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Resolutions of general canonical curves on rational normal scrolls

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Abstract. Let $\mathrm{C} \subset \mathbb{P}^{g-1}$ be a general curve of genus g, and let k be a positive integer such that the Brill–Noether number $\rho(g, k, 1) \geq 0$ and $g > k + 1$. The aim of this short note is to study the relative canonical resolution of C on a rational normal scroll swept out by a $g^1_k = |\mathrm{L}|$ with $\mathrm{L} \in \mathrm{W}^1_k(\mathrm{C})$ general. We show that the bundle of quadrics appearing in the relative canonical resolution is unbalanced if and only if $\rho > 0$ and $\left(k - \rho - \frac{7}{2}\right)^2 - 2k + \frac{23}{4} > 0$.

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1. Introduction. Let $\mathrm{C} \subset \mathbb{P}^{g-1}$ be a canonical curve of genus g that admits a complete base point free g^1_k, then the g^1_k sweeps out a rational normal scroll X of dimension $d = k - 1$ and degree $f = g - k + 1$. One can resolve the curve $\mathrm{C} \subset \mathbb{P}(\mathscr{E})$, where $\mathbb{P}(\mathscr{E})$ is the $\mathbb{P}^{d-1}$-bundle associated to the scroll X. Schreyer showed in [11] that this so-called *relative canonical resolution* is of the form

$$0 \to \pi^*\mathrm{N}_{k-2}(-k) \to \pi^*\mathrm{N}_{k-3}(-k+2) \to \cdots \to \pi^*\mathrm{N}_1(-2) \to \mathscr{O}_{\mathbb{P}(\mathscr{E})} \to \mathscr{O}_{\mathrm{C}} \to 0,$$

where $\pi : \mathrm{C} \to \mathbb{P}^1$ is the map induced by the g^1_k and $\mathrm{N}_i = \bigoplus_{j=1}^{\beta_i} \mathscr{O}_{\mathbb{P}^1}(a_j^{(i)})$.

To determine the splitting type of these N_i is an open problem. If C is a general canonical curve with a g^1_k such that the genus g is large compared to k, it is conjectured that the bundles N_i are balanced, which means that $\max |a_j^{(i)} - a_l^{(i)}| \leq 1$. This is known to hold for $k \leq 5$ (see e.g. [7] or [3]). Bujokas and Patel [5] gave further evidence to the conjecture by showing that all N_i are balanced if $g = n \cdot k + 1$ for $n \geq 1$ and the bundle N_1 is balanced if $g \geq (k-1)(k-3)$.

The aim of this short note is to provide a range in which the first syzygy bundle N_1, hence the relative canonical resolution, is unbalanced for a general pair (C, g^1_k) with non-negative Brill–Noether number $\rho(g,k,1)$. Our main theorem is the following.

Main Theorem. *Let* $\mathrm{C} \subset \mathbb{P}^{g-1}$ *be a general canonical curve, and let* k *be a positive integer such that* $\rho := \rho(g,k,1) \geq 0$ *and* $g > k+1$. *Let* $\mathrm{L} \in W^1_k(\mathrm{C})$ *be a general point inducing a* $g^1_k = |\mathrm{L}|$. *Then the bundle* N_1 *in the relative canonical resolution of* C *is unbalanced if and only if* $\left(k - \rho - \frac{7}{2}\right)^2 - 2k + \frac{23}{4} > 0$ *and* $\rho > 0$.

After introducing the relative canonical resolution, we prove the above theorem in Sect. 3. The strategy for the proof is to study the birational image C' of C under the residual mapping $|\omega_\mathrm{C} \otimes \mathrm{L}^{-1}|$. Quadratic generators of C' correspond to special generators of $\mathrm{C} \subset \mathbb{P}(\mathscr{E})$ whose existence forces N_1 to be unbalanced in the case $\rho > 0$. Under the generality assumptions on C and L, one obtains a sharp bound for which pairs (k, ρ), the curve C' has quadratic generators. Finally in Sect. 4, we state a more precise conjecture about the splitting type of the bundles in the relative canonical resolution.

