Arch. Math. 108 (2017), 85–97

0003-889X/17/010085-13
published online August 27, 2016
DOI 10.1007/s00013-016-0961-8

Archiv der Mathematik

Higher integrability for nonlinear parabolic equations of p-Laplacian type

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Abstract. In this paper we give a new alternative proof of the local higher integrability in Orlicz spaces of the gradient for weak solutions of quasilinear parabolic equations of p-Laplacian type

$$u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \operatorname{div}\left(|\mathbf{f}|^{p-2}\mathbf{f}\right) \quad \text{in } \Omega\times(0,T]$$

for any $p>0$. Moreover, we point out that our results are homogeneous regularity estimates in Orlicz spaces and improve the known results for such equations by using some new techniques. Actually, our results can be extended to the global estimates and cover a more general class of degenerate/singular parabolic problems of p-Laplacian type.

Mathematics Subject Classification. 35B45, 35K55.

Keywords. Higher integrability, Regularity, Orlicz, Nonlinear, Gradient, Parabolic, p-Laplacian.

1. Introduction. Assume that $p>1$. We are concerned with the following quasilinear parabolic equation of p-Laplacian type

$$u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \operatorname{div}\left(|\mathbf{f}|^{p-2}\mathbf{f}\right) \quad \text{in } \Omega_T=\Omega\times(0,T], \tag{1.1}$$

where Ω is an open bounded domain in $\mathbb{R}^n$ and $\mathbf{f}=(f^1,...,f^n)$ is a given vector field. Actually, our results can cover a more general class of degenerate/singular parabolic problems of p-Laplacian type

$$u_t - \operatorname{div}\mathbf{a}(\nabla u, z) = \operatorname{div}\left(|\mathbf{f}|^{p-2}\mathbf{f}\right) \text{ in } \Omega_T. \tag{1.2}$$

In the elliptic case

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \operatorname{div}(|\mathbf{f}|^{p-2}\mathbf{f}) \text{ in } \Omega \tag{1.3}$$

and the general case, $W^{1,q}$ regularity has been extensively studied by many authors with different assumptions on the coefficients and domains (see [1, 7, 11, 12, 14, 15, 17]). Moreover, in [8] we have extended $W^{1,q}$ regularity in the setting of the general Orlicz spaces

$$|\mathbf{f}|^p \in L^\phi(\Omega) \ \Rightarrow \ |\nabla u|^p \in L^\phi(\Omega)$$

with the estimate

$$\int_\Omega \phi(|\nabla u|^p)dx \le C \int_\Omega \phi(|\mathbf{f}|^p)dx \tag{1.4}$$

for weak solutions of (1.3) and the general case with $u = 0$ on $\partial\Omega$. We would like to point out that if $\phi(t) = t^q$ for $q \ge 1$, then (1.4) can be reduced to the classical L^q estimates.

Different from the elliptic case (1.3), (1.1) is not homogeneous even if $\mathbf{f} \equiv 0$, which is one of the most common difficulties. Kinnunen and Lewis [13] obtained a reverse Hölder-inequality of the gradient for (1.1) and the general case. Moreover, Acerbi and Mingione [2] obtained $L^q_{loc}(q \ge 1)$ estimates for (1.1) and the general case

$$|\mathbf{f}|^p \in L^q_{loc}(\Omega_T) \ \Rightarrow \ |\nabla u|^p \in L^q_{loc}(\Omega_T) \text{ for any } q \ge 1$$

with the estimate

$$\left(\int_{Q_1} |\nabla u|^{pq} dz\right)^{\frac{1}{q}} \le C(n,p,q) \left[\int_{Q_2} |\nabla u|^p dz + \left(\int_{Q_2} |\mathbf{f}|^{pq} + 1 dz\right)^{\frac{1}{q}}\right]^d, \tag{1.5}$$

where $Q_r = B_r \times (-r^2, r^2]$ and

$$1 \le d := \begin{cases} p/2 & \text{for } p \ge 2, \\ 2p/[p(n+2) - 2n] & \text{for } 2n/(n+2) < p < 2. \end{cases} \tag{1.6}$$

Furthermore, Byun, Ok, and Ryu [6] proved the global L^q estimtes for (1.2)

$$|\mathbf{f}|^p \in L^q(\Omega_T) \ \Rightarrow \ |\nabla u|^p \in L^q(\Omega_T) \quad \text{for any } q \ge 1$$

with the estimate

$$\int_{\Omega_T} |\nabla u|^{pq} dz \le C \left(\int_{\Omega_T} |\mathbf{f}|^{pq} dz + 1\right)^d.$$

