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# On Hamiltonian minimal submanifolds in the space of oriented geodesics in real space forms

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**Abstract.** We prove that a deformation of a hypersurface in an $(n+1)$-dimensional real space form $\mathbb{S}^{n+1}_{p,1}$ induces a Hamiltonian variation of the normal congruence in the space $\mathbb{L}(\mathbb{S}^{n+1}_{p,1})$ of oriented geodesics. As an application, we show that every Hamiltonian minimal submanifold in $\mathbb{L}(\mathbb{S}^{n+1})$ (resp. $\mathbb{L}(\mathbb{H}^{n+1})$) with respect to the (para-)Kähler Einstein structure is locally the normal congruence of a hypersurface $\Sigma$ in $\mathbb{S}^{n+1}$ (resp. $\mathbb{H}^{n+1}$) that is a critical point of the functional $\mathcal{W}(\Sigma) = \int_\Sigma \left(\Pi_{i=1}^n |\epsilon + k_i^2|\right)^{1/2}$, where $k_i$ denote the principal curvatures of $\Sigma$ and $\epsilon \in \{-1, 1\}$. In addition, for $n = 2$, we prove that every Hamiltonian minimal surface in $\mathbb{L}(\mathbb{S}^3)$ (resp. $\mathbb{L}(\mathbb{H}^3)$), with respect to the (para-)Kähler conformally flat structure, is the normal congruence of a surface in $\mathbb{S}^3$ (resp. $\mathbb{H}^3$) that is a critical point of the functional $\mathcal{W}'(\Sigma) = \int_\Sigma \sqrt{H^2 - K + 1}$ (resp. $\mathcal{W}'(\Sigma) = \int_\Sigma \sqrt{H^2 - K - 1}$ ), where $H$ and $K$ denote, respectively, the mean and Gaussian curvature of $\Sigma$.

**Mathematics Subject Classification.** 53D12, 53A35.

**Keywords.** Kähler structures, Hamiltonian minimal submanifolds, Space of oriented geodesics, Real space forms.

**1. Introduction.** The space $\mathbb{L}(M)$ of oriented geodesics of a pseudo-Riemannian manifold $(M, g)$ has been of great interest for the last three decades and has been studied by different authors (see for example [1,2,5–8,14,15]). When $(M, g)$ is a Riemannian symmetric space of rank one, Alekseevsky et al. have described in [1] all possible metrics defined on $\mathbb{L}(M)$ that are invariant under the isometry group of $g$.

In the case where $(M, g)$ is a real $(n+1)$-dimensional space form $\mathbb{S}^{n+1}_{p,1}$ of signature $(p, n+1-p)$ with constant sectional curvature one, Anciaux has shown in [2] that $\mathbb{L}(\mathbb{S}^{n+1}_{p,1})$ admits a Kähler or a para-Kähler structure $(G, \mathbb{J}, \Omega)$, where $\mathbb{J}$ is the complex or paracomplex structure and $\Omega$ is the symplectic structure such that the metric $G$ is Einstein and is invariant under the isometry group of $g$. In the same work, for $n = 2$, Anciaux has proved that $\mathbb{L}(\mathbb{M})$ admits an extra Kähler or para-Kähler structure $(G', \mathbb{J}', \Omega)$, where $\mathbb{J}'$ is the complex or paracomplex structure, such that the invariant metric $G'$ is of neutral signature, locally conformally flat, and is invariant under the isometry group of $g$.

The submanifold theory of $\mathbb{L}(M)$ gives interesting information about the submanifold theory of $M$. For example, the normal congruence (or image under the Gauss map, understood to be a map into the space of oriented geodesics $\mathbb{L}(M)$) of a one-parameter family of parallel hypersurfaces in $M$ is a Lagrangian submanifold (the induced symplectic structure vanishes identically) of the corresponding space of geodesics (see for example [2]). In particular, the normal congruence $L(\Sigma)$ of a Weingarten surface $\Sigma$ in $\mathbb{S}^3_{p,1}$ (its principal curvatures are functionally related) is flat with respect to the metric $G'$ induced on $L(\Sigma)$ [2].

