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On discrete universality of the Riemann zeta-function with respect to uniformly distributed shifts

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Abstract. The Voronin universality theorem asserts that a wide class of analytic functions can be approximated by shifts $\zeta(s+i\tau)$, $\tau \in \mathbb{R}$, of the Riemann zeta-function. In the paper, we obtain a universality theorem on the approximation of analytic functions by discrete shifts $\zeta(s+ix_k h)$, $k \in \mathbb{N}$, $h > 0$, where $\{x_k\} \subset \mathbb{R}$ is such that the sequence $\{ax_k\}$ with every real $a \neq 0$ is uniformly distributed modulo 1, $1 \leq x_k \leq k$ for all $k \in \mathbb{N}$ and, for $1 \leq k, m \leq N$, $k \neq m$, the inequality $|x_k - x_m| \geq y_N^{-1}$ holds with $y_N > 0$ satisfying $y_N x_N \ll N$.

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1. Introduction. It is well known [1,4,5,7,11,16,17] that the Riemann zeta-function $\zeta(s)$, $s = \sigma + it$, is universal in the Voronin sense, that is, its shifts $\zeta(s+i\tau)$, $\tau \in \mathbb{R}$, approximate a wide class of analytic functions. For precise statements of universality theorems, it is convenient to use the following notation. Denote by $\mathcal{K}$ the class of compact subsets of the strip $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ with connected complements, and by $H_0(K)$, $K \in \mathcal{K}$, the class of continuous non-vanishing functions on K which are analytic in the interior of K. Moreover, let meas$A$ and $\#B$ stand for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$ and the cardinality of a set B. Then the universality property of continuous type for $\zeta(s)$ is described in the following theorem.

Theorem 1.1. *Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\epsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \epsilon \right\} > 0.$$

Thus, the set of shifts $\zeta(s+i\tau)$ approximating a given analytic function $f(s) \in H_0(K)$ with an accuracy ϵ has a positive lower density.

If τ takes values from a discrete set, then analogues of Theorem 1.1 are called discrete universality theorems for $\zeta(s)$. The simplest of them deals with the set $\{kh : k \in \mathbb{N}_0\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $h > 0$ is a fixed number, i.e., τ takes values from an arithmetic progression.

Theorem 1.2. *Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $h > 0$ and $\epsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \sup_{s\in K} |\zeta(s+ikh) - f(s)| < \epsilon \right\} > 0.$$

Theorem 1.1 is an improved version of the original Voronin theorem obtained in [17]. The discrete universality for zeta-functions was proposed by A. Reich. In [15] he obtained a theorem of such a kind for Dedekind zeta-functions. Theorems 1.1 and 1.2 with a slightly different form of the set K were presented in [1]. A proof of Theorem 1.1 can be found [7]. Theorem 1.2 was extended in [3] for the sequence $\{k^\alpha h : k \in \mathbb{N}_0\}$ with a fixed α, $0 < \alpha < 1$. For this, the uniform distribution modulo 1 of the sequence $\{k^\alpha : k \in \mathbb{N}_0\}$ was applied. We recall that the sequence $\{z_k : k \in \mathbb{N}\} \subset \mathbb{R}$ is called uniformly distributed modulo 1 if, for every $I = [a, b) \subset [0, 1)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_I(\{z_k\}) = \text{length}(I),$$

where χ_I is the indicator function of the interval I, and $\{u\}$ denotes the fractional part of $u \in \mathbb{R}$.

The aim of this paper is a generalization of the mentioned theorem from [3]. In what follows, we suppose that $N \to \infty$. We consider the class $\mathfrak{X}$ of sequences $\{x_k : k \in \mathbb{N}\} \subset \mathbb{R}$ satisfying the following hypotheses:

1. $\{a x_k\}$ is uniformly distributed modulo 1 for all real $a \neq 0$;
2. $1 \le x_k \le k$ for all $k \in \mathbb{N}$;
3. for $1 \le k, m \le N$, $k \neq m$, the inequality

$$|x_k - x_m| \ge \frac{1}{y_N}$$

   holds with $y_N > 0$ satisfying $y_N x_N \ll N$.