Moreover, we [23] proved the following results in Orlicz spaces for weak solutions of (1.1)

$$|\mathbf{f}|^p \in L^\phi_{loc}(\Omega_T) \ \Rightarrow \ |\nabla u|^p \in L^\phi_{loc}(\Omega_T) \tag{1.7}$$

with the estimate

$$\int_{Q_1} \phi(|\nabla u|^p) dz \le C(n,p,\phi) \left\{ \phi\left[\left(\int_{Q_2} |\nabla u|^p + |\mathbf{f}|^p dz + 1\right)^d\right] + \int_{Q_2} \phi\left(|\mathbf{f}|^p\right) dz \right\}. \tag{1.8}$$

We would like to point out that if $\phi(t) = t^q$ for $q \geq 1$, then (1.8) can be reduced to L^q estimates (1.5).

In this paper we shall give a new alternative proof of (1.7) for weak solutions of (1.1). Actually, we shall prove the following homogeneous regularity estimates in Orlicz spaces for weak solutions of (1.1)

$$\left\| |\nabla u|^p \right\|_{L^\phi(Q_1)} \leq C(n,p,\phi)\left[\|u\|^p_{L^p(Q_2)} + \|u\|^2_{L^2(Q_2)} + \| \, |\mathbf{f}|^p \, \|_{L^\phi(Q_2)}\right]. \quad (1.9)$$

Especially when $\phi(t) = t^q$ for $q \geq 1$, then (1.9) can be reduced to the classical L^q estimtes

$$\left(\int_{Q_1} |\nabla u|^{pq} dz\right)^{\frac{1}{q}} \leq C(n,p,q)\left[\int_{Q_2} |u|^p + |u|^2 \, dz + \left(\int_{Q_2} |\mathbf{f}|^{pq} dz\right)^{\frac{1}{q}}\right].$$

As usual, the solutions of (1.1) are taken in a weak sense.

Definition 1.1. Assume that $\mathbf{f} \in L^p_{loc}(\Omega_T)$. A function $u \in L^p_{loc}(0,T;W^{1,p}_{loc}(\Omega)) \cap L^\infty_{loc}(0,T;L^2_{loc}(\Omega))$ is a local weak solution of (1.1) in Ω_T if for any compact subset $\mathcal{K}$ of Ω and for any subinterval $[t_1,t_2]$ of $(0,T]$

$$\int_{\mathcal{K}} u\varphi dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_{\mathcal{K}} \left\{-u\varphi_t + |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\right\} dxdt = -\int_{t_1}^{t_2}\int_{\mathcal{K}} |\mathbf{f}|^{p-2}\mathbf{f}\cdot\nabla\varphi dxdt$$

for any $\varphi \in W^{1,2}_{loc}(0,T;L^2(\mathcal{K})) \cap L^p_{loc}(0,T;W^{1,p}_0(\mathcal{K}))$.

Orlicz spaces have been studied as the generalization of Sobolev spaces since they were introduced by Orlicz [18] (see [3,4,9,16,22,23]). The theory of Orlicz spaces plays a crucial role in many fields of mathematics including geometry, probability, stochastic, Fourier analysis, and PDE (see [19]).

We denote Φ by

$$\Phi = \left\{\phi : [0,+\infty) \longrightarrow [0,+\infty) \;\middle|\; \phi \text{ is increasing and convex}\right\}. \quad (1.10)$$

Moreover, a function $\phi \in \Phi$ is said to be a Young function if

$$\lim_{t\to 0+} \phi(t)/t = \lim_{t\to+\infty} t/\phi(t) = 0.$$

Definition 1.2. A Young function ϕ is said to satisfy the Δ_2 condition, denoted by $\phi \in \Delta_2$, if there exists a positive constant K such that

$$\phi(2t) \leq K\phi(t) \quad \text{for every } t > 0.$$

Moreover, a Young function ϕ is said to satisfy the ∇_2 condition, denoted by $\phi \in \nabla_2$, if there exists a number $a > 1$ such that

$$\phi(t) \leq \frac{\phi(at)}{2a} \quad \text{for every } t > 0.$$

Remark 1.3. Let ϕ be a Young function. Then $\phi \in \Delta_2 \cap \nabla_2$ if and only if there exist constants $A_2 \geq A_1 > 0$ and $\alpha_1 \geq \alpha_2 > 1$ such that

$$A_1\left(\frac{s}{t}\right)^{\alpha_2} \leq \frac{\phi(s)}{\phi(t)} \leq A_2\left(\frac{s}{t}\right)^{\alpha_1} \quad \text{for } 0 < t \leq s. \quad (1.11)$$

