Arch. Math. 106 (2016), 591–600

0003-889X/16/060591-10
published online April 6, 2016
DOI 10.1007/s00013-016-0899-x

Archiv der Mathematik

Effect of drift of the generalized Brownian motion process: an example for the analytic Feynman integral

Seung Jun Chang and Jae Gil Choi

Abstract. In the theory of the analytic Feynman integral, the integrand is a functional of the standard Brownian motion process. In this note, we present an example of a bounded functional which is not Feynman integrable. The bounded functionals discussed in this note are defined in sample paths of the generalized Brownian motion process.

Mathematics Subject Classification. 28C20, 60J65, 46G12.

Keywords. Generalized Brownian motion process, Generalized analytic Feynman integral, Fresnel-type class.

1. Introduction. The purpose of this note is to illustrate an effect of drift of the generalized Brownian motion process (GBMP). To do this, we discuss the theory of analytic Feynman integrals. Frankly speaking, in order to emphasize an effect of drift of GBMPs, we present an example of a bounded functional which is not analytic Feynman integrable on the function space $C_{a,b}[0,T]$. The function space $C_{a,b}[0,T]$ is a probability space induced by a GBMP.

Let $W \equiv C_0[0,T]$ denote one-parameter Wiener space; this is the space of all real-valued continuous functions x on $[0,T]$ with $x(0)=0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$ and let $\mathfrak{m}$ be the Wiener measure. Then, as is well known, $(C_0[0,T],\mathcal{M},\mathfrak{m})$ is a complete measure space. The coordinate process $\mathcal{W}$ on $C_0[0,T]\times[0,T]$ defined by $(x,t)\stackrel{\mathcal{W}}{\longrightarrow}\mathcal{W}_t(x)=x(t)$ is a standard Brownian motion process (SBMP).

Next, let $H \equiv C_0'[0,T]$ be the class of absolutely continuous functions x from $[0,T]$ to $\mathbb{R}$ for which $x(0)=0$ and with $Dx \equiv dx/dt \in L^2[0,T]$, and let $\mathcal{D}$ be the non-existent Lebesgue measure on H. In the heuristic setting of [26], the Feynman path integral of a functional F on H is

$$\int_H^{\mathrm{f}_q} F(x)d\mathfrak{m}(x) = \int_H F(x)\frac{1}{Z_q}\exp\left\{\frac{iq}{2}\|x\|_H^2\right\}\mathcal{D}(x),$$

where $\mathcal{D}$ is the heuristic version of Lebesgue measure, $q \in \mathbb{R}\backslash\{0\}$, and Z_q is taken to be a normalization constant for which $\frac{1}{Z_q}\exp\{\frac{iq}{2}\|x\|_H^2\}\mathcal{D}(x)$ is a probability measure on H. As is widely known, there is no true measure $\mathcal{D}$ and Z_q is, in fact, infinite. To head towards a rigorous definition, let $(H, W, \mathfrak{m})$ be the abstract Wiener space with $H \hookrightarrow W$. For each $\lambda > 0$, let us use the usual informal expression for Wiener measure with variance λ^{-1} given by

$$d\mathfrak{m}_\lambda(x) = \frac{1}{Z_\lambda}\exp\left\{-\frac{\lambda}{2}\|x\|_H^2\right\}\mathcal{D}(x).$$

Then a heuristic calculation shows that

$$\begin{aligned}\int_H F(x)d\mathfrak{m}_\lambda(x) &= \int_H F(x)\frac{1}{Z_\lambda}\exp\left\{-\frac{\lambda}{2}\|x\|_H^2\right\}\mathcal{D}(x)\\ &= \int_H F(x)\frac{1}{Z_\lambda}\exp\left\{-\frac{1}{2}\left\|\sqrt{\lambda}x\right\|_H^2\right\}\mathcal{D}(x)\\ &= \int_H F(\lambda^{-1/2}x)\frac{1}{Z_1}\exp\left\{-\frac{1}{2}\|x\|_H^2\right\}\mathcal{D}(x)\\ &= \int_H F(\lambda^{-1/2}x)d\mathfrak{m}(x).\end{aligned}$$

Thus, we should expect that the Feynman path integral of F on W is given by

$$\int_W^{\mathrm{f}_q} F(x)d\mathfrak{m}(x) = \lim_{\lambda\to -iq}\int_W F(\lambda^{-1/2}x)d\mathfrak{m}(x),$$

where one must first assume that $\lambda \to \int_W F(\lambda^{-1/2}x)d\mathfrak{m}(x)$ has an '*analytic continuation*' in the right-half complex plane and that the above limit exists appropriately. This description illustrates the '*analytic Feynman integral*' on the classical Wiener space $C_0[0, T]$.

