction.

Therefore it finally follows that $r = 3$ and by (2.1), we have

$$\dim \mathcal{G} = \dim \mathcal{W} = \dim \mathcal{W}^\vee = 4g-15. \tag{2.2}$$

□

Remark 2.4. As was mentioned earlier, Hrito Iliev proved the irreducibility of $\mathcal{H}_{g,g,3}$ for $g \geq 13$; cf. [10, Theorem 3.1]. In doing so, he used the fact that $\mathcal{G}^2_d$ has a unique component whose general element is birationally very ample on the correspoinding curve if $\rho(d,g,2) > 0$; cf. [2, Theorem 1.1 & Proposition 2.1]. In our proof of Theorem 2.1 we use Lemma 1.3 as well as Lemma (2.3) instead, which will take care of all the possible cases including the unknown cases $g \leq 12$.

The irreducibility of $\mathcal{H}_{g,g,3}$ follows easily as an immediate consequence of Lemma 2.3 together with Lemma 1.3.

Proof of Theorem 2.1. Retaining the same notations as before, let $\widetilde{\mathcal{G}}$ be the union of irreducible components $\mathcal{G}$ of $\mathcal{G}^3_g$ whose general element corresponds to a pair $(p,\mathcal{D})$ such that $\mathcal{D}$ is a very ample linear series on $C := \xi^{-1}(p)$. Let a $\widetilde{\mathcal{W}}^\vee$ be the union of the components $\mathcal{W}^\vee$ of $\mathcal{W}^2_{g-2}$, where $\mathcal{W}^\vee$ consists of the residual series of elements in a component $\mathcal{G}$ of $\widetilde{\mathcal{G}}$. We also let $\mathcal{G}'$ be the union of irreducible components of $\mathcal{G}^2_{g-2}$ whose general element is a birationally very ample and base-point-free linear series. We recall that, by Lemma 1.3 and Proposition 1.1(2), $\mathcal{G}'$ is irreducible and $\dim \mathcal{G}' = 3(g-2)+g-9 = 4g-15$. By Lemma 2.3 (or (2.2)),

$$\dim \mathcal{W}^\vee = \dim \mathcal{G} = 4g-15 = \dim \mathcal{G}'. \tag{2.3}$$

Since a general element of any component $\mathcal{W}^\vee \subset \widetilde{\mathcal{W}}^\vee \subset \mathcal{W}^2_{g-2}$ is a base-point-free, birationally very ample, and complete net by Lemma 2.3, there is a natural rational map $\widetilde{\mathcal{W}}^\vee \overset{\kappa}{\dashrightarrow} \mathcal{G}'$ with $\kappa(|\mathcal{D}|) = \mathcal{D}$ which is clearly injective on an open subset $\widetilde{\mathcal{W}}^{\vee o}$ of $\widetilde{\mathcal{W}}^\vee$ consisting of those which are base-point-free, birationally very ample, and complete nets. Therefore the rational map κ is dominant by (2.3). We also note that there is a natural rational map $\mathcal{G}' \overset{\iota}{\dashrightarrow} \widetilde{\mathcal{W}}^\vee$ with $\iota(\mathcal{D}) = |\mathcal{D}|$, which is an inverse to κ (wherever it is defined). Therefore it follows that $\widetilde{\mathcal{W}}^\vee$ is birationally equivalent to the irreducible locus $\mathcal{G}'$, hence $\widetilde{\mathcal{W}}^\vee$ is irreducible and so is $\widetilde{\mathcal{G}}$. Since $\mathcal{H}_{g,g,3}$ is a $\mathbb{P}\mathrm{GL}(r+1)$ bundle over an open subset of $\widetilde{\mathcal{G}}$, $\mathcal{H}_{g,g,3}$ is irreducible. □

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