Definition 1.4. Let ϕ be a Young function. Then the Orlicz class $K^{\phi}(\Omega)$ is the set of all measurable functions $g : \Omega \to \mathbb{R}$ satisfying

$$\int_{\Omega} \phi(|g|)\, dx < \infty.$$

The Orlicz space $L^{\phi}(\Omega)$ is the linear hull of $K^{\phi}(\Omega)$. Moreover, we define the Luxemburg norm

$$\|g\|_{L^{\phi}(\Omega)} = \inf\left\{\lambda > 0 : \int_{\Omega} \phi\left(\frac{|g|}{\lambda}\right) dx \le 1\right\}. \tag{1.12}$$

Let us state the main result of this work.

Theorem 1.5. *Assume that $p > 1$ and $\phi \in \Delta_2 \cap \nabla_2$. Let u be a local weak solution of (1.1) in Ω_T and $|\mathbf{f}|^p \in L^{\phi}_{loc}(\Omega_T)$. Then we have*

$$|\nabla u|^p \in L^{\phi}_{loc}(\Omega_T)$$

with the estimate (1.9).

Remark 1.6. We remark that the $\Delta_2 \cap \nabla_2$ condition is optimal (see [21]).

2. Proof of the main result. In this section, we shall finish the proof of the main result, Theorem 1.5. We first give the following local L^p estimate and comparison result.

Lemma 2.1. *Assume that u is a local weak solution of (1.1) in Ω_T and v is the weak solution of*

$$\begin{cases} v_t - div\big(|\nabla v|^{p-2}\nabla v\big) = 0 & \text{in } Q_2, \\ v = u & \text{on } \partial_p Q_2 \end{cases} \tag{2.1}$$

with $Q_2 \subset \Omega_T$. Then we have

1. $$\int_{Q_1} |\nabla u|^p dz \le C\left(\int_{Q_2} |u|^p + |u|^2 dz + \int_{Q_2} |\mathbf{f}|^p dz\right). \tag{2.2}$$

2. $$\int_{Q_2} |\nabla v|^p dz \le C\left(\int_{Q_2} |\nabla u|^p dz + \int_{Q_2} |\mathbf{f}|^p dz\right). \tag{2.3}$$

Proof. 1. We may as well select the test function $\varphi = \zeta^p u$, where $\zeta(x,t) \in C_0^{\infty}(\mathbb{R}^{n+1})$ is a cut-off function satisfying

$$0 \le \zeta \le 1, \quad \zeta \equiv 1 \text{ in } Q_1, \quad \zeta \equiv 0 \text{ in } \mathbb{R}^n/Q_2, \quad \text{and} \quad |\zeta_t| + |\nabla\zeta| \le C.$$

Then by Definition 1.1, we have

$$
\begin{aligned}
&\frac{1}{2}\int_{B_2} |u(x,4)|^2 \zeta^p(x,4)dx + \int_{Q_2} \zeta^p |\nabla u|^p\, dz \\
&\quad = \frac{p}{2}\int_{Q_2} \zeta^{p-1}\zeta_t u^2 dz - p\int_{Q_2} \zeta^{p-1} u\, |\nabla u|^{p-2}\, \nabla u\cdot\nabla\zeta dz \\
&\qquad - \int_{Q_2} \zeta^p\, |\mathbf{f}|^{p-2}\, \mathbf{f}\cdot\nabla u + p\zeta^{p-1} u\, |\mathbf{f}|^{p-2}\, \mathbf{f}\cdot\nabla\zeta dz.
\end{aligned}
$$

Using Young's inequality with $\tau > 0$ and the definition of ζ, we deduce that

$$
\int_{Q_2} \zeta^p |\nabla u|^p\, dz \le 2\tau \int_{Q_2} \zeta^p |\nabla u|^p + C(\tau)\int_{Q_2} |u|^2 + |u|^p + |\mathbf{f}|^p\, dz,
$$

which finishes the proof of (2.2) by choosing $\tau = 1/4$.