The concept of the '*analytic Feynman integral*' on the Wiener space $C_0[0, T]$ was introduced by Cameron in [1]. In [3], Cameron and Storvick introduced a Banach algebra $\mathcal{S}$ of the analytic Feynman integrable functionals. The functionals in $\mathcal{S}$ are defined as a stochastic Fourier transform of complex measures on $L^2[0, T]$, and are bounded on $C_0[0, T]$. Other classes of the analytic Feynman integrable functionals on $C_0[0, T]$ can be found in [2,19,22–24,27–29]. But the '*analytic Feynman integral'* cannot be interpreted as the integration in standard measure theory.

On the other hand, in [5–7,9,14,15], the authors defined the generalized analytic Feynman integral and the generalized analytic Fourier–Feynman transform on the function space $C_{a,b}[0, T]$, and studied their properties and related topics. The function space $C_{a,b}[0, T]$, induced by a GBMP, was introduced by Yeh in [30], and was used extensively in [4,8,10–13,20]. There have also been several attempts to construct financial mathematical theories using this process, see [18,21,25].

A GBMP on a probability space (Ω, Σ, P) and a time interval $[0, T]$ is a Gaussian process $Y \equiv \{Y_t\}_{t\in[0,T]}$ such that $Y_0 = c$ almost surely for some constant $c \in \mathbb{R}$, and for any set $\{t_0, t_1, \ldots, t_n\} \subset [0, T]$ with $0 = t_0 < t_1 < \cdots < t_n \leq T$ and any Borel set $B \subset \mathbb{R}^n$, the measure $P(I_{t_1,\ldots,t_n,B})$ of the cylinder set $I_{t_1,\ldots,t_n,B}$ of the form $I_{t_1,\ldots,t_n,B} = \big\{\omega \in \Omega : (Y_{t_1}(\omega), \ldots, Y_{t_n}(\omega)) \in B\big\}$ is given by

$$\left((2\pi)^n \prod_{j=1}^{n} \big(b(t_j) - b(t_{j-1})\big)\right)^{-1/2} \times \int_B \exp\left\{-\frac{1}{2}\sum_{j=1}^{n} \frac{((\eta_j - a(t_j)) - (\eta_{j-1} - a(t_{j-1})))^2}{b(t_j) - b(t_{j-1})}\right\} d\eta_1 \cdots d\eta_n$$

where $\eta_0 = c$, $a(t)$ is a continuous real-valued function on $[0, T]$, and $b(t)$ is an increasing continuous real-valued function on $[0, T]$. Thus, the GBMP Y is determined by the continuous functions $a(\cdot)$ and $b(\cdot)$. For more details, see [30,31]. Note that when $c = 0$, $a(t) \equiv 0$, and $b(t) = t$ on $[0, T]$, the GBMP reduces to a SBMP.

In this note, we set $c = a(0) = b(0) = 0$. Then the function space $C_{a,b}[0, T]$ induced by the GBMP Y determined by the $a(\cdot)$ and $b(\cdot)$ can be considered as the space of continuous sample paths of Y, see [4–16,20], and one can see that for each $t \in [0, T]$,

$$e_t(x) \sim N\big(a(t), b(t)\big),$$

where $e_t : C_{a,b}[0, T] \times [0, T] \to \mathbb{R}$ is the coordinate evaluation map given by $e_t(x) = x(t)$ and $N(m, \sigma^2)$ denotes the normal distribution with mean m and variance σ^2. We are obliged to point out that a SBMP is stationary in time and is free of drift, whereas a GBMP is generally not stationary in time and is subject to a drift $a(t)$.

