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Representable spaces have the polynomial Daugavet property

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Abstract. In this note we prove that every Banach space that is representable in a compact Hausdorff topological space in the sense of (J Funct Anal 254:2294–2302, 2008) has the polynomial Daugavet property. As an application we provide new examples of Banach spaces enjoying the polynomial Daugavet property.

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1. Introduction. We consider the space $\mathcal{L}(X, Y)$ of bounded linear operators from a Banach space X to a Banach space Y, both of them over the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, endowed with the usual supremum (operator) norm. If $X = Y$, we simply write $\mathcal{L}(X)$. A Banach space X is said to have the *Daugavet property* if every continuous linear rank-one operator $T \in \mathcal{L}(X)$ satisfies

$$\|\mathrm{Id} + T\| = 1 + \|T\|,$$

which is known as the *Daugavet equation* (of course, Id denotes the identity operator on X). This equation was first studied by Daugavet [8] in 1963 for compact linear operators on the space $C[0, 1]$ of continuous functions on the interval $[0, 1]$. Since then the search for Banach spaces having the Daugavet property became a fashionable subject in modern Banach space theory (see, e.g., [1–3, 12, 17]). Classical examples of Banach spaces with this property are $C(K)$ and $L_1(\mu)$ for every perfect compact Hausdorff space K and every atomless σ-finite measure μ.

The Daugavet equation has been studied for nonlinear operators as well. One example is the Dauvaget equation for Lipschitz operators studied in [11].

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In this note we are interested in the Daugavet equation for polynomials, a study that began with the consideration, by Choi et al. [6] in 2007, of the Daugavet equation for bounded functions from the unit ball of a Banach space into the space. More precisely, let B_X denote the closed unit ball of the Banach space X and let $\ell_\infty(B_X, X)$ denote the Banach space of all bounded mappings $\Phi\colon B_X \longrightarrow X$ endowed with the supremum norm. According to [6], a function $\Phi \in \ell_\infty(B_X, X)$ satisfies the *Daugavet equation* if

$$\|\mathrm{Id} + \Phi\| = 1 + \|\Phi\|. \tag{DE}$$

This equation has been studied for polynomials on X by various authors, see, e.g., [6,7,13,15,16]. The following notion emerged from these studies: a Banach space X is said to have the *polynomial Daugavet property* if the restriction to B_X of every weakly compact polynomial $P\colon X \longrightarrow X$ satisfies (DE). For background on polynomials on Banach spaces, the reader is referred to [10,14]. The main examples of Banach spaces having the polynomial Daugavet property are the following: the space $C_b(\Omega, X)$ of bounded X-valued continuous functions on a perfect completely regular space Ω, the Lebesgue–Bochner spaces $L_\infty(\mu, X)$ and $L_1(\mu, X)$ of X-valued measurable functions when μ is an atomless σ-finite measure, and the spaces $C_w(K, X)$ of X-valued weakly continuous functions and $C_{w^*}(K, X^*)$ of X^*-valued weak* continuous functions when K is a perfect compact Hausdorff space (X^* denotes de topological dual of X).

New examples of Banach spaces with the Daugavet property were provided in 2008 by Becerra Guerrero and Rodríguez-Palacios [4]. They introduced the notion of a Banach space that is representable in a compact Hausdorff topological space (cf. Definition 2.1) and proved that all such Banach spaces have the Daugavet property. This allowed them to present new examples of Banach spaces satisfying the Daugavet property.

The purpose of this note is to extend this result of Becerra Guerrero and Rodríguez-Palacios by proving that every representable Banach space has the polynomial Daugavet property as well. This will allow us to present new examples of Banach spaces with the polynomial Daugavet property.

2. Main result and applications. We start this section by introducing the definition of a representable space according to [4]. By S_X we denote the unit sphere of the Banach space X. In the case of the scalar field, we use the symbol $\mathbb{T}$, that is, $\mathbb{T} = \{\omega \in \mathbb{K} : |\omega| = 1\}$.