Our theorem and conjecture are motivated by experiments using the computer algebra software *Macaulay2* [8] and the package `RelativeCanonicalResolution.m2` [4].

2. Relative canonical resolutions. In this section we briefly summarize the connections between pencils on canonical curves and rational normal scrolls in order to define the relative canonical resolution. Furthermore, we give a closed formula for the degrees of the bundles N_i appearing in the relative canonical resolution. Most of this section follows Schreyer's article [11].

Definition 2.1. Let $e_1 \geq e_2 \geq \cdots \geq e_d \geq 0$ be integers, $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}(e_1) \oplus \cdots \oplus \mathscr{O}_{\mathbb{P}^1}(e_d)$, and let $\pi : \mathbb{P}(\mathscr{E}) \to \mathbb{P}^1$ be the corresponding $\mathbb{P}^{d-1}$-bundle.

A *rational normal scroll* $\mathrm{X} = \mathrm{S}(e_1, \ldots, e_d)$ of type $(e_1, \ldots, e_d)$ is the image of

$$j : \mathbb{P}(\mathscr{E}) \to \mathbb{P}\mathrm{H}^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)) = \mathbb{P}^r,$$

where $r = f + d - 1$ with $f = e_1 + \cdots + e_d \geq 2$.

In [10] it is shown that the variety X defined above is a non-degenerate d-dimensional variety of minimal degree $\deg(\mathrm{X}) = f = r - d + 1 = \mathrm{codim}(\mathrm{X}) + 1$. If $e_1, \ldots, e_d > 0$, then $j : \mathbb{P}(\mathscr{E}) \to X \subset \mathbb{P}H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)) = \mathbb{P}^r$ is an isomorphism. Otherwise, it is a resolution of singularities. Since $\mathrm{R}^i j_* \mathscr{O}_{\mathbb{P}(\mathscr{E})} = 0$, it is convenient to consider $\mathbb{P}(\mathscr{E})$ instead of X for cohomological considerations.

It is furthermore known, that the Picard group $\mathrm{Pic}(\mathbb{P}(\mathscr{E}))$ is generated by the ruling $\mathrm{R} = [\pi^* \mathscr{O}_{\mathbb{P}^1}(1)]$ and the hyperplane class $\mathrm{H} = [j^* \mathscr{O}_{\mathbb{P}^r}(1)]$ with intersection products

$$\mathrm{H}^d = f, \quad \mathrm{H}^{d-1} \cdot \mathrm{R} = 1, \quad \mathrm{R}^2 = 0.$$

Now let $C \subset \mathbb{P}^{g-1}$ be a canonically embedded curve of genus g, and let further

$$g^1_k = \{D_\lambda\}_{\lambda \in \mathbb{P}^1} \subset |D|$$

be a pencil of divisors of degree k. If we denote by $\overline{D_\lambda} \subset \mathbb{P}^{g-1}$ the linear span of the divisor, then

$$X = \bigcup_{\lambda \in \mathbb{P}^1} \overline{D_\lambda} \subset \mathbb{P}^{g-1}$$

is a $(k-1)$-dimensional rational normal scroll of degree $f = g - k + 1$. Conversely, if X is a rational normal scroll of degree f containing a canonical curve, then the ruling on X cuts out a pencil of divisors $\{D_\lambda\} \subset |D|$ such that $h^0(C, \omega_C \otimes \mathscr{O}_C(D)^{-1}) = f$.

Theorem 2.2 [11, Corollary 4.4]. *Let* C *be a curve with a base point free* g^1_k, *and let* $\mathbb{P}(\mathscr{E})$ *be the projective bundle associated to the scroll X, swept out by the* g^1_k.

(a) $C \subset \mathbb{P}(\mathscr{E})$ *has a resolution* $F_\bullet$ *of type*

$$0 \to \pi^* N_{k-2}(-k) \to \pi^* N_{k-3}(-k+2) \to \cdots \to \pi^* N_1(-2) \to \mathscr{O}_{\mathbb{P}(\mathscr{E})} \to \mathscr{O}_C \to 0$$

with $\pi^* N_i = \sum_{j=1}^{\beta_i} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(a_j^{(i)} R)$ *and* $\beta_i = \frac{i(k-2-i)}{k-1}\binom{k}{i+1}$.

