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Symmetric periodic solutions in the Sitnikov problem

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Abstract. An elementary method to prove the existence of odd periodic solutions with a prescribed number of zeros is presented. In some cases it is also possible to prove the uniqueness of this solution. The method combines shooting arguments with Sturm comparison theory and can be applied to a large class of nonlinear oscillators. In particular, this class includes the Sitnikov problem, a well-known restricted three body problem.

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1. Introduction. The Sitnikov problem describes the motion of a particle of negligible mass attracted by two equal masses $m_1 = m_2 = \frac{1}{2}$. The primaries m_1 and m_2 move on the plane (x, y), following an elliptic motion with eccentricity $e \in [0, 1[$ and major axis of length $2a = 1$. In these conditions the minimal period of the elliptic motion is 2π if it is assumed that the gravitational constant is normalized to $G = 1$. The small particle lies on the perpendicular axis $(0, 0, z)$ and is influenced by the gravitational forces produced by the primaries. The motion of the third body is given by the non-autonomous equation

$$\ddot{z} + \frac{z}{(z^2 + r(t, e)^2)^{3/2}} = 0, \tag{1}$$

where $r(t, e)$ is the distance from the primaries to the center of mass; that is,

$$r(t, e) = \frac{1 - e\cos u}{2}, \quad u - e\sin u = t.$$

This is perhaps the simplest model in Celestial Mechanics presenting a complicated behaviour, and different aspects of the dynamics of this equation

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have been analyzed by Sitnikov, Alekseev, and Moser. More information can be found in [6] and in the more recent paper [1]. The existence of symmetric (even or odd) periodic solutions has been discussed in the papers [2–4,7]. In [2] methods of local analysis are employed, and they lead to results which are valid only for small eccentricity e. The paper [3] presents a very complete description of the set of symmetric periodic solutions based on numerical computations. The papers [4,7] deal with arbitrary eccentricity from a theoretical perspective by using the global continuation method due to Leray and Schauder. The purpose of this note is to show that it is also possible to obtain theoretical results for all values of the eccentricity using only very elementary tools: the shooting method and Sturm oscillation theory. The main observation will be a principle which seems to be valid for a large class of equations: given a family of symmetric solutions with a controlled oscillatory behavior, the solution of the family with least energy is periodic. More precisely, we fix integers $N \geq 0$ and $m \geq 1$ and consider the class $\mathcal{S}_{N,m}$ of solutions of the equation (1) with initial conditions

$$z(0) = 0, \quad \dot{z}(0) = v > 0,$$

and having at most N zeros in the interval $]0, m\pi[$. If it is assumed that the quantity $\frac{1}{2}v^2$ reaches a minimum in $\mathcal{S}_{N,m}$, then the corresponding solution will be an odd $2m\pi$-periodic solution. In the next pages we will show the validity of this principle for a rather general class of equations. This suggests that the same strategy could be applied to other problems such as the restricted N-body problem considered in [9] or the curved Sitnikov problem analyzed in [8]. We conclude this introduction with some concrete results derived from this method. They are valid for any value of the eccentricity $e \in [0, 1[$.

Theorem 1. *For each integer $m \geq 1$ there exists a unique solution $z(t)$ of (1) satisfying the conditions*

$$z(t + 2m\pi) = z(t), \quad z(-t) = -z(t), \quad t \in \mathbb{R}, \tag{2}$$

$$z(t) > 0, \quad t \in]0, m\pi[. \tag{3}$$

In particular, this result implies the existence of a periodic solution with minimal period $2m\pi$ for each $m \geq 1$. The existence of a solution in the conditions of the above theorem can also be proved using the methods developed in [4], but the uniqueness seems to be a new conclusion. In [3] there appear some diagrams drawing the initial conditions of symmetric periodic solutions as functions of e. According to this description, the set of odd periodic solutions seems to be organized in families labelled by the number of oscillations. These families can emanate from the integrable case $e = 0$, from the center of mass $z = 0$ or even from the singular case $e = 1$. Sometimes these families are simple curves. As a consequence of the previous theorem, we will prove that the family of odd periodic solutions with less oscillations is a simple curve. The experiments in [3] also show that some families emanating from $e = 0$ are not simple curves and present a pitchfork bifurcation. Therefore we should not expect uniqueness for the odd periodic solutions vanishing on the interval