2. Noting that u and v are the weak solutions of (1.1) and (2.1), respectively, we may as well select the test function $\varphi = v - u$. Then a direct calculation shows the resulting expression as

$$
I_1 = I_2 + I_3,
$$

where

$$
\begin{aligned}
I_1 &= \frac{1}{2}\int_{B_2} |v(x,4) - u(x,4)|^2 dx + \int_{Q_2} |\nabla v|^p dz \ge \int_{Q_2} |\nabla v|^p dz,\\
I_2 &= \int_{Q_2} |\nabla v|^{p-2}\nabla v\cdot\nabla u dz + \int_{Q_2} |\nabla u|^{p-2}\nabla u\cdot\nabla v - |\nabla u|^p dz,\\
I_3 &= \int_{Q_2} |\mathbf{f}|^{p-2}\mathbf{f}\cdot\nabla v - |\mathbf{f}|^{p-2}\mathbf{f}\cdot\nabla u dz.
\end{aligned}
$$

Estimate of I_2 and I_3. Using Young's inequality with $\tau > 0$ and the definition of ζ, we deduce that

$$
|I_2| \le 2\tau \int_{Q_2} |\nabla v|^p\, dz + C(\tau)\int_{Q_2} |\nabla u|^p\, dz
$$

and

$$
|I_3| \le \tau \int_{Q_2} |\nabla v|^p\, dz + C(\tau)\int_{Q_2} |\nabla u|^p + |\mathbf{f}|^p\, dz \quad \text{for any } \tau > 0.
$$

Combining the estimates of I_i $(1 \le i \le 3)$, we deduce that

$$
\int_{Q_2} |\nabla v|^p dz \le 3\tau \int_{Q_2} |\nabla v|^p dz + C\int_{Q_2} |\nabla u|^p + |\mathbf{f}|^p\, dz.
$$

Selecting $\tau = 1/4$, we deduce that (2.3) is true. This finishes our proof. □

In this work we shall use the Hardy–Littlewood maximal function which controls the local behavior of a function.

Definition 2.2. Let h be a locally integrable function. The Hardy–Littlewood maximal function $\mathcal{M}h(z)$ is defined as

$$\mathcal{M}h(z) = \sup_{r>0} \fint_{Q_r(z)} |h(y,s)| dyds.$$

Moreover, if h is not defined outside Ω_T, then $\mathcal{M}h(z) = \mathcal{M}\left(h\,\chi_{\Omega_T}\right)(z)$.

Lemma 2.3 (see [16]) Let $\phi \in \Delta_2 \cap \nabla_2$ and $g \in L^\phi(\Omega_T)$. *Then we have*

1. $\displaystyle\int_{\Omega_T} \phi(|g|)dz = \int_0^\infty |\{z \in \Omega_T : |g| > \lambda\}|\, d\left[\phi(\lambda)\right].$
2. $\displaystyle\int_{\Omega_T} \phi\left(|g|\right) dz \leq \int_{\Omega_T} \phi\left(\mathcal{M}\left(|g|\right)\right) dz \leq C(n,\phi) \int_{\Omega_T} \phi\left(|g|\right) dz.$

We will use the following the modified Vitali covering lemma.

Lemma 2.4 (see [5,20]) *Assume that C and D are measurable sets with $C \subset D \subset Q_1 \subset \mathbb{R}^{n+1}$ and*

$$|C| < \epsilon |Q_1| \quad \text{for } \epsilon > 0,$$

and for all $z \in Q_1$ and for all $r \in (0,1]$ with $|C \cap Q_r(z)| \geq \epsilon |Q_r(z)|$,

$$Q_r(z) \cap Q_1 \subset D.$$

Then we have

$$|C| \leq 10^{n+2} \epsilon |D|.$$

Next, we shall prove the following important result.

Lemma 2.5. *For any $\epsilon > 0$, there exists a small $\delta = \delta(\epsilon) > 0$ such that if u is a local weak solution of (1.1) in Ω_T with $Q_2 \subset \Omega_T$,*

$$\fint_{Q_2} |\nabla u|^p dz \leq 1 \quad \text{and} \quad \fint_{Q_2} |\mathbf{f}|^p dz \leq \delta, \tag{2.4}$$

then there exist a weak solution v of (2.1) and $N_0 > 1$ such that

$$\fint_{Q_2} |\nabla (u-v)|^p dz \leq \epsilon \tag{2.5}$$

and

$$\sup_{Q_{\frac{3}{2}}} |\nabla v|^p \leq N_0. \tag{2.6}$$

Proof. Actually, the conclusion (2.6) follows from (2.4), (2.5), and [10, Chapter 8, Theorem 5.1 and 5.2]. Noting that both u and v are the weak solutions of (1.1) and (2.1), respectively, we may as well select the test function $\varphi = v - u$. Then a direct calculation shows the resulting expression as