(b) *The complex* $F_\bullet$ *is self dual, i.e.,* $\mathscr{H}om(F_\bullet, \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-kH + (f-2)R)) \cong F_\bullet$

According to [7], the resolution $F_\bullet$ above is called the *relative canonical resolution*.

Remark 2.3. A generalization of Theorem 2.2 can be found in [6] for covers $\pi : Y \to Y'$ of degree k of equal dimensional varieties. In [6], the authors used the Tschirnhausen bundle $\mathscr{E}_T$ defined by

$$0 \to \mathscr{O}_{Y'} \to \pi_*(\mathscr{O}_Y) \to \mathscr{E}_T^\vee \to 0$$

to construct relative resolutions. Note that for covers of $\mathbb{P}^1$, $\mathscr{E}_T = \mathscr{E} \otimes \mathscr{O}_{\mathbb{P}^1}(2)$ and therefore, the degrees of the syzygy bundles N_i in [6] differ slightly from the ones given in Proposition 2.9.

Definition 2.4. We say that a bundle of the form $\sum_{j=1}^{\beta} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(nH + a_j R)$ is *balanced* if $\max_{i,j} |a_j - a_i| \leq 1$. The relative canonical resolution is called balanced if all bundles occurring in the resolution are balanced.

Remark 2.5. To determine the splitting type of the bundle $\mathscr{E}$, one can use [11, (2.5)]. It follows that the $\mathbb{P}^1$-bundle $\mathscr{E}$ associated to the scroll is always balanced for a Petri-general curve C with a g^1_k if $\rho(g, k, 1) \geq 0$.

If C is a general k-gonal curve and the degree k map to $\mathbb{P}^1$ is determined by a unique g^1_k, then it follows by [2] that $\mathscr{E}$ is balanced as well.

Remark 2.6. If all $a_j^{(i)} \geq -1$, one can resolve the $\mathscr{O}_{\mathbb{P}(\mathscr{E})}$-modules occurring in the relative canonical resolution of C by Eagon-Northcott type complexes. An iterated mapping cone gives a possibly non-minimal resolution of the curve

$C \subset \mathbb{P}^{g-1}$. In [11], Schreyer used this method to classify all possible Betti tables of canonical curves up to genus 8. An implementation of this construction can be found in the *Macaulay2*-package [4].

We will give a lower bound on the integers $a_j^{(1)}$ appearing in the resolution $F_\bullet$.

Proposition 2.7. *Let* C *be a general canonically embedded curve of genus g, and let $k \geq 4$ be an integer such that $\rho(g,k,1) \geq 0$ and $g > k+1$. Let further* $L \in W_k^1(C)$ *be a general point inducing a complete base point free g_k^1. Then with notation as in Theorem 2.2, all twists $a_j^{(1)}$ of the bundle* N_1 *are non-negative.*

Proof. As usual, we denote by $\mathbb{P}(\mathscr{E})$ the $\mathbb{P}^1$-bundle induced by the g_k^1. We consider the relative canonical resolution of $C \subset \mathbb{P}(\mathscr{E})$. Twisting of the relative canonical resolution by 2H and pushing forward to $\mathbb{P}^1$, we get an isomorphism $\pi_*(\mathscr{I}_{C/\mathbb{P}(\mathscr{E})}(2H)) \cong N_1 = \bigoplus_{j=1}^{\beta_1} \mathscr{O}_{\mathbb{P}^1}(a_j^{(1)})$. Then, all twists $a_j^{(1)}$ are non-negative if and only if

$$\begin{aligned} h^1(\mathbb{P}^1, N_1(-1)) &= h^1(\mathbb{P}^1, \pi_*(\mathscr{I}_{C/\mathbb{P}(\mathscr{E})}(2H-R))) \\ &= h^1(\mathbb{P}(\mathscr{E}), \mathscr{I}_{C/\mathbb{P}(\mathscr{E})}(2H-R)) = 0. \end{aligned}$$

We consider the long exact cohomology sequence

$$\begin{aligned} 0 \to H^0(\mathbb{P}(\mathscr{E}), \mathscr{I}_{C/\mathbb{P}(\mathscr{E})}(2H-R)) &\to H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(2H-R)) \\ &\to H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_C(2H-R)) \to \\ &\to H^1(\mathbb{P}(\mathscr{E}), \mathscr{I}_{C/\mathbb{P}(\mathscr{E})}(2H-R)) \to \cdots \end{aligned}$$

obtained from the standard short exact sequence.