Theorem 1.3. *Suppose that the sequence $\{x_k : k \in \mathbb{N}\} \in \mathfrak{X}$. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $h > 0$ and $\epsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N} \# \left\{ 1 \le k \le N : \sup_{s\in K} |\zeta(s+ix_k h) - f(s)| < \epsilon \right\} > 0.$$

As it was noted above, the functions of the class $H_0(K)$, $K \in \mathcal{K}$, can be approximated by shifts $\zeta(s+ik^\alpha h)$ with $0 < \alpha < 1$ and $h > 0$ [3]. We observe that the sequence $\{k^\alpha\}$, $0 < \alpha < 1$, is an element of the class $\mathfrak{X}$.

Really, the sequence $\{ak^\alpha\}$ is uniformly distributed modulo 1 (Exercise 3.10 in [6]). Moreover, it is not difficult to see that, for $1 \le k \le N-1$,

$$(k+1)^\alpha - k^\alpha \ge \frac{\alpha}{2N^{1-\alpha}}.$$

Therefore, in this case, we can take $y_N = \frac{\alpha}{2N^{1-\alpha}}$. Thus, Theorem 5 of [3] is a particular case of Theorem 1.3.

The properties of the sequence $\{k^\alpha\}$, $0 < \alpha < 1$, were also applied in [9], where a joint discrete universality theorem for Dirichlet L-functions $L(s,\chi)$ was obtained. Denote by $\mathcal{P}$ the set of all prime numbers. Assuming that the set

$$\{(h_1 \log p : p \in \mathcal{P}), \dots, (h_r \log p : p \in \mathcal{P})\},$$

$h_1 > 0, \dots, h_r > 0$, is linearly independent over the field of rational numbers, it was proved [9] that if $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$, $j = 1, \dots, r$, then, for every $\epsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N} \# \left\{ 1 \le k \le N : \sup_{1 \le j \le r} \sup_{s \in K_j} |L(s + ik^\alpha h_j, \chi_j) - f_j(s)| < \epsilon \right\} > 0.$$

It is also known [6] that the sequence $\{ak^{\beta_1} \log^{\beta_2} k\}$ with $0 < \beta_1 < 1$ and $\beta_2 > 0$ is uniformly distributed modulo 1 for all real $a \neq 0$. In [10], a discrete universality theorem for the periodic Hurwitz zeta-function

$$\zeta(s, \alpha; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s}, \quad \sigma > 1,$$

where $\mathfrak{a} = \{a_m\}$ is a periodic sequence of complex numbers, and $0 < \alpha < 1$ is a fixed parameter, on the approximation of analytic functions by shifts $\zeta(s + ihk^{\beta_1} \log^{\beta_2} k, \alpha; \mathfrak{a})$ was proved. Properties of elementary functions show that, for $2 \le k \le N-1$,

$$(k+1)^{\beta_1} \log^{\beta_2}(k+1) - k^{\beta_1} \log^{\beta_2} k \ge \frac{c \log^{\beta_2} N}{N^{1-\beta_1}}.$$

with some $c > 0$. Therefore, $\{k^{\beta_1} \log^{\beta_2} k\} \in \mathfrak{X}$ with $0 < \beta_1 < 1$, $\beta_2 > 0$.

We note that at the moment many universality theorems for various zeta-functions are known. In our opinion, the above remarks suggest that the results of [9,10], and of other works can be generalized in accordance with Theorem 1.3. Recently, very interesting results in this direction were obtained by Ł. Pańkowski [14]. Among other important results, he proved the following joint discrete universality theorem for Dirichlet L-functions. Assume that $\chi_1, \dots, \chi_n$ are arbitrary Dirichlet characters, $\alpha_1, \dots, \alpha_n \in \mathbb{R}$, $a_1, \dots, a_n$ non-negative real numbers and, $b_1, \dots, b_n$ such that

$$b_j \in \begin{cases} \mathbb{R} & \text{if } a_j \notin \mathbb{Z}, \\ (-\infty, 0] \cup (1, +\infty) & \text{if } a_j \in \mathbb{N}, \end{cases}$$

and $a_j \neq a_k$ or $b_j \neq b_k$ if $k \neq j$. Let $K \in \mathcal{K}$ and $f_1, \dots, f_n \in H_0(K)$. Then, for every $\epsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N} \# \left\{ 2 \leq k \leq N : \max_{1\leq j\leq n} \max_{s\in K} \left| L(s + i\alpha_j k^{a_j} \log^{b_j} k, \chi) - f_j(s) \right| < \epsilon \right\} > 0.$$

We observe that the author does not require that the characters $\chi_1, \dots, \chi_n$ would be pairwise non-equivalent, that is, he proved joint universality theorems for dependent L-functions.