$$I_1 + I_2 = I_3,$$

where

$$I_1 = \frac{d}{dt}\int_{Q_2} \frac{|v-u|^2}{2}dz = \int_{B_2} \frac{|v(x,4)-u(x,4)|^2}{2}dx \geq 0,$$

$$I_2 = \int_{Q_2} \left[|\nabla v|^{p-2}\nabla v - |\nabla u|^{p-2}\nabla u\right]\cdot\nabla(v-u)dz,$$

$$I_3 = \int_{Q_2} |\mathbf{f}|^{p-2}\mathbf{f}\cdot\nabla(v-u)dz.$$

Estimate of I_2. We divide into two cases.

Case 1 $p \geq 2$. Using the elementary inequality

$$\left[|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right]\cdot(\xi-\eta) \geq C(p)|\xi-\eta|^p \quad \text{for every } \xi, \quad \eta \in \mathbb{R}^n,$$

we have

$$I_2 \geq C\int_{Q_2} |\nabla(u-v)|^p dz.$$

Case 2 $1 < p < 2$. Using the elementary inequality

$$|\xi-\eta|^p \leq C(p)\tau^{(p-2)/p}\left[|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right]\cdot(\xi-\eta) + \tau|\eta|^p$$
$$\text{for every } \xi,\eta\in\mathbb{R}^n \quad \text{and} \quad \tau\in(0,1),$$

we have

$$I_2 + \tau^{2/p}\int_{Q_2}|\nabla u|^p dz \geq C(\tau)\int_{Q_2}|\nabla(u-v)|^p dz.$$

Estimate of I_3. Using Young's inequality with τ, we have

$$I_3 \leq \tau\int_{Q_2}|\nabla(u-v)|^p dz + C(\tau)\int_{Q_2}|\mathbf{f}|^p dz.$$

Combining all the estimates of I_i $(1 \leq i \leq 3)$, we obtain

$$C(\tau)\int_{Q_2}|\nabla(u-v)|^p dz \leq \tau\int_{Q_2}|\nabla(u-v)|^p dz + \tau^{2/p}\int_{Q_2}|\nabla u|^p dz + C(\tau)\int_{Q_2}|\mathbf{f}|^p dz.$$

Selecting a small constant $\tau > 0$ such that $0 < \delta \ll \tau < 1$, and then using (2.4), we conclude that

$$\fint_{Q_2}|\nabla(u-v)|^p dz \leq C\tau^{2/p} + C(\tau)\delta = \epsilon$$

by selecting δ satisfying the last inequality above. This completes our proof. □

Furthermore, we shall give the following result.

Lemma 2.6. *There is a constant $N_1 > 1$ so that for any $\epsilon > 0$, there exists a small $\delta = \delta(\epsilon) > 0$ such that if u is a local weak solution of (1.1) in Ω_T with $Q_4 \subset \Omega_T$ and*

$$Q_1 \cap \{z \in Q_1 : \mathcal{M}\left(|\nabla u|^p\right)(z) \le 1\} \cap \{z \in Q_1 : \mathcal{M}\left(|\mathbf{f}|^p\right)(z) \le \delta\} \neq \phi, \quad (2.7)$$

then we have

$$|\{z \in Q_1 : \mathcal{M}\left(|\nabla u|^p\right)(z) > N_1\}| < \epsilon\,|Q_1|. \quad (2.8)$$

Proof. From (2.7), there exists $z_1 \in Q_1$ satisfying

$$\mathcal{M}\left(|\nabla u|^p\right)(z_1) \le 1 \quad \text{and} \quad \mathcal{M}\left(|\mathbf{f}|^p\right)(z_1) \le \delta, \quad (2.9)$$

which implies that

$$\fint_{Q_2} |\nabla u|^p dz \le \left(\frac{3}{2}\right)^{n+2} \fint_{Q_3(z_1)} |\nabla u|^p dz \le \left(\frac{3}{2}\right)^{n+2}$$

$$\text{and} \quad \fint_{Q_2} |\mathbf{f}|^p dz \le \left(\frac{3}{2}\right)^{n+2} \delta,$$

since $Q_2 \subset Q_3(z_1) \subset Q_4 \subset \Omega_T$. Thus, using Lemma 2.5, for any $\eta > 0$ there exists $\delta = \delta(\eta) > 0$ and a corresponding weak solution v of (2.1) such that

$$\fint_{Q_2} |\nabla u - \nabla v|^p dz \le \eta \quad \text{and} \quad \sup_{Q_{\frac{3}{2}}} |\nabla v|^p \le N_0. \quad (2.10)$$