Definition 2.1. Let K be a compact Hausdorff space. A Banach space X is said to be *K-representable* if there exists a family $(X_k)_{k\in K}$ of Banach spaces such that X is (linearly isometric to) a closed $C(K)$-submodule of the $C(K)$-module $\prod_{k\in K}^{\infty} X_k$ in such a way that, for every $x \in S_X$ and every $\varepsilon > 0$, the set $\{k \in K : \|x(k)\| > 1 - \varepsilon\}$ is infinite. X is said to be *representable* if it is K-representable for some compact Hausdorff space K.

We need three ingredients to prove that every representable space has the polynomial Daugavet property. The first one is the following characterization of this property due to Choi et al. [7].

Proposition 2.2. ([7, Proposition 6.3]) *Let X be a Banach space, and suppose that for all $x, z \in S_X$, $\omega \in \mathbb{T}$, and $\varepsilon > 0$, there exists a sequence $(z_n)_{n=1}^{\infty}$ in X such that*

(i) *the series $\sum_{n=1}^{\infty} z_n$ is weakly unconditionally Cauchy;*

(ii) $\limsup_n \|z + z_n\| \leq 1$*;*

(iii) $\|x + \omega(z + z_n)\| > 2 - \varepsilon$ *for every* $n \in \mathbb{N}$*.*

Then X has the polynomial Daugavet property.

The second ingredient is the following topological standard:

Lemma 2.3. *Let K be a compact Hausdorff space, and let S be an infinite subset of K. Then there exists a sequence $(k_n)_{n\in\mathbb{N}}$ in S together with a sequence $(V_n)_{n\in\mathbb{N}}$ of pairwise disjoint nonempty open subsets of K such that k_n belongs to V_n for every $n \in \mathbb{N}$.*

The third ingredient is the following characterization of weakly unconditional Cauchy series:

Theorem 2.4. ([9, Theorem V.6]) *The following statements regarding a formal series $\sum_{n=1}^{\infty} x_n$ in a Banach space are equivalent:*

(i) $\sum_{n=1}^{\infty} x_n$ *is weakly unconditionally Cauchy.*

(ii) *There is a constant $C > 0$ such that for any sequence $(t_n)_{n=1}^{\infty} \in \ell_\infty$,*

$$\sup_n \left\| \sum_{k=1}^{n} t_k x_k \right\| \leq C \sup_n |t_n|.$$

Now we use the argument of [4, Lemma 2.4] and the three previous ingredients to prove the main result of this note.

Theorem 2.5. *Every representable Banach space has the polynomial Daugavet property.*

Proof. Let K be a compact Hausdorff space such that the Banach space X is K-representable. Let $(X_k)_{k\in K}$ be a family of Banach spaces as in Definition 2.1. Fix $x, z \in S_X$, $\omega \in \mathbb{T}$, and $0 < \varepsilon < 2$. Regarding X as a closed $C(K)$-submodule of the $C(K)$-module $\prod_{k\in K}^{\infty} X_k$, we can write $x = (x(k))_{k\in K}$ and $z = (z(k))_{k\in K}$ with $x(k), z(k) \in X_k$. By definition, the set

$$V = \left\{ k \in K : \|x(k)\| > 1 - \frac{\varepsilon}{2} \right\}$$

is infinite, hence by Lemma 2.3 there exist a sequence $(k_n)_{n\in\mathbb{N}}$ in V and a sequence $(V_n)_{n\in\mathbb{N}}$ of pairwise disjoint nonempty open subsets of K, such that k_n belongs to V_n for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, apply Urysohn's Lemma to find a continuous function $f_n : K \longrightarrow [0, 1]$ such that $f_n(k_n) = 1$ and $f_n(k) = 0$ for every $k \in (K \setminus V_k)$. Defining

$$z_n = f_n \cdot (\omega^{-1} x - z),$$

we have $z_n \in X$ because X is a $C(K)$-submodule of $\prod_{k\in K}^{\infty} X_k$. Since the supports of the functions $(f_n)_{n\in\mathbb{N}}$ are pairwise disjoint, by Theorem 2.4 we conclude that the series $\sum_{n=1}^{\infty} z_n$ is weakly unconditionally Cauchy. Moreover, for every $n \in \mathbb{N}$,

$$\begin{aligned}\|[z+z_n](k)\| &= \left\|(1-f_n(k))\,z(k)+f_n(k)\omega^{-1}x(k)\right\| \\ &\leq (1-f_n(k))\|z(k)\|+f_n(k)\|x(k)\| \leq 1,\end{aligned}$$

for every $k \in K$, so $\|z+z_n\| \leq 1$. Also, for every $n \in \mathbb{N}$,

$$\begin{aligned}\|x+\omega(z+z_n)\| &\geq \|x(k_n)+\omega\left(z(k_n)+z_n(k_n)\right)\| = \|2x(k_n)\| > 2\left(1-\frac{\varepsilon}{2}\right) \\ &= 2-\varepsilon.\end{aligned}$$