$]0, m\pi[$. Next we present an existence result for this case. The variational equation at the center of mass $z = 0$ will play an important role; it is the equation of Hill's type

$$\ddot{\xi} + \frac{1}{r(t,e)^3}\xi = 0. \tag{4}$$

Theorem 2. *Assume that $m \geq 1$ and $N \geq 0$ are given integers. Then the following statements are equivalent:*

(i) *There exists a solution of (1) satisfying the conditions in (2) and having exactly N zeros in the interval $]0, m\pi[$*

(ii) *The solution $\xi(t)$ of (4) with initial conditions $\xi(0) = 0$, $\dot{\xi}(0) = 1$ has more than N zeros in $]0, m\pi[$.*

The crucial role played by the linear equation (4) was already recognized in [4], and the new observation is that this equation leads to necessary and sufficient conditions for the periodic problem with prescribed oscillation associated to (1). As observed in [7] the change of independent variable $\xi = \xi(u)$, where u is the eccentric anomaly, transforms the equation (4) in an equation of Ince type. This is a well-known class of equations, and we refer to the book [5] and to [7] for more information. The study of the oscillatory properties of (4) together with the above result should lead to some explicit conditions for existence. For instance, the inequality $\frac{1}{r(t,e)^3} > 1$ is valid for every t, and it implies that $\xi(t)$ has at least m zeros in $]0, m\pi[$. Hence there exists an odd periodic solution of period $2m\pi$ and having exactly N zeros for each $N = 0, 1, \ldots, m-1$.

The equation (1) is invariant under the transformation $(t, z) \mapsto (-t, -z)$, and so the conditions given by (2) are equivalent to the boundary conditions $z(0) = z(m\pi) = 0$ when we are dealing with solutions of this equation. For this reason the rest of the paper will be concerned with the Dirichlet problem

$$\ddot{z} + D(t,z)z = 0, \quad z(0) = 0, z(L) = 0, \tag{5}$$

where $D = D(t, z)$ is a general function satisfying certain properties enjoyed by $(z^2 + r(t,e)^2)^{-3/2}$ and $L > 0$ is an arbitrary parameter.

2. Minimal solutions with prescribed oscillation. From now on we assume that $D = D(t, z)$ is a function in the class $C^{0,1}([0, L] \times \mathbb{R})$. This means that the partial derivative $\partial_z D(t, z)$ exists everywhere and the functions $(t, z) \mapsto D(t, z)$ and $(t, z) \mapsto \partial_z D(t, z)$ are continuous on $[0, L] \times \mathbb{R}$. We also assume that

$$D(t,z) < D(t,0), \quad z \neq 0 \tag{6}$$

$$|D(t,z)| \leq \frac{C}{1+|z|}, \quad (t,z) \in [0,L] \times \mathbb{R}, \tag{7}$$

where $C > 0$ is a fixed constant.

The condition (7) implies that all the solutions of the equation

$$\ddot{z} + D(t,z)z = 0 \tag{8}$$

are defined on the interval $[0, L]$. Given $v \in \mathbb{R}$, the solution of (8) satisfying $z(0) = 0$, $\dot{z}(0) = v$ will be denoted by $z(t, v)$. In particular, $z(t, 0) = 0$ and the uniqueness of the Cauchy problem associated to (8) implies that the zeros of

$z(\cdot, v)$ are simple if $v \neq 0$. Therefore the number of zeros in the interval $]0, L[$ must be finite. It will be indicated by $\nu(v)$. We will obtain some properties of the integer-valued function $v \mapsto \nu(v)$.