The vanishing of $H^1(\mathbb{P}(\mathscr{E}), \mathscr{I}_{C/\mathbb{P}(\mathscr{E})}(2H-R))$ is equivalent to the surjectivity of the map

$$H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(2H-R)) \longrightarrow H^0(C, \mathscr{O}_C(2H-R)).$$

From the commutative diagram

$$\begin{array}{ccc} H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(2H-R)) & \longrightarrow & H^0(C, \mathscr{O}_C(2H-R)) \\ \uparrow & & \uparrow \eta \\ H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H)) \otimes H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(H-R)) & \xrightarrow{\cong} & H^0(C, \mathscr{O}_C(H)) \otimes H^0(C, \mathscr{O}_C(H-R)), \end{array}$$

we see that it suffices to show the surjectivity of η.

Note that the system $|H-R|$ on C is $\omega_C \otimes L^{-1}$. The residual line bundle $\omega_C \otimes L^{-1} \in W_{2g-2-k}^{g-k}(C)$ is general since L is general. Hence, the residual morphism induced by $|\omega_C \otimes L^{-1}|$ is birational for $g-k \geq 2$ by [9, Section 0.b (4)].

We may apply [1, Theorem 1.6] and get a surjection

$$\bigoplus_{q \geq 0} \operatorname{Sym}_q \left(H^0(C, \omega_C \otimes L^{-1})\right) \otimes H^0(C, \omega_C) \longrightarrow \bigoplus_{q \geq 0} H^0\left(C, \omega_C \otimes (\omega_C \otimes L^{-1})^q\right),$$

i.e., the $\operatorname{Sym}(H^0(C, \omega_C \otimes L^{-1}))$-module $\bigoplus_{q \in \mathbb{Z}} H^0(C, \omega_C \otimes (\omega_C \otimes L^{-1})^q)$ is generated in degree 0. In particular, this implies the surjectivity of η. □

Remark 2.8. Using the projective normality of $\mathrm{C} \subset \mathbb{P}(\mathscr{E})$, one can show that all twists $a_j^{(1)}$ of N_1 are greater or equal to -1. There exist several examples where N_1 has negative twists (see [11]). We conjecture that all $a_j^{(i)} \geq -1$ and in general $a_j^{(i)} \geq 0$.

It is known that the degrees of the bundles N_i can be computed recursively. However, we did not find a closed formula for the degrees in the literature.

Proposition 2.9. *The degree of the bundle* N_i *of rank* $\beta_i = \frac{k}{i+1}(k-2-i)\binom{k-2}{i-1}$ *in the relative canonical resolution* $\mathrm{F}_\bullet$ *is*

$$\deg(\mathrm{N}_i) = \sum_{j=1}^{\beta_i} a_j^{(i)} = (g-k-1)(k-2-i)\binom{k-2}{i-1}.$$

For $i = 1, 2$ *one obtains* $\deg(\mathrm{N}_1) = (k-3)(g-k-1)$ *and* $\deg(\mathrm{N}_2) = (k-4)(k-2)(g-k-1)$.

Proof. The degrees of the bundles N_i can be computed by considering the identity

$$\chi(\mathscr{O}_{\mathrm{C}}(\nu)) = \sum_{i=0}^{k-2} (-1)^i \chi(\mathrm{F}_i(\nu)). \tag{1}$$

If $b \geq -1$, we have

$$h^i(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(a\mathrm{H} + b\mathrm{R})) = \begin{cases} h^i(\mathbb{P}^1, \mathrm{S}_a(\mathscr{E})(b)), & \text{for } a \geq 0 \\ 0, & \text{for } -k < a < 0 \\ h^{k-i}(\mathbb{P}^1, \mathrm{S}_{-a-k}(\mathscr{E})(f-2-b)), & \text{for } a \leq -k \end{cases}$$

where $f = \deg(\mathscr{E}) = g-k+1$. As in the construction of the bundles in [6, Proof of Step B, Theorem 2.1], one obtains that the degree of N_i is independent of the splitting type of the bundle. Hence, we assume that $a_j^{(i)} \geq -1$ and therefore, we can apply the above formula to all terms in $\mathrm{F}_\bullet$.