Let $(M, J, g, \omega)$ be a (para-)Kähler manifold, and let $\phi : \Sigma \to M$ be a Lagrangian immersion. A normal vector field $X$ is called *Hamiltonian* if $X = J\nabla u$, where $J$ is the (para-)complex structure and $\nabla u$ is the gradient of $u \in C^\infty(\Sigma)$ with respect to the non-degenerate induced metric $\phi^* g$. We say that a variation $(\phi_t)$ of $\phi$ is a *Hamiltonian variation* if its velocity $X = \partial_t|_{t=0}\phi_t$ is a Hamiltonian vector field with the additional condition that the function $u$ is compactly supported. The Lagrangian immersion $\phi$ is said to be *Hamiltonian minimal* or *H-minimal* if it is a critical point of the volume functional with respect to Hamiltonian variations. The first variation formula of the volume functional implies that a Hamiltonian minimal submanifold is characterised by the equation $\mathrm{div} JH = 0$, where $H$ denotes the mean curvature vector of $\phi$ and div is the divergence operator with respect to the induced metric [11]. Further study of $H$-minimal submanifolds can be found at the following articles [4, 9, 10, 12].

Palmer showed in [13] that a smooth variation of a hypersurface in the sphere $\mathbb{S}^{n+1}$ induces a Hamiltonian variation of the Gauss map in $\mathbb{L}(\mathbb{S}^{n+1})$. An analogous result for the 3-dimensional Euclidean space $\mathbb{E}^3$ has been shown by Anciaux et al. in [3]. Following similar computations as those performed in [13], we prove that any smooth variation of a hypersurface in the real space form $\mathbb{S}^{n+1}_{p,1}$ induces a Hamiltonian variation in the symplectic manifold $(L^{\pm}(\mathbb{S}^{n+1}_{p,1}), \Omega)$. In particular, we prove the following:

**Theorem 1.1.** *Let $\phi_t$, $t \in (-\epsilon, \epsilon)$, be a smooth one-parameter deformation of an immersion $\phi := \phi_{t=0}$ of the $n$-dimensional oriented manifold $\Sigma$ in the real space form $\mathbb{S}^{n+1}_{p,1}$. Then the corresponding Gauss maps $\Phi_t$ form a Hamiltonian variation with respect to the symplectic manifold $(L^{\pm}(\mathbb{S}^{n+1}_{p,1}), \Omega)$.*

It has been proved in [3] that every Hamiltonian minimal surface in $\mathbb{L}(\mathbb{E}^3)$ (resp. $\mathbb{L}(\mathbb{E}^3_1)$, where $\mathbb{E}^3_1$ denotes the Lorentzian 3-space) is the Gauss map of a surface $S$ in $\mathbb{E}^3$ (resp. $\mathbb{E}^3_1$) that is a critical point of the functional $\mathcal{F} = \int_S \sqrt{H^2 - K} dA$, where $H$ and $K$ denote the mean curvature and the Gauss curvature, respectively. In this article, we extend this result for the case of the space $\mathbb{L}(\mathbb{S}^{n+1}_{p,1})$ of oriented geodesics in an $(n+1)$-dimensional real space form. In particular, we consider the (para-)Kähler–Einstein structure $(G, \mathbb{J}, \Omega)$ and the locally conformally flat (para-)Kähler structure $(G', \mathbb{J}', \Omega)$ both endowed on $\mathbb{L}(\mathbb{S}^{n+1}_{p,1})$. Then, as an application of Theorem 1.1, we prove the following:

**Theorem 1.2.** *Let $\phi : \Sigma^n \to \mathbb{S}^{n+1}_{p,1}$ be a real diagonalizable hypersurface in $\mathbb{S}^{n+1}_{p,1}$ and let $\Phi$ be the Gauss map of $\phi$. Then, away of umbilic points, we have the following statements:*

(i) *The Gauss map $\Phi$ is a Hamiltonian minimal submanifold with respect to the (para-)Kähler Einstein structure $(G, \mathbb{J})$ if and only if the immersion $\phi$ is a critical point of the functional*

$$\mathcal{W}(\phi) = \int_\Sigma \sqrt{\Pi_{i=1}^n |\epsilon + k_i^2|}\, dV,$$

*where $k_1, \ldots, k_n$ are the principal curvatures of $\phi$ and $\epsilon$ denotes the length of the normal vector field of $\phi$.*