2. Probabilistic model. Denote by $\mathcal{B}(X)$ the Borel σ-field of the space X. Consider the set

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \{s \in \mathbb{C} : |s| = 1\}$ for all primes p. The infinite-dimensional torus Ω, with the product topology and operation of pointwise multiplication, is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H can be defined, and we obtain the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the projection of an element $\omega \in \Omega$ to the circle γ_p, and define

$$\zeta(s, \omega) = \prod_p \left(1 - \frac{\omega(p)}{p^s}\right)^{-1}.$$

Then it is proved [7] that the infinite product over primes converges uniformly on compact subsets of the strip D for almost all $\omega \in \Omega$. Therefore, denoting by $H(D)$ the space of analytic functions on D endowed with the topology of uniform convergence on compacta, we have that $\zeta(s, \omega)$ is the $H(D)$-valued random element defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Let P_ζ be the distribution of $\zeta(s, \omega)$, i.e.,

$$P_\zeta(A) = m_H\left(\omega \in \Omega : \zeta(s, \omega) \in A\right), \quad A \in \mathcal{B}(H(D)).$$

The aim of this section is the following limit theorem.

Theorem 2.1. *Suppose that $\{x_k : k \in \mathbb{N}\} \in \mathfrak{X}$. Then, for every $h > 0$,*

$$P_N(A) \overset{def}{=} \frac{1}{N} \# \left\{1 \leq k \leq N : \zeta(s + ix_k h) \in A\right\}, \quad A \in \mathcal{B}(H(D)),$$

converges weakly to P_ζ as $N \to \infty$.

The main ingredient of the proof of Theorem 2.1 is a limit theorem on the torus Ω. Define

$$Q_N(A) = \frac{1}{N} \#\{1 \leq k \leq N : (p^{-ix_k h} : p \in \mathcal{P}) \in A\}, \quad A \in \mathcal{B}(\Omega).$$

Lemma 2.2. *Suppose that the sequence $\{ax_k\}$ with every real $a \neq 0$ is uniformly distributed modulo 1. Then, for every $h > 0$, Q_N converges weakly to the Haar measure m_H as $N \to \infty$.*

Proof. The Weyl criterion is a powerful tool for checking that a sequence is uniformly distributed modulo 1. This criterion says [6] that a sequence $\{z_k\}$ is uniformly distributed modulo 1 if and only if, for every $m \in \mathbb{Z}\backslash\{0\}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\infty} \mathrm{e}^{2\pi i z_k m} = 0.$$

To prove Lemma 2.2, we consider the Fourier transform $g_N(\underline{k})$, $\underline{k} = (k_p : k_p \in \mathbb{Z},\ p \in \mathcal{P})$. It is well known that

$$\begin{aligned} g_N(\underline{k}) &= \int_\Omega \prod_p \omega^{k_p}(p) \mathrm{d}Q_N = \frac{1}{N} \sum_{k=1}^{N} \prod_p p^{-ix_k k_p h} \\ &= \frac{1}{N} \sum_{k=1}^{N} \exp\left\{ -ix_k h \sum_p k_p \log p \right\}, \end{aligned} \tag{2.1}$$

where only a finite number of integers k_p are distinct from zero. Obviously, if $\underline{k} = \underline{0}$, then

$$g_N(\underline{k}) = 1. \tag{2.2}$$

It is well known that the logarithms of prime numbers are linearly independent over the field of rational numbers. Therefore, if $\underline{k} \neq \underline{0}$, then

$$\sum_p k_p \log p \neq 0.$$

Thus, by hypothesis 1 for the class $\mathfrak{X}$, we have that, in the case $\underline{k} \neq \underline{0}$, the sequence

$$\left\{ -\frac{1}{2\pi} x_k h \sum_p k_p \log p \right\}$$

is uniformly distributed modulo 1. Hence, in view of the Weyl criterion,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \exp\left\{ -ix_k h \sum_p k_p \log p \right\} = 0.$$

This, (2.1), and (2.2) show that

$$\lim_{N\to\infty} g_N(\underline{k}) = \begin{cases} 1 & \text{if } \ \underline{k} = \underline{0}, \\ 0 & \text{if } \ \underline{k} \neq \underline{0}. \end{cases} \tag{2.3}$$