As mentioned above, the SBMP used in [1–3,19,22–24,27–29] is stationary in time and is free of drift. However, the stochastic process used in this paper, as well as in [4–16,18,20,21,25,30], is non-stationary in time and is subject to a drift because

$$\mathbb{E}[Y_s(w)Y_t(w)] = \min\{b(s), b(t)\} + a(s)a(t),$$

see [31, Theorem 17.1]. It turns out, as noted in [5, Remark 3.1] and [14, Remark 4.2], that the inclusion of a drift term $a(t)$ makes establishing the existence of the generalized analytic Feynman integral of functionals on $C_{a,b}[0, T]$ very difficult.

In this note, we present an example of a bounded functional which is not generalized analytic Feynman integrable on the function space $C_{a,b}[0, T]$.

2. The function space $C_{a,b}[0,T]$. In this section, we first present a brief background and some well-known results about the function space $C_{a,b}[0,T]$ induced by GBMP.

Let $a(t)$ be an absolutely continuous real-valued function on $[0,T]$ with $a(0)=0$ and $a'(t)\in L^2[0,T]$, and let $b(t)$ be a strictly increasing, continuously differentiable real-valued function with $b(0)=0$ and $b'(t)>0$ for each $t\in[0,T]$. The generalized Brownian motion process Y determined by $a(t)$ and $b(t)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t)=\min\{b(s),b(t)\}$. For more details, see [5,10,14,30,31]. By [31, Theorem 14.2], the probability measure μ induced by Y, taking a separable version, is supported by $C_{a,b}[0,T]$ (which is equivalent to the Banach space of continuous functions x on $[0,T]$ with $x(0)=0$ under the sup norm). Hence, $(C_{a,b}[0,T],\mathcal{B}(C_{a,b}[0,T]),\mu)$ is the function space induced by Y where $\mathcal{B}(C_{a,b}[0,T])$ is the Borel σ-field of $C_{a,b}[0,T]$. We then complete this function space to obtain the measure space $(C_{a,b}[0,T],\mathcal{W}(C_{a,b}[0,T]),\mu)$ where $\mathcal{W}(C_{a,b}[0,T])$ is the set of all Wiener measurable subsets of $C_{a,b}[0,T]$.

Remark 2.1. The function space $C_{a,b}[0,T]$ reduces to the Wiener space $C_0[0,T]$, considered in papers [1–3,19,22–24,27–29] if and only if $a(t)\equiv 0$ and $b(t)=t$ for all $t\in[0,T]$.

A subset B of $C_{a,b}[0,T]$ is said to be scale-invariant measurable provided ρB is $\mathcal{W}(C_{a,b}[0,T])$-measurable for all $\rho>0$, and a scale-invariant measurable set N is said to be a scale-invariant null set provided $\mu(\rho N)=0$ for all $\rho>0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere(s-a.e.). A functional F is said to be scale-invariant measurable provided F is defined on a scale-invariant measurable set and $F(\rho\,\cdot)$ is $\mathcal{W}(C_{a,b}[0,T])$-measurable for every $\rho>0$. If two functionals F and G defined on $C_{a,b}[0,T]$ are equal s-a.e., we write $F\approx G$. Note that the relation "$\approx$" is an equivalence relation.

Let $L^2_{a,b}[0,T]$ (see [5,14]) be the space of functions on $[0,T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue–Stieltjes measures on $[0,T]$ induced by $a(\cdot)$ and $b(\cdot)$, i.e.

$$L^2_{a,b}[0,T]=\left\{v:\int_0^T v^2(s)db(s)<+\infty \text{ and } \int_0^T v^2(s)d|a|(s)<+\infty\right\}$$

where $|a|(\cdot)$ denotes the total variation function of $a(\cdot)$. Then $L^2_{a,b}[0,T]$ is a separable Hilbert space with inner product defined by

$$(u,v)_{a,b}=\int_0^T u(t)v(t)dm_{|a|,b}(t)\equiv\int_0^T u(t)v(t)d[b(t)+|a|(t)],$$

where $m_{|a|,b}$ denotes the Lebesgue–Stieltjes measure induced by $|a|(\cdot)$ and $b(\cdot)$. In particular, note that $\|u\|_{a,b}\equiv\sqrt{(u,u)_{a,b}}=0$ if and only if $u(t)=0$ a.e. on $[0,T]$. For more details, see [5,14].