(ii) *For $n = 2$, the Gauss map $\Phi$ is a Hamiltonian minimal surface in $(L^\pm(\mathbb{S}^3_{p,1}), G', \mathbb{J}')$ if and only if the surface $\phi$ is a critical point of the functional*

$$\mathcal{W}'(\phi) = \int_\Sigma |k_1 - k_2|\, dA,$$

*where $k_1, k_2$ denote the principal curvatures of $\phi$.*

**2. Preliminaries.** For $n \geq 1$, consider the Euclidean space $\mathbb{R}^{n+2}$ endowed with the canonical pseudo-Riemannian metric of signature $(p, n+2-p)$, where $0 \leq p \leq n+2$:

$$\langle \cdot, \cdot \rangle_p = -\sum_{i=1}^{p} dx_i^2 + \sum_{i=p+1}^{n+2} dx_i^2.$$

Define the $(n+1)$-dimensional real space form

$$\mathbb{S}^{n+1}_{p,1} = \{x \in \mathbb{R}^{n+2} |\, \langle x, x \rangle_p = 1\},$$

and let $\iota : \mathbb{S}^{n+1}_{p,1} \hookrightarrow (\mathbb{R}^{n+2}, \langle \cdot, \cdot \rangle_p)$ be the canonical inclusion. The induced metric $\iota^* \langle ., . \rangle_p$ has signature $(p, n+1-p)$ and is of constant sectional curvature $K = 1$.

Following the notations of [2], we denote by $L^+(\mathbb{S}^{n+1}_{p,1})$ (resp. $L^-(\mathbb{S}^{n+1}_{p,1})$) the set of spacelike (resp. timelike) oriented geodesics of $\mathbb{S}^{n+1}_{p,1}$, that is,

$$L^\pm(\mathbb{S}^{n+1}_{p,1}) = \{x \wedge y \in \Lambda^2(\mathbb{R}^{n+2}) \mid y \in T_x\mathbb{S}^{n+1}_{p,1}, \langle y, y \rangle_p = \epsilon\},$$

where $\epsilon = 1$ (resp. $\epsilon = -1$) corresponds to $L^+(\mathbb{S}^{n+1}_{p,1})$ (resp. $L^-(\mathbb{S}^{n+1}_{p,1})$). If $\Lambda^2(\mathbb{R}^{n+2})$ is equipped with the flat pseudo-Riemannian metric:

$$\langle\langle x_1 \wedge y_1, x_2 \wedge y_2 \rangle\rangle = \langle x_1, x_2 \rangle_p \langle y_1, y_2 \rangle_p - \langle x_1, y_2 \rangle_p \langle x_2, y_1 \rangle_p\,,$$

we denote by $G$ the metric $\langle\langle \cdot, \cdot \rangle\rangle$ induced by the inclusion map $i : L^\pm(\mathbb{S}^{n+1}_{p,1}) \hookrightarrow \Lambda^2(\mathbb{R}^{n+2})$. On the other hand, in $L^\pm(\mathbb{S}^{n+1}_{p,1})$ a complex (paracomplex) structure $\mathbb{J}$ can be defined as follows:

Let $J$ be the canonical complex (paracomplex) structure in the oriented plane $x\wedge y\in L^{\pm}(\mathbb{S}^{n+1}_{p,1})$ defined by $Jx=y$ and $Jy=-\epsilon x$. Thus, $J^2=-\epsilon Id$. A tangent vector to $i(L^{\pm}(\mathbb{S}^{n+1}_{p,1}))$ at the point $x\wedge y$ is of the form $x\wedge X+y\wedge Y$, where $X,Y\in(x\wedge y)^{\perp}$ in $\Lambda^2(\mathbb{R}^{n+2})$. The complex (paracomplex) structure $\mathbb{J}$ is defined by:

$$\mathbb{J}(x\wedge X+y\wedge Y):=(Jx)\wedge X+(Jy)\wedge Y=y\wedge X+\epsilon x\wedge Y.$$

The metric $G$ and the (para-)complex structure $\mathbb{J}$ are invariant under the natural action of the isometry group of $\mathbb{S}^{n+1}_{p,1}$. For $n\geq 3$, it has been shown in [1] that $G$ is the unique invariant metric under the natural action of $SO(n+2-p,p)$.