Since the right-hand side of (2.3) is the Fourier transform of the Haar measure m_H, the lemma follows by a continuity theorem for probability measures on compact groups. □

The next step of the proof of Theorem 2.1 is a limit theorem for absolutely convergent Dirichlet series. Let $\theta > \frac{1}{2}$ be a fixed number, and, for $m, n \in \mathbb{N}$,

$$v_n(m) = \exp\left\{ -\left(\frac{m}{n}\right)^\theta \right\}.$$

Define

$$\zeta_n(s) = \sum_{m=1}^{\infty} \frac{v_n(m)}{m^s}.$$

Then the series for $\zeta_n(s)$ is absolutely convergent for $\sigma > \frac{1}{2}$ [7]. For $A \in \mathcal{B}(H(D))$, define

$$P_{N,n}(A) = \frac{1}{N} \# \left\{ 1 \le k \le N : \zeta_n(s + ix_k h) \in A \right\},$$

and consider the weak convergence of $P_{N,n}$ as $N \to \infty$.

We take the function $u_n : \Omega \to H(D)$ defined by the formula

$$u_n(\omega) = \sum_{m=1}^{\infty} \frac{v_n(m)\omega(m)}{m^s},$$

where $\omega(p)$ is extended to the set $\mathbb{N}$ by the formula

$$\omega(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \omega^l(p), \quad m \in \mathbb{N}.$$

Since the series for $\zeta_n(s)$ is absolutely convergent for $\sigma > \frac{1}{2}$, we have that the function u_n is a continuous one. These remarks together with Lemma 2.2 and standard arguments give the following limit theorem.

Lemma 2.3. *Suppose that the sequence $\{ax_k\}$ with every real $a \neq 0$ is uniformly distributed modulo 1. Then $P_{N,n}$ converges weakly to the measure $\widehat{P}_n = m_H u_n^{-1}$ as $N \to \infty$. The measure $\widehat{P}_n$ is defined by $\widehat{P}_n(A) = m_H \left(u_n^{-1} A \right)$, $A \in \mathcal{B}(H(D))$.*

The most complicated part of the proof of Theorem 2.1 consists of the arguments which allow to pass from $\zeta_n(s)$ to $\zeta(s)$. For this, other assumptions on the class $\mathfrak{X}$ will be applied. We start with discrete moments of the Riemann zeta-function. First, we recall the Gallagher lemma which relates discrete and continuous mean squares of continuous functions.

Lemma 2.4. *Let T_0 and $T \ge \delta > 0$ be real numbers, and $\mathcal{T}$ be a finite set in the interval $[T_0 + \frac{\delta}{2}, T_0 + T - \frac{\delta}{2}]$. Define*

$$N_\delta(x) = \sum_{\substack{t \in \mathcal{T} \\ |t - x| < \delta}} 1,$$

and let $S(x)$ be a complex-valued continuous function on $[T_0, T + T_0]$ having a continuous derivative on $(T_0, T + T_0)$. Then

$$\sum_{t \in \mathcal{T}} N_\delta^{-1}(t) |S(t)|^2 \le \frac{1}{\delta} \int_{T_0}^{T_0+T} |S(x)|^2 \mathrm{d}x + \left(\int_{T_0}^{T_0+T} |S(x)|^2 \mathrm{d}x \int_{T_0}^{T_0+T} |S'(x)|^2 \mathrm{d}x \right)^{\frac{1}{2}}.$$

The proof of the lemma can be found in [13, Lemma 1.4.].

Denote by $\rho(g_1, g_2)$ the metric in $H(D)$ which induces its topology of uniform convergence on compacta, see [7].

Lemma 2.5. *Suppose that* $\{x_k\} \in \mathfrak{X}$. *Then, for every* $h > 0$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \rho\left(\zeta(s + ix_k h), \zeta_n(s + ix_k h)\right) = 0.$$

Proof. By hypothesis 3 of the class $\mathfrak{X}$, we find with $\delta = \frac{1}{y_N}$ that

$$N_\delta(x_k) = \sum_{\substack{m=1 \\ |x_k - x_m| < \frac{1}{y_N}}}^{N} 1 = 1.$$