Our first task will be to find an upper bound of ν. To this end we introduce the linear equation

$$\ddot{\xi} + D(t, 0)\xi = 0 \tag{9}$$

and denote by ν_0 the number of zeros in $]0, L[$ of the solution of (9) with initial conditions $\xi(0) = 0$, $\dot{\xi}(0) = 1$.

Property I: $\nu(v) \leq \nu_0$ for each $v \neq 0$.

The function $z(t, v)$ is a solution of the linear equation

$$\ddot{\eta} + D(t, z(t, v))\eta = 0.$$

From the assumption (6) we deduce that $D(t, z(t, v)) < D(t, 0)$ excepting at the zeros of $z(\cdot, v)$. Then we can compare this equation with (9) to deduce that $z(t, v)$ cannot have more zeros than $\xi(t)$ in the interval $]0, L[$.

In the next step we prove that the jumps of the integer-valued function ν are of one unit and they can only appear at the solutions of the Dirichlet problem.

Property II: Given $w \neq 0$, there exists $\delta > 0$ such that $\nu(w) \leq \nu(v) \leq \nu(w) + 1$ if $|v - w| \leq \delta$. Assuming in addition that $z(L, w) \neq 0$, the identity $\nu(w) = \nu(v)$ holds if $|v - w| \leq \delta$.

To prove this property we need a preliminary result on the preservation of the number of zeros in the passage to the limit with respect to the C^1 topology. Given a function $f : [0, L] \to \mathbb{R}$, the cardinality of the set $\{t \in]0, L[: \ f(t) = 0\}$ will be denoted by $n(f)$. In particular, $\nu(v) = n(z(\cdot, v))$.

Lemma 3. *Assume that $\{f_k\}$ is a sequence of functions in $C^1[0, L]$ satisfying $f_k(0) = 0$ for each k and*

$$f_k \to f, \quad \dot{f}_k \to \dot{f}$$

uniformly in $[0, L]$, where f is another function in $C^1[0, L]$ with the property

$$f(t)^2 + \dot{f}(t)^2 > 0$$

for each $t \in [0, L]$. Then, for large k,

$$n(f) \leq n(f_k) \leq n(f) + 1.$$

In addition, if $f(L) \neq 0$, then $n(f) = n(f_k)$ for large k.

The proof of this result is elementary and is left to the reader. We are ready to prove Property II. It is enough to consider a sequence $\{v_k\}$ converging to w and obtain the conclusions on the number of zeros for k large enough. We define $f_k(t) = z(t, v_k)$. Note that, by the continuous dependence with respect to the initial conditions, this sequence converges in $C^1[0, L]$ to the function with simple zeros $f(t) = z(t, w)$. Then Property II is a direct consequence of Lemma 3.

Next we compute the limit of $\nu(v)$ as $v \to 0$.

Property III: There exists $v_* > 0$ such that $\nu(v) = \nu_0$ if $|v| \le v_*$.

First notice that (9) is the variational equation of (8) around the trivial solution $z = 0$. Therefore, after differentiating with respect to initial conditions, we deduce that

$$\frac{1}{v} z(t, v) \to \xi(t), \quad \frac{1}{v} \dot{z}(t, v) \to \dot{\xi}(t)$$

as $v \to 0$, uniformly in $t \in [0, L]$. Moreover, the zeros of $\xi(t)$ are simple, and so the previous Lemma is applicable. We conclude that, for small $|v| > 0$,

$$n(\xi) = \nu_0 \le n(z(\cdot, v)) = \nu(v) \le n(\xi) + 1 = \nu_0 + 1.$$

The identity $\nu(v) = \nu_0$ now follows from Property I.

The last property is concerned with the limit of $\nu(v)$ as $|v| \to \infty$.

Property IV: There exists $v^* > 0$ such that $\nu(v) = 0$ if $|v| \ge v^*$.