Now we shall prove that

$$\begin{aligned} &\left\{z \in Q_1 : |\nabla u|^p(z) > N_1 =: \max\{2^p N_0, 8^{n+2}\}\right\} \\ &\quad \subset \{z \in Q_1 : |\nabla u - \nabla v|^p(z) > N_0\}. \end{aligned} \quad (2.11)$$

Let $z_2 \in \{z \in Q_1 : \mathcal{M}\left(|\nabla u - \nabla v|^p\right)(z) \le N_0\}$. Then we divide into two cases.

Case 1 $r \le \frac{1}{2}$. Then we have $Q_r(z_2) \subset Q_{\frac{3}{2}}$. From (2.10) we have

$$\begin{aligned} \fint_{Q_r(z_2)} |\nabla u|^p dz &\le 2^{p-1} \fint_{Q_r(z_2)} |\nabla u - \nabla v|^p + |\nabla v|^p dz \\ &\le 2^{p-1} N_0 + 2^{p-1} N_0 =: 2^p N_0. \end{aligned}$$

Case 2 $r > \frac{1}{2}$. Then we have $z_1, z_2 \in Q_1 \subset Q_{4r}(z_2) \subset Q_{8r}(z_1)$. From (2.9) we find that

$$\fint_{Q_r(z_2)} |\nabla u|^p dz \le 8^{n+2} \fint_{Q_{8r}(z_1)} |\nabla u|^p dz \le 8^{n+2}.$$

Thus, from *Cases 1* and *2* we conclude that the desired result (2.11) is true. Finally, (2.10) and (2.11) imply that

$$\begin{aligned} |\{z \in Q_1 : \mathcal{M}\left(|\nabla u|^p\right)(z) > N_1\}| &\le |\{z \in Q_1 : \mathcal{M}\left(|\nabla u - \nabla v|^p\right)(z) > N_0\}| \\ &\le C \int_{Q_1} |\nabla u - \nabla v|^p dz < C\eta = \epsilon \end{aligned}$$

by choosing η small enough satisfying the last inequality. Thus we complete the proof. □

Moreover, we can obtain the following result by a scaling and normalization argument.

Lemma 2.7. *There is a constant* $N_1 > 1$ *so that for any* $\epsilon > 0$ *and* $r \leq 1$, *there exists a small* $\delta = \delta(\epsilon) > 0$ *such that if* u *is a local weak solution of (1.1) in* Ω_T *with* $Q_{4r} \subset Q_4 \subset \Omega_T$ *and*

$$\left|\{z \in Q_r : \mathcal{M}\left(|\nabla u|^p\right)(z) > \lambda N_1\}\right| \geq \epsilon \left|Q_r\right| \quad \text{for any } \lambda > 0, \tag{2.12}$$

then we have

$$Q_r \subset \{z \in Q_r : \mathcal{M}\left(|\nabla u|^p\right)(z) > \lambda\} \cup \{z \in Q_r : \mathcal{M}\left(|\mathbf{f}|^p\right)(z) > \lambda\delta\}. \tag{2.13}$$

Proof. We divide into three cases.

Case 1 $r = 1$ and $\lambda = 1$. The result can follow directly from Lemma 2.6.

Case 2 $0 < r < 1$ and $\lambda = 1$. We rescale by defining

$$w(x,t) = \frac{1}{r}u(rx, r^2 t) \quad \text{and} \quad \mathbf{g}(z) = \mathbf{f}(rx, r^2 t).$$

Then w is a local weak solution of

$$w_t - \operatorname{div}\left(|\nabla w|^{p-2}\nabla w\right) = \operatorname{div}\left(|\mathbf{g}|^{p-2}\mathbf{g}\right) \quad \text{in } \Omega'_T \supset Q_4,$$

where $\frac{1}{r}\Omega_T = \frac{1}{r}\Omega \times \left(0, \frac{1}{r^2}T\right) = \Omega'_T$. Therefore, we can get the result directly from Lemma 2.6.

Case 3 $0 < r \leq 1$ and $\lambda > 0$. We can get the desired result if we consider

$$\frac{1}{\lambda}u_t - \frac{1}{\lambda}\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \frac{1}{\lambda}\operatorname{div}\left(|\mathbf{f}|^{p-2}\mathbf{f}\right) \quad \text{in } \Omega_T.$$

Thus, we finish the proof. □

Next, we shall prove the following important result.