Therefore, taking into account the well-known estimates, for a fixed $\sigma \in \left(\frac{1}{2}, 1\right)$,

$$\int_1^T |\zeta(\sigma + it)|^2 \, \mathrm{d}t = O(T)$$

and

$$\int_1^T |\zeta'(\sigma + it)|^2 \, \mathrm{d}t = O(T),$$

we find using Lemma 2.4 and hypotheses 2 and 3 of the class $\mathfrak{X}$, that, for $\frac{1}{2} < \sigma < 1$,

$$\sum_{k=1}^{N} |\zeta(\sigma + ix_k h + it)|^2 \le \frac{y_N}{h} \int_{x_1 h}^{x_N h} |\zeta(\sigma + i\tau + it)|^2 \mathrm{d}\tau$$
$$+ \left(\int_{x_1 h}^{x_N h} |\zeta(\sigma + i\tau + it)|^2 \mathrm{d}\tau \int_{x_1 h}^{x_N h} |\zeta'(\sigma + i\tau + it)|^2 \mathrm{d}\tau \right)^{\frac{1}{2}}$$
$$\ll \frac{y_N}{h}\left(x_N h + |t|\right) + x_N h + |t| \ll y_N x_N + y_N |t| \ll N\left(1 + |t|\right). \tag{2.4}$$

Now let K be an arbitrary compact subset of D. Then, using estimate (2.4) and applying the contour integration, we find similarly to the proof of Theorem 4.1 of [8] that

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \sup_{s\in K} |\zeta(s + ix_k h) - \zeta_n(s + ix_k h)| = 0.$$

This and the definition of the metric ρ prove the lemma. □

Proof of Theorem 2.1. On a certain probability space $(\widehat{\Omega}, \mathcal{F}, \mu)$, define the random variable η_N by the formula

$$\mu(\eta_N = x_k h) = \frac{1}{N}, \quad k = 1, \ldots, N.$$

Let $X_{N,n}$ be the $H(D)$-valued random element given by

$$X_{N,n} = X_{N,n}(s) = \zeta_n(s + i\eta_N),$$

and let $\widehat{X}_n$ be the $H(D)$-valued random element with the distribution $\widehat{P}_n$, where $\widehat{P}_n$ is defined in Lemma 2.3. Then, using the convergence in distribution, we have by Lemma 2.3 that

$$X_{N,n} \xrightarrow[N\to\infty]{\mathcal{D}} \widehat{X}_n, \tag{2.5}$$

Next we consider the family of probability measures $\left\{\widehat{P}_n : n \in \mathbb{N}\right\}$, and prove its tightness. Indeed, using the absolute convergence of the series for $\zeta_n(s)$, we find that, for $\sigma > \frac{1}{2}$,

$$\sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{T} \int_1^T |\zeta_n(\sigma+it)|^2 \,\mathrm{d}t = \sup_{n\in\mathbb{N}} \sum_{m=1}^{\infty} \frac{v_n^2(m)}{m^{2\sigma}} \le \sum_{m=1}^{\infty} \frac{1}{m^{2\sigma}} \le C < \infty.$$

This, Lemma 2.4, and the Cauchy inequality yield the bound

$$\sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} |\zeta_n(\sigma+ix_kh)| \le C_1 < \infty$$

for $\sigma > \frac{1}{2}$. Therefore, the integral Cauchy formula implies the bound

$$\sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \sup_{s\in K_l} |\zeta_n(s+ix_kh)| \le B_l, \tag{2.6}$$

where $\{K_l : l \in \mathbb{N}\}$ is a sequence of compact subsets of D such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$, and if $K \subset D$ is a compact, then $K \subset K_l$ for some l. The sequence $\{K_l\}$ occurs in the definition of the metric ρ.

Let $\epsilon > 0$ be an arbitrary fixed number, and $M_l = B_l \epsilon^{-1} 2^l$. Then, in view of (2.6), we have that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\limsup_{N\to\infty} \mu\left(\sup_{s\in K_l} |X_{N,n}(s)| > M_l\right) \\
&\quad = \limsup_{N\to\infty} \frac{1}{N} \#\left\{1 \le k \le N : \sup_{s\in K_l} |\zeta_n(s+ix_kh)| > M_l\right\} \\
&\quad \le \limsup_{N\to\infty} \frac{1}{NM_l} \sum_{k=1}^{N} \sup_{s\in K_l} |\zeta_n(s+ix_kh)| \le \frac{\epsilon}{2^l}
\end{aligned}$$

for $l \in \mathbb{N}$. Therefore, by (2.5),

$$\mu\left(\sup_{s\in K_l} \left|\widehat{X}_n(s)\right| > M_l\right) \le \frac{\epsilon}{2^l} \tag{2.7}$$

for all $n \in \mathbb{N}$ and $l \in \mathbb{N}$. Now let

$$K = K(\epsilon) = \left\{g \in H(D) : \sup_{s\in K_l} |g(s)| \le M_l,\, l \in \mathbb{N}\right\}.$$