The function $z(t, v)$ is a solution of the integral equation

$$z(t, v) = vt - \int_0^t (t - s) D(s, z(s, v)) z(s, v) ds.$$

The condition (7) leads to the estimate

$$|z(t, v) - vt| \le C \frac{L^2}{2},$$

valid for every $t \in [0, L]$ and $v \in \mathbb{R}$. Similarly, the identity

$$\dot{z}(t, v) = v - \int_0^t D(s, z(s, v)) z(s, v) ds$$

implies that

$$|\dot{z}(t, v) - v| \le CL.$$

Hence, if we let $|v| \to \infty$, the function $f_v(t) = \frac{1}{v} z(t, v)$ converges in $C^1[0, L]$ to $f(t) = t$. This last function satisfies the condition of simple zeros, and so Lemma 3 is applicable. For large $|v|$, $\nu(v) = n(f) = 0$.

After the study of the properties of ν, we are ready for the minimization procedure. Given an integer N with $0 \le N < \nu_0$, we consider the set

$$\mathcal{S}_N = \{v \in]0, \infty[: \ \nu(v) \le N\}.$$

The Property IV implies that $\mathcal{S}_N$ is non-empty, and we can define

$$w_N = \inf \mathcal{S}_N.$$

As a consequence of Property III, we know that 0 is not an accumulation point of $\mathcal{S}_N$, that is, $w_N > 0$. The definition of infimum implies that $\nu(v) > N$ if $v < w_N$ and there exists a sequence $v_k \to w_N$ with $v_k \ge w_N$ and $\nu(v_k) \le N$. From Property II we know that $\nu(v_k) \ge \nu(w_N)$ for k large enough, and so

$\nu(w_N) \leq N$. This implies that w_N is really a minimum. On the other hand, the inequality $N < \nu(v) \leq \nu(w_N) + 1$, valid for $0 < w_N - v < \delta$, implies that $\nu(w_N) = N$. In particular, we have proved that ν has a jump at $v = w_N$, and so Property II implies that $z(t, w_N)$ is a solution of the Dirichlet problem.

We sum up the above discussions in a result on the existence of solutions of the Dirichlet problem.

Proposition 4. *Assume that the function D satisfies the conditions* (6) *and* (7). *Then the Dirichlet problem* (5) *has a solution with $N \geq 0$ zeros in $]0, L[$ if and only if $\nu_0 > N$.*

We already know that $z(t, w_N)$ is the searched solution when the condition $\nu_0 > N$ is assumed. Let us prove that this condition is also necessary for the existence. Assume that $z(t)$ is a solution of (5) with N zeros in $]0, L[$. We apply Sturm theory and compare the equation (9) and

$$\ddot{\eta} + D(t, z(t))\eta = 0 \tag{10}$$

with corresponding solutions $\xi(t)$ and $z(t)$. The basic conclusion of the theory says that $\xi(t)$ must have at least one zero in each interval lying between two consecutive zeros of $z(t)$. This implies that the number of zeros of $\xi(t)$ in $]0, L[$ is greater than N. The proof of Proposition 4 is now complete. Theorem 2 follows as a consequence.

3. Uniqueness of positive solution. A solution of the Dirichlet problem (5) will be called positive if $z(t) > 0$ for each $t \in]0, L[$. Next we present a result on the uniqueness of this class of solutions.

Proposition 5. *Assume that D is a function of class $C^{0,1}$ satisfying*

$$\partial_z D(t, z) < 0 \quad \textit{for each} \quad t \in [0, L] \quad \textit{and} \quad z > 0. \tag{11}$$

Then (5) *has at most one positive solution*

Proof. Assume by contradiction that $z(t)$ and $z^*(t)$ are two different positive solutions. The function $\omega(t) = z(t) - z^*(t)$ is a non-trivial solution of the equation

$$\ddot{\omega} + \mathcal{D}(t)\omega = 0 \tag{12}$$

with

$$\mathcal{D}(t) = D(t, z(t)) + z^*(t) \int_0^1 \partial_z D(t, \lambda z(t) + (1 - \lambda) z^*(t)) d\lambda.$$

The positivity of $z(t)$ and $z^*(t)$ together with the condition (11) imply that $D(t, z(t)) > \mathcal{D}(t)$ for each $t \in]0, L[$. This allows us to use the comparison theorem for the linear equations (10) and (12). Since $\omega(t)$ vanishes at $t = 0$ and $t = L$, any non-trivial solution of (10) should have at least one zero in the interval $]0, L[$. This applies in particular to $z(t)$, leading to a contradiction with the positivity of this solution.