We compute the degree of N_n by induction. The base case is straightforward. We twist the relative canonical resolution by $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(n+1)$ and compute the Euler characteristic of each term. By the Riemann–Roch Theorem, $\chi(\mathscr{O}_{\mathrm{C}}(n+1)) = (2n+1)g - (2n+1)$. Applying the above formula yields

$$\chi(\mathrm{F}_i(n+1)) = \begin{cases} \binom{k-1+n}{k-2} + f\binom{k-1+n}{k-1}, & \text{for } i = 0 \\ (\deg(\mathrm{N}_i) + \beta_i)\binom{k-2+n-i}{k-2} + \beta_i f\binom{k-2+n-i}{k-1}, & \text{for } n \geq i \geq 1 \\ 0, & \text{for } i \geq n+1 \end{cases}$$

Substituting all formulas in (1), we get

$$\begin{aligned}(2n+1)g-(2n+1) = &\binom{k-1+n}{k-2}+f\binom{k-1+n}{k-1}\\ &+\sum_{i=1}^{n-1}(-1)^i\left((\deg(\mathrm{N}_i)+\beta_i)\binom{k-2+n-i}{k-2}\right.\\ &\left.+\beta_i f\binom{k-2+n-i}{k-1}\right)\\ &+(-1)^n(\deg(N_n)+\beta_n).\end{aligned}$$

Using the induction step, the alternating sums simplify to

$$\begin{aligned}\sum_{i=1}^{n-1}(-1)^i\deg(\mathrm{N}_i)\binom{k-2+n-i}{k-2} &= (f-2)(2n+1-nk)\\ &\quad+(-1)^{n+1}(f-2)(k-2-n)\binom{k-2}{n-1}\\ \sum_{i=1}^{n-1}(-1)^i\beta_i\binom{k-2+n-i}{k-2} &= k-\binom{k-1+n}{k-2}\\ &\quad+(-1)^{n+1}\frac{k}{n+1}(k-2-n)\binom{k-2}{n-1}\\ \sum_{i=1}^{n-1}(-1)^i\beta_i f\binom{k-2+n-i}{k-1} &= nkf-f\binom{k-1+n}{k-1}\end{aligned}$$

and we get the desired formula for $\deg(\mathrm{N}_n)$. □

3. The bundle of quadrics. Let $\mathrm{C}\subset\mathbb{P}^{g-1}$ be a general canonically embedded genus g curve, and let k be a positive integer such that the Brill–Noether number $\rho:=\rho(g,k,1)$ is non-negative and $g>k+1$. Let $\mathrm{L}\in\mathrm{W}^1_k(\mathrm{C})$ general. Then, we denote by X the rational normal scroll swept out by the $g^1_k=|\mathrm{L}|$ and by $\mathbb{P}(\mathscr{E})\to\mathrm{X}$ the projective bundle associated to X. By Remark 2.5, the bundle $\mathscr{E}$ on $\mathbb{P}^1$ is of the form

$$\mathscr{E}=\bigoplus_{i=1}^{k-1-\rho}\mathscr{O}_{\mathbb{P}^1}(1)\oplus\bigoplus_{i=1}^{\rho}\mathscr{O}_{\mathbb{P}^1}.$$