Let

$$C'_{a,b}[0,T] = \left\{ w \in C_{a,b}[0,T] : w(t) = \int_0^t z(s)db(s) \text{ for some } z \in L^2_{a,b}[0,T] \right\}.$$

For $w \in C'_{a,b}[0,T]$, with $w(t) = \int_0^t z(s)db(s)$ for $t \in [0,T]$, let $D : C'_{a,b}[0,T] \to L^2_{a,b}[0,T]$ be defined by the formula

$$Dw(t) = z(t) = \frac{w'(t)}{b'(t)}. \tag{2.1}$$

Then $C'_{a,b} \equiv C'_{a,b}[0,T]$ with inner product

$$(w_1, w_2)_{C'_{a,b}} = \int_0^T Dw_1(t)Dw_2(t)db(t)$$

is a separable Hilbert space.

One can see that $L^2_{a,b}[0,T]$ and $C'_{a,b}[0,T]$ are (topologically) homeomorphic under the operator D given by Eq. (2.1). The inverse operator of D is given by

$$(D^{-1}z)(t) = \int_0^t z(s)db(s), \quad t \in [0,T].$$

In this note, in addition to the conditions put on $a(t)$ above, we now add the condition

$$\int_0^T |a'(t)|^2 d|a|(t) < +\infty. \tag{2.2}$$

Then the function $a : [0,T] \to \mathbb{R}$ satisfies the condition (2.2) if and only if $a(\cdot)$ is an element of $C'_{a,b}[0,T]$. Under the condition (2.2), we observe that for each $w \in C'_{a,b}[0,T]$ with $Dw = z$,

$$(w, a)_{C'_{a,b}} = \int_0^T Dw(t)Da(t)db(t) = \int_0^T z(t)da(t).$$

Let $\{e_n\}_{n=1}^{\infty}$ be a complete orthonormal set in $(C'_{a,b}[0,T], \|\cdot\|_{C'_{a,b}})$ such that the De_n's are of bounded variation on $[0,T]$. For $w \in C'_{a,b}[0,T]$ and $x \in C_{a,b}[0,T]$, we define the Paley–Wiener–Zygmund (PWZ) stochastic integral $(w,x)^{\sim}$ as follows:

$$(w,x)^{\sim} = \lim_{n\to\infty} \int_0^T \sum_{j=1}^n (w, e_j)_{C'_{a,b}} De_j(t)dx(t)$$

if the limit exists. The limit in defining the PWZ stochastic integral $(w,x)^{\sim}$ is essentially independent of the choice of the complete orthonormal set. We note that for each $w \in C'_{a,b}[0,T]$, the PWZ stochastic integral $(w,x)^{\sim}$ exists for s-a.e. $x \in C_{a,b}[0,T]$. If $Dw = z \in L^2_{a,b}[0,T]$ is of bounded variation on $[0,T]$, then the PWZ stochastic integral $(w,x)^\sim$ equals the Riemann–Stieltjes integral $\int_0^T z(t)dx(t)$. Furthermore, for each $w \in C'_{a,b}[0,T]$, $(w,x)^\sim$ is a Gaussian random variable with mean $(w,a)_{C'_{a,b}}$ and variance $\|w\|^2_{C'_{a,b}}$. Also we note that for $w, x \in C'_{a,b}[0,T]$, $(w,x)^\sim = (w,x)_{C'_{a,b}}$.

Next (see [5,14]), we state the definition of the generalized analytic Feynman integral on the function space $C_{a,b}[0,T]$.