Given a solution $z(t)$ of the equation (8), we consider the corresponding variational equation

$$\ddot{\delta} + [D(t, z(t)) + z(t)\partial_z D(t, z(t))]\delta = 0. \tag{13}$$

A solution of the Dirichlet problem (5) is called non-degenerate if the trivial solution $\delta \equiv 0$ is the only solution of (13) with

$$\delta(0) = \delta(L) = 0. \tag{14}$$

The same argument as in the previous proof shows that the positive solution of (5) is non-degenerate when it exists.

The function $D(t, z)$ corresponding to the Sitnikov problem satisfies also the condition (11), and we can now complete the proof of Theorem 1. The uniqueness follows from Proposition 5, and the existence is a consequence of Proposition 4. Indeed, as noticed in the Introduction, the inequality $\nu_0 \geq m$ holds for the equation (4) and $L = m\pi$.

We finish the paper with some considerations on the dependence with respect to the eccentricity in Theorem 1. Let $Z(t, e)$ be the solution given by this Theorem, we prove that this function is analytic in $(t, e) \in \mathbb{R} \times [0, 1[$. Let $z(t, v, e)$ be the solution of (1) satisfying $z(0) = 0$, $\dot{z}(0) = v$. This function is analytic in the three variables, and if we define $\varphi(e) = \dot{Z}(0, e)$, then $Z(t, e) = z(t, \varphi(e), e)$. We prove that the function $\varphi(e)$ is analytic at every point of $[0, 1[$. To this end we fix $e_* \in [0, 1[$ and consider the implicit function problem

$$F(\psi(e), e) = 0, \quad \psi(e_*) = \varphi(e_*) \tag{15}$$

where $F(v, e) := z(m\pi, v, e)$. The solution $Z(t, e_*)$ is non-degenerate for the Dirichlet problem, and so

$$\partial_v F(\varphi(e_*), e_*) = \partial_v z(m\pi, \varphi(e_*), e_*) = \delta(m\pi) \neq 0,$$

where $\delta(t)$ is the solution of (13) satisfying $\delta(0) = 0$ and $\dot{\delta}(0) = 1$. Here we are assuming that $D(t, z) = (r(t, e_*)^2 + z^2)^{-3/2}$ and $z(t) = Z(t, e_*)$. In consequence the problem (15) has an analytic solution $\psi(e)$ defined on some interval $]e_* - \epsilon, e_* + \epsilon[$. The function $\chi(t, e) = z(t, \psi(e), e)$ is a solution of the Dirichlet problem on $[0, m\pi]$. Moreover, it coincides with $Z(t, e)$ at $e = e_*$. The function $Z(t, e_*)$ vanishes only at $t = 0$ and $t = L$ and the derivative is not zero at these instants. From here we deduce that $\chi(t, e)$ must be positive on $]0, L[$ if e is close enough to e_*. The uniqueness of positive solution implies that $Z(t, e) = \chi(t, e)$ when e is close to e_*. Hence $\varphi(e) = \psi(e)$ in a neighborhood of e_*, and so φ is real analytic.

The family given by Theorem 1 can be described in the plane (v, e) as an analytic and simple curve, namely

$$\mathcal{C}_{0,m} = \{(\varphi(e), e) : \ e \in [0, 1[\}.$$

It would be interesting to prove that some of the families $\mathcal{C}_{k,m}$ with $k \geq 1$ oscillations in $]0, m\pi[$ are curves with multiple points. □

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