By Theorem 2.2, the resolution of the ideal sheaf $\mathscr{I}_{\mathrm{C}/\mathbb{P}(\mathscr{E})}$ is of the form

$$0\longleftarrow\mathscr{I}_{\mathrm{C}/\mathbb{P}(\mathscr{E})}\longleftarrow\mathrm{Q}:=\sum_{j=1}^{\beta_1}\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2\mathrm{H}+a_j^{(1)}\mathrm{R})\longleftarrow\dots$$

where $\beta_1=\frac{1}{2}k(k-3)$. We denote Q the bundle of quadrics. By Proposition 2.9, the degree of $\mathrm{N}_1=\pi_*(\mathrm{Q})$ is precisely

$$\deg(\mathrm{N}_1)=\sum_{j=1}^{\beta_1}a_j^{(1)}=(k-3)(g-k-1).$$

By Proposition 2.7, all a_i are non-negative. Since each summand of Q corresponds to a non-zero global section of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(2\mathrm{H}-a_j^{(1)}\mathrm{R})$, we get $2\cdot e_1-a_j^{(1)}\geq 0$. Hence $a_j^{(1)}\leq 2$ for all j. It follows that the bundle of quadrics Q is of the following form

$$\mathrm{Q}=\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2\mathrm{H})^{\oplus l_0}\oplus\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2\mathrm{H}+\mathrm{R})^{\oplus l_1}\oplus\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2\mathrm{H}+2\mathrm{R})^{\oplus l_2}.$$

We will describe the possible generators of $\mathscr{I}_{\mathrm{C}/\mathbb{P}(\mathscr{E})}$ in $\mathrm{H}^0(\mathbb{P}(\mathscr{E}),\mathscr{O}_{\mathbb{P}(\mathscr{E})}(2\mathrm{H}-2\mathrm{R}))$. Therefore, we consider the residual line bundle $\omega_{\mathrm{C}}\otimes\mathrm{L}^{-1}$ with

$$h^0(\mathrm{C},\omega_{\mathrm{C}}\otimes\mathrm{L}^{-1})=f=g-k+1\quad\text{and}\quad\deg(\omega_{\mathrm{C}}\otimes\mathrm{L}^{-1})=2g-k-2.$$

By [9, Section 0.b (4)], $|\omega_{\mathrm{C}}\otimes\mathrm{L}^{-1}|$ induces a birational map for $g>k+1$.

Lemma 3.1. *Let $\mathrm{C}'\subset\mathbb{P}^{g-k}$ be the birational image of* C *under the residual linear system $|\omega_{\mathrm{C}}\otimes\mathrm{L}^{-1}|$. There is a one-to-one correspondence between quadratic generators of $\mathrm{C}'\subset\mathbb{P}^{g-k}$ and quadratic generators of $\mathrm{C}\subset\mathbb{P}(\mathscr{E})$ contained in $\mathrm{H}^0(\mathbb{P}(\mathscr{E}),\mathscr{O}_{\mathbb{P}(\mathscr{E})}(2\mathrm{H}-2\mathrm{R}))$.*

Proof. Since $\rho\geq 0$, the scroll X is a cone over the Segre variety $\mathbb{P}^1\times\mathbb{P}^{g-k}$. Let $p:\mathbb{P}(\mathscr{E})\longrightarrow\mathbb{P}^{g-k}$ be the projection on the second factor. An element q of $\mathrm{H}^0(\mathbb{P}(\mathscr{E}),\mathscr{O}_{\mathbb{P}(\mathscr{E})}(2\mathrm{H}-2\mathrm{R}))$ corresponds to a global section of $\mathrm{H}^0(\mathbb{P}^1,S_2(\mathscr{E})\otimes\mathscr{O}_{\mathbb{P}^1}(-2))$ which does not depend on the fiber over $\mathbb{P}^1$. Hence, the image of $\mathrm{V}(q)$ under the projection yields a quadric containing C′. Conversely, the pullback under the projection p of a quadratic generator of $\mathrm{C}'\subset\mathbb{P}^{g-k}$ does not depend on the fiber and has therefore to be contained in $\mathrm{H}^0(\mathbb{P}(\mathscr{E}),\mathscr{O}_{\mathbb{P}(\mathscr{E})}(2\mathrm{H}-2\mathrm{R}))$. □

We are now interested in a bound on k and ρ such that the curve C′ lies on a quadric.