Then the set K is compact in $H(D)$, and, by (2.7),

$$\mu\left(X_n(s) \in K\right) \geq 1-\epsilon$$

for all $n \in \mathbb{N}$. In other words, $\widehat{P}_n(K) \geq 1-\epsilon$ for all $n \in \mathbb{N}$. Thus, the tightness of $\{\widehat{P}_n\}$ is proved.

The tightness of $\{\widehat{P}_n\}$ implies its relative compactness. Therefore, there exists a sequence $\{\widehat{P}_{n_l}\} \subset \{\widehat{P}_n\}$ such that P_{n_l} converges weakly to a certain probability measure P on $(H(D), \mathcal{B}(H(D))$ as $l \to \infty$. In other words,

$$\widehat{X}_{n_l} \xrightarrow[l\to\infty]{\mathcal{D}} P. \tag{2.8}$$

Define one more $H(D)$-valued random element $X_N = X_N(s) = \zeta(s+i\eta_N)$. Then, using Lemma 2.5, we obtain that, for every $\epsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{N\to\infty} \mu\left(\rho\left(X_N(s), X_{N,n}(s)\right) \geq \epsilon\right) \\
&\leq \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N\epsilon} \sum_{k=1}^{N} \rho\left(\zeta(s+ix_kh), \zeta_n(s+ix_kh)\right) = 0.
\end{aligned} \tag{2.9}$$

Now an application of [2, Theorem 4.2] and (2.5), (2.8), and (2.9) lead to

$$X_N \xrightarrow[N\to\infty]{\mathcal{D}} P. \tag{2.10}$$

This means that P_N converges weakly to P as $N \to \infty$. On the other hand, (2.10) shows that the limit measure P is independent on the sequence $\{P_{n_l}\}$. Therefore,

$$\widehat{X}_n \xrightarrow[n\to\infty]{\mathcal{D}} P,$$

i.e., $\widehat{P}_n$ converges weakly to P as $n \to \infty$.

In [7], it was proved that

$$\frac{1}{T} \text{ meas } \left\{\tau \in [0,T] : \zeta(s+i\tau) \in A\right\}, \quad A \in \mathcal{B}(H(D)),$$

as $T \to \infty$, also converges weakly to the limit measure P of $\widehat{P}_n$, and that $P = P_\zeta$. Therefore, P_N converges weakly to P_ζ as $N \to \infty$. □

3. Proof of Theorem 1.3. We recall the Mergelyan theorem on the approximation of analytic functions by polynomials [12].

Lemma 3.1. *Let $K \subset D$ be a compact set with connected complement, and $f(s)$ be a continuous function on K which is analytic in the interior of K. Then, for every $\epsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s\in K} |g(s) - p(s)| < \epsilon.$$

Proof of Theorem 1.3. By Lemma 3.1, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} \left| f(s) - \mathrm{e}^{p(s)} \right| < \frac{\epsilon}{2}. \tag{3.1}$$

In [7], it is obtained that the support of the measure P_ζ is the set $\{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}$. Let

$$G = \left\{ g \in H(D) : \sup_{s \in K} \left| g(s) - \mathrm{e}^{p(s)} \right| < \frac{\epsilon}{2} \right\}.$$

Since $\mathrm{e}^{p(s)} \neq 0$, we have that G is an open neighborhood of an element of the support of P_ζ. Therefore, $P_\zeta(G) > 0$. Hence, in view of Theorem 2.1 and Theorem 2.1 of [2],

$$\liminf_{N \to \infty} \frac{1}{N} \# \{1 \leq k \leq N : \zeta(s + ix_k h) \in G\} \geq P_\zeta(G) > 0.$$

This and the definition of the set G give the inequality

$$\liminf_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leq k \leq N : \sup_{s \in K} \left| \zeta(s + ix_k h) - \mathrm{e}^{p(s)} \right| < \frac{\epsilon}{2} \right\} > 0.$$

Combining this with (3.1) completes the proof of the theorem. □

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