Definition 2.2. Let $F : C_{a,b}[0,T] \to \mathbb{C}$ be a scale-invariant measurable functional such that for each $\lambda > 0$, the function space integral

$$J(\lambda) = \int_{C_{a,b}[0,T]} F(\lambda^{-1/2}x)d\mu(x)$$

exists and is finite. If there exists a function $J^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic function space integral of F over $C_{a,b}[0,T]$ with parameter λ, and for $\lambda \in \mathbb{C}_+$ we write

$$\int_{C_{a,b}[0,T]}^{\mathrm{an}_\lambda} F(x)d\mu(x) = J^*(\lambda).$$

Let q be a nonzero real number and let F be a functional such that the analytic function space integral $\int_{C_{a,b}[0,T]}^{\mathrm{an}_\lambda} F(x)d\mu(x)$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the generalized analytic Feynman integral of F with parameter q and we write

$$\int_{C_{a,b}[0,T]}^{\mathrm{anf}_q} F(x)d\mu(x) = \lim_{\substack{\lambda \to -iq \\ \lambda \in \mathbb{C}_+}} \int_{C_{a,b}[0,T]}^{\mathrm{an}_\lambda} F(x)d\mu(x).$$

3. On the Fresnel-type class $\mathcal{F}(C_{a,b}[0,T])$. Let $\mathcal{M}(C'_{a,b}[0,T])$ be the space of complex-valued, countably additive (and hence finite) Borel measures on $C'_{a,b}[0,T]$. $\mathcal{M}(C'_{a,b}[0,T])$ is a Banach algebra under the total variation norm and with convolution as multiplication.

We define the Fresnel-type class $\mathcal{F}(C_{a,b}[0,T])$ of functionals on $C_{a,b}[0,T]$ as the space of all stochastic Fourier transforms of elements of $\mathcal{M}(C'_{a,b}[0,T])$; that is, $F \in \mathcal{F}(C_{a,b}[0,T])$ if and only if there exists a measure f in $\mathcal{M}(C'_{a,b}[0,T])$ such that

$$F(x) = \int_{C'_{a,b}[0,T]} \exp\{i(w,x)^\sim\}df(w) \tag{3.1}$$

for s-a.e. $x \in C_{a,b}[0,T]$. More precisely, since we shall identify functionals which coincide s-a.e. on $C_{a,b}[0,T]$, $\mathcal{F}(C_{a,b}[0,T])$ can be regarded as the space of all s-equivalence classes of functionals of the form (3.1).

The Fresnel-type class $\mathcal{F}(C_{a,b}[0,T])$ is a Banach algebra with norm

$$\|F\| = \|f\| = \int_{C'_{a,b}[0,T]} d|f|(w).$$

In fact, the correspondence $f \mapsto F$ is injective, carries convolution into pointwise multiplication, and is a Banach algebra isomorphism where f and F are related by (3.1). For more details, see [4,6].

For each real $q \in \mathbb{R}\backslash\{0\}$, $(-iq)^{1/2}$ denotes the principal square root of $-iq$; that is, $(-iq)^{1/2}$ is always chosen to have positive real part, so that $\mathrm{Re}(-iq)^{-1/2} = \mathrm{Re}(1/(-iq)^{1/2}) > 0$.

Theorem 3.1 below is a simple modification of the results [14, Eq. (4.3)] and [17, Eqs. (40) and (49)]. The condition (3.2) below will guarantee the existence of the right hand side of Eq. (3.3) below.

Theorem 3.1. *Let q_0 be a positive real number and let $F \in \mathcal{F}(C_{a,b}[0,T])$ be given by Eq. (3.1) whose associated measure f satisfies the condition*

$$\int_{C'_{a,b}[0,T]} \exp\left\{(2q_0)^{-1/2}\|w\|_{C'_{a,b}}\|a\|_{C'_{a,b}}\right\}d|f|(w) < +\infty. \tag{3.2}$$

Then, for all real q with $|q| > q_0$, the generalized analytic Feynman integral of F exists and is given by the formula

$$\int_{C_{a,b}[0,T]}^{\mathrm{anf}_q} F(x)d\mu(x) = \int_{C'_{a,b}[0,T]} \exp\left\{-\frac{i}{2q}\|w\|^2_{C'_{a,b}} + i(-iq)^{-1/2}(w,a)_{C'_{a,b}}\right\}df(w). \tag{3.3}$$