The 2-form $\Omega$ defined by $\Omega(\cdot,\cdot)=\epsilon G(\mathbb{J}\cdot,\cdot)$ is a symplectic structure on $L^{\pm}(\mathbb{S}^{n+1}_{p,1})$ and in particular:

**Proposition 2.1** [2]. *The quadraple $(L^{+}(\mathbb{S}^{n+1}_{p,1}),G,\mathbb{J},\Omega)$ is a $2n$-dimensional Kähler manifold with signature $(2p,2n-2p)$, while $(L^{-}(\mathbb{S}^{n+1}_{p,1}),G,\mathbb{J},\Omega)$ is a $2n$-dimensional para-Kähler manifold. In both cases, the metric $G$ is Einstein with constant scalar curvature $S=2\epsilon n^2$.*

We now consider the case of $L^{\pm}(\mathbb{S}^{3}_{p,1})\subset\Lambda^2(\mathbb{R}^4)$. The orthogonal $(x\wedge y)^{\perp}$ of an oriented plane $x\wedge y\in L^{\pm}(\mathbb{S}^{3}_{p,1})$ is also a plane in $\mathbb{R}^4$ and is oriented in such a way that its orientation is combatible with the orientation of the plane $x\wedge y$. Then, it is possible to define a canonical complex or paracomplex structure $J'$, depending of whether the metric $\langle\langle .,.\rangle\rangle$ induced on $(x\wedge y)^{\perp}$ is positive or indefinite. In this case, we may define a complex or paracomplex structure $\mathbb{J}'$ on $L^{\pm}(\mathbb{S}^{3}_{p,1})$ by

$$\mathbb{J}'(x\wedge X+y\wedge Y):=x\wedge(J'X)+y\wedge(J'Y).$$

The pseudo-Riemannian metric $G'$ on $L^{\pm}(\mathbb{S}^{3}_{p,1})$ is given by:

$$G'(\cdot,\cdot):=\Omega(\cdot,J'\cdot)=-\epsilon G(\cdot,\mathbb{J}\circ\mathbb{J}'\cdot).$$

Furthermore,

**Proposition 2.2** [2]. *The quadraples $(L^{\pm}(\mathbb{S}^{3}_{p,1}),G',\mathbb{J}',\Omega)$ are 4-dimensional (para-)Kähler manifolds. The metric $G'$ is of neutral signature $(2,2)$, scalar flat, locally conformally flat, and is invariant under the natural action of $SO(4-p,p)$.*

Let $\phi:\Sigma^n\to\mathbb{S}^{n+1}_{p,1}$ be an immersion of an $n$-dimensional orientable manifold into the real space form $\mathbb{S}^{n+1}_{p,1}$, and let $N$ be the unit normal vector of the hypersurface $S=\phi(\Sigma)$. The set $\bar{S}$ of geodesics that are orthogonal to $S$, oriented in the direction of $N$, is called *the normal congruence* or *the Gauss map* of $S$. Then,

**Proposition 2.3** [2]. *Let $\phi$ be an immersion of an orientable manifold $\Sigma^n$ in $\mathbb{S}^{n+1}_{p,1}$ with unit normal vector field $N$. Then the Gauss map of $S=\phi(\Sigma)$ is the image of the map $\Phi:\Sigma^n\to L^{\pm}(\mathbb{S}^{n+1}_{p,1})$ defined by $\Phi=\phi\wedge N$. When $\Phi$ is an immersion, it is Lagrangian. Conversely, let $\Phi:\Sigma^n\to L^{\pm}(\mathbb{S}^{n+1}_{p,1})$ be an* *immersion of a simply connected $n$-manifold. Then $\bar{S} := \Phi(\Sigma)$ is the Gauss map of an immersed hypersurface of $\mathbb{S}^{n+1}_{p,1}$ if and only if $\Phi$ is Lagrangian.*

**3. Hamiltonian minimal submanifolds.** Throughout the article, the induced metric of a hypersurface in $\mathbb{S}^{n+1}_{p,1}$ and the induced metric of a Lagrangian submanifold in $L^{\pm}(\mathbb{S}^{n+1}_{p,1})$ is assumed to be non-degenerate.