Lemma 2.8. *Assume that* u, δ, N_1 *satisfy the same conditions as those in Lemma 2.7. Assume further*

$$\left|\left\{z \in Q_1 : \frac{1}{\lambda_0}\mathcal{M}\left(|\nabla u|^p\right)(z) > N_1\right\}\right| < \epsilon\left|Q_1\right| \quad \textit{for any } \lambda_0 > 0. \tag{2.14}$$

Then for any $\lambda \geq 1$ *we have*

$$\begin{aligned}
&\left|\left\{z \in Q_1 : \frac{1}{\lambda_0}\mathcal{M}\left(|\nabla u|^p\right)(z) > \lambda N_1\right\}\right| \\
&\quad \leq 10^{n+2}\epsilon\left(\left|\left\{z \in Q_1 : \frac{1}{\lambda_0}\mathcal{M}\left(|\nabla u|^p\right)(z) > \lambda\right\}\right|\right. \\
&\qquad \left. + \left|\left\{z \in Q_1 : \frac{1}{\lambda_0}\mathcal{M}\left(|\mathbf{f}|^p\right)(z) > \lambda\delta\right\}\right|\right).
\end{aligned}$$

Proof. Without loss of generality, we may as well assume that $\lambda_0 = 1$. If not, we can replace $|\nabla u|^p, |\mathbf{f}|^p$ by $\frac{|\nabla u|^p}{\lambda_0}, \frac{|\mathbf{f}|^p}{\lambda_0}$. Define

$$C = \{z \in Q_1 : \mathcal{M}(|\nabla u|^p)(z) > \lambda N_1\}$$

and

$$D = \{z \in Q_1 : \mathcal{M}(|\nabla u|^p)(z) > \lambda\} \cup \{z \in Q_1 : \mathcal{M}(|\mathbf{f}|^p)(z) > \lambda\delta\}$$

for any $\lambda \geq 1$. Then $C \subset D \subset Q_1$ and

$$|C| \leq |\{z \in Q_1 : \mathcal{M}(|\nabla u|^p)(z) > N_1\}| < \epsilon |Q_1|$$

in view of (2.14). Furthermore, from Lemmas 2.4 and 2.7 we find that

$$|C| \leq 10^{n+2}\epsilon |D|,$$

which finishes our proof. □

Finally, we are set to prove the main result, Theorem 1.5.

Proof. Let

$$\lambda_0 = \frac{1}{\delta}\left(\|u\|^p_{L^p(Q_2)} + \|u\|^2_{L^2(Q_2)} + \| |\mathbf{f}|^p \|_{L^\phi(Q_2)}\right) \quad \text{for some small } \delta \in (0,1). \tag{2.15}$$

It follows from Lemma 2.1 that

$$\int_{Q_1} \frac{1}{\lambda_0}|\nabla u|^p dz \leq C\left\{\int_{Q_2} \frac{1}{\lambda_0}\left(|u|^p + |u|^2\right) dz + \int_{Q_2} \frac{1}{\lambda_0}|\mathbf{f}|^p dz\right\}. \tag{2.16}$$

Moreover, if $g \in L^\phi(Q_1)$, then from (1.11) we find that

$$\begin{aligned} \int_{Q_1} \frac{1}{\lambda_0}|g| dz &= \int_{\left\{z\in Q_1 : \frac{1}{\lambda_0}|g| \leq 1\right\}} \frac{1}{\lambda_0}|g| dz + \int_{\left\{z\in Q_1 : \frac{1}{\lambda_0}|g| \geq 1\right\}} \left(\frac{|g|}{\lambda_0}\right)^{\alpha_2} dz \\ &\leq |Q_1| + \frac{1}{A_1\phi(1)}\int_{Q_1} \phi\left(\frac{1}{\lambda_0}|g|\right) dz. \end{aligned} \tag{2.17}$$