Lemma 3.2. *For a general curve* C *and a general line bundle $\mathrm{L}\in\mathrm{W}^1_k(\mathrm{C})$, the curve $\mathrm{C}'\subset\mathbb{P}^{g-k}$ lies on a quadric if and only if the pair (k,ρ) satisfies the inequality*

$$\left(k-\rho-\frac{7}{2}\right)^2-2k+\frac{23}{4}>0.$$

Proof. By [12], the map

$$\mathrm{H}^0(\mathbb{P}^{g-k},\mathscr{O}_{\mathbb{P}^{g-k}}(2))\to\mathrm{H}^0(\mathrm{C}',\mathscr{O}_{\mathrm{C}'}(2))$$

has maximal rank for a general curve C and a general line bundle $\omega_{\mathrm{C}}\otimes\mathrm{L}^{-1}$. Using the long exact cohomology sequence to the short exact sequence

$$0\to\mathscr{I}_{\mathrm{C}'}(2)\to\mathscr{O}_{\mathbb{P}^{g-k}}(2)\to\mathscr{O}_{\mathrm{C}'}(2)\to 0,$$

we see that C′ lies on a quadric if and only if

$$h^0(\mathbb{P}^{g-k},\mathscr{O}_{\mathbb{P}^{g-k}}(2))-h^0(\mathrm{C}',\mathscr{O}_{\mathrm{C}'}(2))>0.$$

We compute the Hilbert polynomial of C′: $h_{\mathrm{C}'}(n)=(2g-k-2)n+1-g$ and get $h_{\mathrm{C}'}(2)=3g-2k-3$. The dimension of the space of quadrics in $\mathbb{P}^{g-k}$ is $\binom{g-k+2}{2}$. Hence,

$$h^0(\mathbb{P}^{g-k},\mathscr{O}_{\mathbb{P}^{g-k}}(2))-h^0(\mathrm{C}',\mathscr{O}_{\mathrm{C}'}(2))=\binom{g-k+2}{2}-3g+2k+3>0.\quad(2)$$

Expressing g in terms of k and ρ, the inequality (2) is equivalent to

$$\left(k-\rho-\frac{7}{2}\right)^2-2k+\frac{23}{4}>0.$$

□

Proof of the Main Theorem. As mentioned above, the bundle $Q=\pi^*N_1$ is of the form $Q=\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H)^{\oplus l_0}\oplus\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+R)^{\oplus l_1}\oplus\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+2R)^{\oplus l_2}$ (see also Proposition 2.7). By Lemma 3.1, the bundle of quadrics is balanced if no quadratic generator of $C'\subset\mathbb{P}^{g-k}$ exists. So, we are done for pairs (k,ρ) with $\left(k-\rho-\frac{7}{2}\right)^2-2k+\frac{23}{4}\leq 0$.

It remains to show that the bundle of quadrics is unbalanced in the case $\rho>0$ for pairs (k,ρ) satisfying the inequality in Lemma 3.2.

Let k and ρ be non-negative integers satisfying the above inequality, and let $l_2=h^0(C',\mathscr{I}_{C'}(2))=\left(k-\rho-\frac{7}{2}\right)^2-2k+\frac{23}{4}$ be the positive dimension of quadratic generators of the ideal of C'. By Lemma 3.1, the bundle Q is now unbalanced if a summand of the type $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H)$ exists. Such a summand exists if and only if the following inequality holds

$$l_0=\beta_1-l_2-l_1=\beta_1-l_2-\left(\sum_{i=1}^{\beta_1}a_i-2\cdot l_2\right)>0. \tag{3}$$

An easy calculation shows that the inequality (3) is equivalent to

$$l_0=\binom{\rho+1}{2}>0.$$

□

For pairs (k,ρ) in the marked region of Fig. 1, the bundle Q is unbalanced.

Remark 3.3. With our presented method, the whole first linear strand of the resolution of $C'\subset\mathbb{P}^{g-k}$ lifts to the resolution of $C\subset\mathbb{P}(\mathscr{E})$. See also Example 4.1.