**3.1. Hamiltonian variations in $L^{\pm}(\mathbb{S}^{n+1}_{p,1})$.** Consider the $2n$-dimensional symplectic manifold $(L^{\pm}(\mathbb{S}^{n+1}_{p,1}), \Omega)$, where $L^{\pm}(\mathbb{S}^{n+1}_{p,1})$ denotes the space of oriented geodesics in the real space form $\mathbb{S}^{n+1}_{p,1}$. Then we prove our first main result:

*Proof of Theorem 1.1* Let $\phi_t : \Sigma^n \to \mathbb{S}^{n+1}_{p,1}$, where $t \in (-t_0, t_0)$ for some $t_0 > 0$, be a smooth variation of an immersion $\phi := \phi_0 : \Sigma^n \to \mathbb{S}^{n+1}_{p,1}$ of an oriented $n$-dimensional manifold $\Sigma$ in $\mathbb{S}^{n+1}_{p,1}$. Let $S_t := \phi_t(\Sigma)$ be the hypersurfaces of $\mathbb{S}^{n+1}_{p,1}$ and $S := \phi(\Sigma)$.

We denote by $N_t$ the 1-parameter family of vector fields such that $N_0 = N$ and $\langle N_t, \phi_t \rangle_p = 0$. Following the notation of [13], there exist a smooth function $f$ on $S$ and a smooth section $Y$ of the tangent bundle $TS$ such that

$$\dot{\phi} = fN + Y,$$

where $\dot{\phi} = \partial_t \phi_t|_{t=0}$. Differentiate the expression $\langle N_t, \phi_t \rangle_p = 0$ with respect to $t$ at $t = 0$, we have

$$\langle \dot{N}, \phi \rangle_p = -\langle N, \dot{\phi} \rangle_p.$$

Then it yields

$$\langle \dot{N}, \phi \rangle_p = -\epsilon f,$$

where $\epsilon = \langle N, N \rangle_p$. For $X_t \in TS_t$, where $X_0 := X \in TS$, we also differentiate with respect to $t$ the expression $\langle N_t, X_t \rangle_p = 0$ at $t = 0$, and we obtain

$$\langle \dot{N}, X \rangle_p = -\epsilon df(X) + \langle dN(Y), X \rangle_p ,$$

which finally gives

$$\dot{N} = -\epsilon \nabla f + dN(Y) - \epsilon f \phi.$$

Let $\Phi : \Sigma^n \to L^{\pm}(\mathbb{S}^{n+1}_{p,1}) : x \mapsto \phi(x) \wedge N(x)$ be the Gauss map of the immersion $\phi : \Sigma^n \to \mathbb{S}^{n+1}_{p,1}$. If $\bar{X} := d\Phi(X)$, from [2], we have that

$$\bar{X} = X \wedge N + AX \wedge \phi.$$

Let $\dot{\Phi} = \partial_t \Phi_t|_{t=0}$ be the velocity of the variation $\Phi_t$ and write $\dot{\Phi} = \dot{\Phi}^{\top} + \dot{\Phi}^{\perp}$, where $\dot{\Phi}^{\top}$ and $\dot{\Phi}^{\perp}$ denote the tangential and the normal component of $\dot{\Phi}$ respectively. Then,

$$\begin{aligned} G(\dot{\Phi}, \mathbb{J}\bar{X}) &= G(Y \wedge N + \epsilon \nabla f \wedge \phi - dN(Y) \wedge \phi,\ \mathbb{J}(X \wedge N + AX \wedge \phi)) \\ &= -df(X), \end{aligned}$$

which implies that

$$\dot{\Phi}^{\perp} = -\mathbb{J}\overline{\nabla} f,$$

and this completes the proof of the Theorem. □

**3.2. Applications.** A *real diagonalizable immersion* is a smooth immersion $\phi$ of a hypersurface $\Sigma^n$ into the (n+1)-dimensional real space form $\mathbb{S}^{n+1}_{p,1}$ such that, locally, the shape operator $A$ can be diagonalized, that is, there exists a local orthonormal frame $(e_1, \dots, e_n)$ and real functions $k_1, \dots, k_n$ where $A = \mathrm{diag}(k_1, \dots, k_n)$. In this case, each vector field $e_i$ is called *a principal direction* with corresponding *principal curvature* $k_i$.