Furthermore, from (2.15), (2.16) and (2.17) we have

$$\begin{aligned} \fint_{Q_1} \frac{1}{\lambda_0}|\nabla u|^p dz &\leq \frac{C}{\lambda_0}\left(\int_{Q_2} |u|^p + |u|^2 dz + \int_{Q_2} |\mathbf{f}|^p dz\right) \\ &\leq C\delta + C\delta\left[1 + \int_{Q_2} \phi\left(\frac{|\mathbf{f}|^p}{\|u\|^p_{L^p(Q_2)} + \|u\|^p_{L^2(Q_2)} + \| |\mathbf{f}|^p \|_{L^\phi(Q_2)}}\right) dz\right] \\ &\leq C\delta \leq \epsilon \end{aligned} \tag{2.18}$$

by taking δ sufficiently small satisfying the last inequality. Furthermore, from (1.12) and (2.18) we find that

$$\left|\left\{z \in Q_1 : \frac{1}{\lambda_0}\mathcal{M}(|\nabla u|^p) > N_1\right\}\right| \leq \frac{1}{N_1}\int_{Q_1} \frac{1}{\lambda_0}|\nabla u|^p dz < \epsilon |Q_1|$$

and

$$\int_{Q_1} \phi\left(\frac{1}{\lambda_0}|\mathbf{f}|^p\right) dz \le C \int_{Q_1} \phi\left(\frac{|\mathbf{f}|^p}{\|u\|^p_{L^p(Q_2)} + \|u\|^p_{L^2(Q_2)} + \|\,|\mathbf{f}|^p\,\|_{L^\phi(Q_2)}}\right) dz \le C. \tag{2.19}$$

Moreover, from Lemmas 2.3 and 2.8 we compute

$$\begin{aligned}
&\int_{Q_1} \phi\left(\frac{1}{\lambda_0}\mathcal{M}\left(|\nabla u|^p\right)\right) dz \\
&\quad = \left\{\int_0^1 + \int_1^\infty\right\} \left|\left\{z \in Q_1 : \frac{1}{\lambda_0}\mathcal{M}\left(|\nabla u|^p\right)(z) > \lambda N_1\right\}\right| d\left[\phi(\lambda N_1)\right] \\
&\quad \le C_1 + \int_1^\infty \left|\left\{z \in Q_1 : \frac{1}{\lambda_0}\mathcal{M}\left(|\nabla u|^p\right)(z) > \lambda N_1\right\}\right| d\left[\phi(\lambda N_1)\right] \\
&\quad \le C_1 + 10^{n+2}\epsilon \int_0^\infty \left|\left\{z \in Q_1 : \frac{1}{\lambda_0}\mathcal{M}\left(|\nabla u|^p\right)(z) > \lambda\right\}\right| d\left[\phi(\lambda N_1)\right] \\
&\qquad + 10^{n+2}\epsilon \int_0^\infty \left|\left\{z \in Q_1 : \frac{1}{\lambda_0}\mathcal{M}\left(|\mathbf{f}|^p\right)(z) > \lambda\delta\right\}\right| d\left[\phi(\lambda N_1)\right] \\
&\quad \le C_1 + C_2\epsilon \int_{Q_1} \phi\left(\frac{1}{\lambda_0}\mathcal{M}\left(|\nabla u|^p\right)\right) dz + C_3 \int_{Q_1} \phi\left(\frac{1}{\lambda_0}\mathcal{M}\left(|\mathbf{f}|^p\right)\right) dz \\
&\quad \le C_1 + C_2\epsilon \int_{Q_1} \phi\left(\frac{1}{\lambda_0}\mathcal{M}\left(|\nabla u|^p\right)\right) dz + C_4 \int_{Q_1} \phi\left(\frac{1}{\lambda_0}|\mathbf{f}|^p\right) dz,
\end{aligned}$$

where $C_2 = C_2(n, \phi, N_1)$ and $C_4 = C_4(n, \phi, \epsilon, N_1)$. Then choosing a suitable ϵ such that $C_2\epsilon < \frac{1}{2}$, thereby determining δ with $0 < \delta < 1$, from (2.19) we obtain

$$\int_{Q_1} \phi\left(\frac{1}{\lambda_0}\mathcal{M}\left(|\nabla u|^p\right)\right) dz \le C.$$

Thus, from the fact that $|\nabla u|^p(z) \le \mathcal{M}\left(|\nabla u|^p\right)(z)$, we have

$$\int_{Q_1} \phi\left(\frac{1}{\lambda_0}|\nabla u|^p\right) dz \le C.$$

Finally, from (1.12) and (2.15) we obtain

$$\left\||\nabla u|^p\right\|_{L^\phi(Q_1)} \le C\left[\|u\|^p_{L^p(Q_2)} + \|u\|^2_{L^2(Q_2)} + \|\,|\mathbf{f}|^p\,\|_{L^\phi(Q_2)}\right],$$

which finishes the proof. □

Acknowledgement. This work is supported in part by the NSFC (11471207) and the Innovation Program of Shanghai Municipal Education Commission (14YZ027).

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Received: 24 May 2016