It is important to note that any functional $F \in \mathcal{F}(C_{a,b}[0,T])$ is bounded on $C_{a,b}[0,T]$ since

$$|F(x)| = \left|\int_{C'_{a,b}[0,T]} \exp\{i(w,x)^\sim\}df(w)\right| \leq \int_{C'_{a,b}[0,T]} \left|\exp\{i(w,x)^\sim\}\right|d|f|(w)$$
$$\leq \int_{C'_{a,b}[0,T]} d|f|(w) = \|f\| < +\infty.$$

However, there is a functional F in $\mathcal{F}(C_{a,b}[0,T])$ which is not generalized analytic Feynman integrable. For each $n \in \mathbb{N}$, let

$$w_n(t) = nb(t) = \int_0^t n db(s)$$

for $t \in [0,T]$. Consider a measure l which is concentrated on the set $\{w_n : n \in \mathbb{N}\}$ and $l(\{w_n\}) = 1/n^2$ for each $n \in \mathbb{N}$. Then l is an element of $\mathcal{M}(C'_{a,b}[0,T])$. Consider the functional $F \in \mathcal{F}(C_{a,b}[0,T])$ given by

$$F(x) = \int_{C'_{a,b}[0,T]} \exp\{(w,x)^\sim\}dl(w)$$

for s-a.e. $x \in C_{a,b}[0,T]$. In this case, by Eq. (3.3) above, we have that for a positive real number $q > 0$,

$$\int_{C_{a,b}[0,T]}^{\text{anf}_{-q}} F(x)d\mu(x) = \int_{C'_{a,b}[0,T]} \exp\left\{\frac{i}{2q}\|w\|^2_{C'_{a,b}} + i(iq)^{-1/2}(w,a)_{C'_{a,b}}\right\} dl(w)$$
$$= \sum_{n=1}^{\infty} \exp\left\{\frac{i}{2q}\|w_n\|^2_{C'_{a,b}} + \left(\frac{1}{\sqrt{2q}} + \frac{i}{\sqrt{2q}}\right)(w_n,a)_{C'_{a,b}}\right\}\frac{1}{n^2}$$
$$= \sum_{n=1}^{\infty} \exp\left\{\frac{i}{2q}n^2 b(T) + \left(\frac{1}{\sqrt{2q}} + \frac{i}{\sqrt{2q}}\right)na(T)\right\}\frac{1}{n^2}. \quad (3.4)$$

Then, we have

$$L \equiv \lim_{n\to\infty} \frac{\left|\exp\left\{\frac{i}{2q}(n+1)^2 b(T) + \left(\frac{1}{\sqrt{2q}} + \frac{i}{\sqrt{2q}}\right)(n+1)a(T)\right\}\frac{1}{(n+1)^2}\right|}{\left|\exp\left\{\frac{i}{2q}n^2 b(T) + \left(\frac{1}{\sqrt{2q}} + \frac{i}{\sqrt{2q}}\right)na(T)\right\}\frac{1}{n^2}\right|}$$
$$= \lim_{n\to\infty} \frac{\exp\left\{\frac{1}{\sqrt{2q}}(n+1)a(T)\right\}}{\exp\left\{\frac{1}{\sqrt{2q}}na(T)\right\}} = \exp\left\{\frac{1}{\sqrt{2q}}a(T)\right\}.$$

If $a(T) > 0$, then $L > 1$ and so, by the d'Alembert ratio test, we see that the series in the last expression of (3.4) diverges.

From this example, we see that the drift term $a(t)$ of the GBMP plays a prominent role in the existence of the generalized analytic Feynman integral for the functionals of the GBMP. Also, it tells us that the concept of the analytic Feynman integral is distinguished from the concept of the integration in standard measure theory.

Acknowledgement. The authors would like to express their gratitude to the editor and the referees for their valuable comments and suggestions which have improved the original paper. This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT and Future Planning (2015R1C1A1A0 1051497) and the Ministry of Education (2015R1D1A1A01058224). Jae Gil Choi worked as a corresponding author.

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Seung Jun Chang and Jae Gil Choi
Department of Mathematics
Dankook University
Cheonan 330-714
Korea
e-mail: jgchoi@dankook.ac.kr

Seung Jun Chang
e-mail: sejchang@dankook.ac.kr

Received: 1 December 2015

Revised: 28 February 2016