**Remark 3.1.** *Note that in the Riemannian case every hypersurface is real diagonalizable, away from umbilic points.*

We are now in position to prove our second result:
*Proof of Theorem 1.2* Consider a smooth immersion of $\phi$ of the $n$-dimensional manifold $\Sigma$ in $\mathbb{S}^{n+1}_{p,1}$ and let $\Phi : \Sigma^n \to L^{\pm}(\mathbb{S}^{n+1}_{p,1})$ be the corresponding Gauss map. The fact that $\phi$ is real diagonalizable implies the existence of an orthonormal frame $(e_1, \dots, e_n)$, with respect to the induced metric $\phi^* g$, such that

$$Ae_i = k_i e_i, \quad i = 1, \dots, n,$$

where $A$ denotes the shape operator of $\phi$. Let $(\phi_t)_{t\in(-t_0,t_0)}$ be a smooth variation of $\phi$ and $(\Phi_t)$ be the corresponding variation of the Gauss map $\Phi$. Real diagonalizability implies that the minimal polynomial of $A$ is the product of distinct linear factors. Using the fact that the variation $(\Phi_t)$ is at least $C^1$-smooth, it is possible to obtain a positive real number $t_1 < t_0$ such that $\phi_t$ is real diagonalizable for every $t \in (-t_1, t_1)$. We may extend all extrinsic geometric quantities such as the shape operator $A$, the principal directions $e_i$, and the principal curvatures $k_i$ to the 1-parameter family of immersions $(\phi_t)$.

(i) From [2], the induced metric $\Phi_t^* G$ is given by $\Phi_t^* G = \epsilon \phi_t^* g + \phi_t^* g(A., A.)$, and thus

$$\Phi_t^* G = \mathrm{diag}(\epsilon_1(\epsilon + k_1^2), \dots, \epsilon_n(\epsilon + k_n^2)),$$

where $\epsilon_i = g(e_i, e_i)$. For every sufficiently small $t > 0$, the volume of every Gauss map $\Phi_t$, with respect to the metric $G$, is

$$\mathrm{Vol}(\Phi_t) = \int_\Sigma \sqrt{|\det \Phi_t^* G|} dV = \mathcal{W}(\phi_t). \tag{1}$$

If $\phi$ is a critical point of the functional $\mathcal{W}$, we have

$$\partial_t(\mathrm{Vol}(\Phi_t)) = 0,$$

for any Hamiltonian variation of $\Phi$. Therefore, $\Phi$ is a Hamiltonian minimal submanifold with respect to the Kähler Einstein structure $(G, J)$. The converse follows directly from (1).

(ii) Assume that $n = 2$. From [2], in terms of the orthonormal frame $(e_1, e_2)$, the induced metric $\Phi^* G'$ is

$$\Phi^* G' = \begin{pmatrix} 0 & \epsilon_2(k_2 - k_1) \\ \epsilon_2(k_2 - k_1) & 0 \end{pmatrix},$$

where $\epsilon_2 = \phi^* g(e_2, e_2)$. Then, the volume of every Gauss map $\Phi_t$, with respect to the metric $G'$, is

$$\mathrm{Vol}'(\Phi_t) = \int\limits_{\Sigma} \sqrt{|\det \Phi_t^* G|} dV = \mathcal{W}'(\phi_t), \tag{2}$$

and thus the second statement of the Theorem follows by a similar argument as the proof of the first statement. □

Let $\phi : \Sigma^n \to \mathbb{S}^{n+1}_{p,1}$ be a hypersurface and $N$ denotes the unit normal vector field. For $\theta \in \mathbb{R}$, consider the immersion $\phi_\theta := \cos\epsilon(\theta)\phi + \sin\epsilon(\theta)N$, where $\epsilon := |N|^2$ and $(\cos\epsilon(\theta), \sin\epsilon(\theta)) = (\cos\theta, \cos\theta)$ if $\epsilon = 1$ while for $\epsilon = -1$ we have $(\cos\epsilon(\theta), \sin\epsilon(\theta)) = (\cosh\theta, \cosh\theta)$. The images $\phi_\theta(\Sigma)$ and $\phi(\Sigma)$ are called parallel hypersurfaces. It is important to mention that parallel hypersurfaces have the same Gauss map [2].