4. Example and open problems.

Example 4.1. Using [4], we construct a nodal curve $C\subset\mathbb{P}^{18}$ of genus 19 with a concrete realization of $L\in W^1_{11}(C)$. The ideal of the scroll X swept out by $|L|$ is given by the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_2 & \dots & x_{16}\\ x_1 & x_3 & \dots & x_{17}\end{pmatrix}.$$

The resolution of the birational image C' of C under the map $|\omega_C\otimes L^{-1}|$ has the following Betti table

	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-
1	-	13	9	-	-	-	-	-
2	-	-	91	259	315	197	56	1
3	-	-	-	-	-	-	-	2

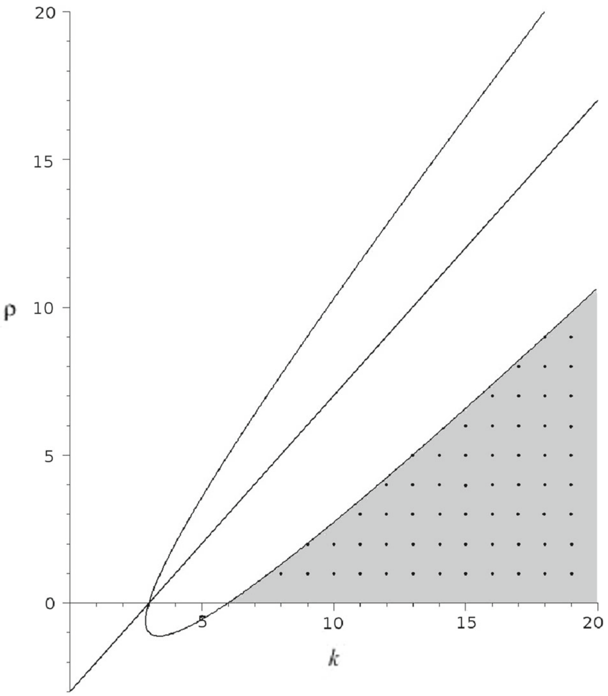

FIGURE 1. *The conic* $\left(k-\rho-\frac{7}{2}\right)^2-2k+\frac{23}{4}=0$ and *the line* $k-\rho-3=0 \Leftrightarrow g=k+1$

Assuming that the relative canonical resolution is as balanced as possible, the first part of the relative canonical resolution is of the following form

$$0 \leftarrow \mathscr{I}_{C/\mathbb{P}(\mathscr{E})} \leftarrow \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+2R)^{\oplus 13} \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H+R)^{\oplus 30} \oplus \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-2H) \end{matrix} \leftarrow \begin{matrix} \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+3R)^{\oplus 9} \\ \oplus \\ \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+2R)^{\oplus 192} \oplus \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-3H+R)^{\oplus 30} \end{matrix} \leftarrow \dots$$

Using the *Macaulay2*-Package [4], our experiments lead to conjecture the following:

Conjecture. (a) *Let* $C \subset \mathbb{P}^{g-1}$ *be a general canonical curve, and let* k *be a positive integer such that* $\rho := \rho(g,k,1) \geq 0$. *Let* $L \in W^1_k(C)$ *be a general point inducing a* $g^1_k = |L|$. *Then for bundles* $N_i = \bigoplus \mathscr{O}_{\mathbb{P}^1}(a^{(i)}_j)$, $i = 2, \dots, \lceil \frac{k-3}{2} \rceil$ *there is the following sharp bound*

$$\max_{j,l} |a^{(i)}_j - a^{(i)}_l| \leq \min\{g-k-1, i+1\}.$$

In particular, if $g-k=2$, *the relative canonical resolution is balanced.*

(b) *For general pairs* (C, g^1_k) *with* $\rho(g,k,1) \leq 0$, *the bundle* N_1 *is balanced.*

Remark 4.2. (a) In order to verify Conjecture (b), it is enough to show the existence of one curve with these properties. With the help of [4], we construct a g-nodal curve on a normalized scroll swept out by a g^1_k and compute the relative canonical resolution. Then, Conjecture (b) is true for

$$(k, \rho) \in \{6, 7, 8, 9\} \times \{-8, -7, \ldots, -1, 0\} \quad \text{where} \quad g = 2k - \rho - 2.$$

(b) We found several examples (e.g. $(g, k) = (17, 7), (19, 8), \ldots$) of g-nodal k-gonal curves where some of the higher syzygy modules $\mathrm{N}_i,\ i \geq 2$ are unbalanced. We believe that the generic relative canonical resolution is unbalanced in these cases.

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