The result follows from Proposition 2.2. □

Next we combine Theorem 2.5 with some results of [4] to provide several new examples of Banach spaces with the polynomial Daugavet property. These examples are listed in the subsequent corollaries.

Corollary 2.6. *Let X be a Banach space, and let Y be a representable Banach space.*

(a) *The complete injective tensor product $X\widehat{\otimes}_\varepsilon Y$ has the polynomial Daugavet property.*
(b) *If M is a closed subspace of $\mathcal{L}(X,Y)$ such that $\mathcal{L}(Y)\circ M \subset M$, then M has the polynomial Daugavet property.*

Proof. By [4, Lemma 2.5 and Corollary 2.6], we know that $X\widehat{\otimes}_\varepsilon Y$ and M are representable Banach spaces. The result follows from Theorem 2.5. □

Now let Y be a nonzero Banach space, and let Z be a subspace of Y^* that is norming for Y, that is, Z is norm-closed in Y^* and

$$\|y\| = \sup\{|\varphi(y)| : \varphi \in B_Z\}$$

for every $y \in Y$. By $\sigma(Y,Z)$ we denote the weak topology on Y relative to its duality with Z. The symbol $\|\cdot\|_Y$ stands for the norm topology on Y.

Corollary 2.7. *Let K be a perfect compact Hausdorff space, Y be a nonzero Banach space, Z be a norming subspace of Y^* for Y, and τ be a vector space topology on Y such that $\sigma(Y,Z) \leq \tau \leq \|\cdot\|_Y$. Then the Banach space $C(K,(Y,\tau))$ of continuous functions $f\colon K \longrightarrow (Y,\tau)$, with the sup norm, has the polynomial Daugavet property.*

Proof. By [4, Theorem 3.1], we know that $C(K,(Y,\tau))$ is a representable Banach space. The result follows from Theorem 2.5. □

For the sake of the reader, we recall the following definition of [5].

Definition 2.8. Let X be a Banach space. A closed subspace J of X is said to be an *L-summand* (*M-summand*, respectively) if there is a closed subspace $J^\perp$ of X such that X is the algebraic direct sum of J and $J^\perp$ and

$$\|x+x^\perp\| = \|x\|+\|x^\perp\| \ (\|x+x^\perp\| = \max\{\|x\|,\|x^\perp\|\}, \text{ respectively})$$

for all $x \in J$ and $x^\perp \in J^\perp$. An L-summand or an M-summand is said to be *minimal* if it is minimal in the sense of the inclusion.

According to [4, Theorem 4.3], every dual Banach space without minimal M-summands is a representable Banach space. Applying Theorem 2.5 once again, we get the

Corollary 2.9. *Every dual Banach space without minimal M-summands has the polynomial Daugavet property.*

And combining [4, Theorem 4.3] with Corollary 2.6(b), we get the

Corollary 2.10. *Let X be a Banach space, Y be a dual Banach space without minimal M-summands, and M be a closed subspace of $\mathcal{L}(X, Y)$ such that $\mathcal{L}(Y) \circ M \subset M$. Then M has the polynomial Daugavet property.*

A final result follows from Corollary 2.10.

Proposition 2.11. *Let X be a Banach space without minimal L-summands, and let Y be a dual Banach space. Then the space $\mathcal{L}(X, Y)$ has the polynomial Daugavet property.*

Proof. Since X is a Banach space without minimal L-summands, a glance at the proof of [4, Corollary 4.5] reveals that X^* has no minimal M-summands. Letting Y_* be a predual of Y, from Corollary 2.10 we know that $\mathcal{L}(Y_*, X^*)$ has the polynomial Daugavet property. The result follows because $\mathcal{L}(X, Y)$ is canonically linearly isometric to $\mathcal{L}(Y_*, X^*)$. □

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