Looking more carefully the relations (1) and (2), we obtain the following symmetry for the functionals $\mathcal{W}$ and $\mathcal{W}'$:

**Corollary 3.2.** *If $\phi_1$ and $\phi_2$ are parallel smooth real diagonalizable immersions of the n-manifold $\Sigma$ in $\mathbb{S}^{n+1}_{p,1}$, then $\mathcal{W}(\phi_1) = \mathcal{W}(\phi_2)$. In the case where $n = 2$, we also have that $\mathcal{W}'(\phi_1) = \mathcal{W}'(\phi_2)$.*

Using the Einstein (para-)Kähler structure $(L^{\pm}(\mathbb{S}^3_{p,1}), G, J)$, we obtain the following Corollary:

**Corollary 3.3.** *Let $\phi : \Sigma^n \to \mathbb{S}^{n+1}_{p,1}$ be a real diagonalizable hypersurface and let $k_1, \ldots k_n$ be the principal curvatures. If $\Phi$ is the Gauss map of $\phi$, then the function $\sum_{i=1}^n \tan\epsilon^{-1}(k_i)$ is harmonic with respect to the induced metric $\Phi^* G$ if and only if $\phi$ is a critical point of the functional $\int_\Sigma \sqrt{\Pi_{i=1}^n |\epsilon + k_i^2|}$.*

*Proof.* Let $\Phi$ be the Gauss map of $\phi$ and consider the Einstein (para-)Kähler structure $(G, J)$. Then, from [2], we know that the mean curvature $\vec{H}$ of $\Phi$ is given by

$$\vec{H} = \frac{\epsilon}{n} J \nabla \left( \sum_{i=1}^n \tan\epsilon^{-1}(k_i) \right),$$

where $\nabla$ denotes the Levi–Civita connection of the induced metric $\Phi^* G$. Then the Corollary follows by the following relation

$$\mathrm{div}(nJ\vec{H}) = \Delta \left( \sum_{i=1}^n \tan\epsilon^{-1}(k_i) \right),$$

where div and $\Delta$ denote the divergence operator and the Laplacian of $\Phi^* G$. □

Using the Remark 3.1, we obtain the following two Corollaries:

**Corollary 3.4.** *Let $\phi : \Sigma^n \to \mathbb{S}^{n+1}$ (resp. in the hyperbolic space $\mathbb{H}^{n+1}$) be a hypersurface in the sphere $\mathbb{S}^{n+1}$. Then the Gauss map $\Phi$ is a Hamiltonian minimal submanifold in the (para-)Kähler Einstein structure $(G, \mathbb{J})$ if and only if the hypersurface $\phi$ is a critical point of the functional $\mathcal{W}(\phi) = \int_\Sigma \sqrt{\Pi_{i=1}^n (1 + k_i^2)}$ (resp. $\mathcal{W}(\phi) = \int_\Sigma \sqrt{\Pi_{i=1}^n |1 - k_i^2|}$), where $k_1, \ldots, k_n$ are the principal curvatures of $\phi$.*

**Corollary 3.5.** *Let $\phi : \Sigma \to \mathbb{S}^3$ (resp. in the hyperbolic space $\mathbb{H}^3$) be a surface in the sphere $\mathbb{S}^3$. Then the Gauss map $\Phi$ is a Hamiltonian minimal submanifold with respect to the Kähler, conformally flat structure $(G', \mathbb{J}')$ if and only if the surface $\phi$ is a critical point of the functional $\mathcal{W}'(\phi) = \int_\Sigma \sqrt{H^2 - K + 1}$ (resp. $\mathcal{W}'(\phi) = \int_\Sigma \sqrt{H^2 - K - 1}$ ), where $H, K$ denote the mean and the Gauss curvature of $\phi$.*

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