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Maximal regularity for non-autonomous evolution equations governed by forms having less regularity

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Abstract. We consider the maximal regularity problem for non-autonomous evolution equations

$$\begin{aligned} &u'(t) + A(t)\,u(t) = f(t), \quad t \in (0,\tau] \\ &u(0) = u_0. \end{aligned} \tag{0.1}$$

Each operator $A(t)$ is associated with a sesquilinear form $a(t)$ on a Hilbert space H. We assume that these forms all have the same domain V. It is proved in Haak and Ouhabaz (Math Ann, doi:10.1007/s00208-015-1199-7, 2015) that if the forms have some regularity with respect to t (e.g., piecewise α-Hölder continuous for some $\alpha > 1/2$) then the above problem has maximal L_p-regularity for all u_0 in the real-interpolation space $(H, \mathscr{D}(A(0)))_{1-1/p,p}$. In this paper we prove that the regularity required there can be improved for a class of sesquilinear forms. The forms considered here are such that the difference $a(t;.,.) - a(s;.,.)$ is continuous on a larger space than the common domain V. We give three examples which illustrate our results.

Mathematics Subject classification. 35K90, 35K50, 35K45, 47D06.

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1. Introduction and main results. Let H and V be real or complex Hilbert spaces such that V is densely and continuously embedded in H. We denote by V' the (anti-)dual of V and by $[\cdot\,|\,\cdot]_H$ the scalar product of H and $\langle\cdot,\cdot\rangle$ the duality pairing $V' \times V$. The latter satisfies (as usual) $\langle v, h\rangle = [v\,|\,h]_H$ whenever $v \in H$ and $h \in V$. By the standard identification of H with H', we then obtain continuous and dense embeddings $V \hookrightarrow H \eqsim H' \hookrightarrow V'$. We denote by $\|\cdot\|_V$

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and $\|\cdot\|_H$ the norms of V and H, respectively. We shall always assume that H is separable.

We consider the non-autonomous evolution equation

$$\begin{cases} u'(t) + A(t)\,u(t) = f(t), & t \in (0,\tau] \\ u(0) = u_0, \end{cases} \tag{P}$$

where each operator $A(t)$, $t \in [0,\tau]$, is associated with a sesquilinear form $\mathfrak{a}(t)$. We assume that $t \mapsto \mathfrak{a}(t;u,v)$ is measurable for all $u, v \in V$ and

[H1] (constant form domain) $\mathscr{D}(\mathfrak{a}(t)) = V$.

[H2] (uniform boundedness) there exists $M > 0$ such that for all $t \in [0,\tau]$ and $u, v \in V$, we have $|\mathfrak{a}(t;u,v)| \le M\|u\|_V\|v\|_V$.

[H3] (uniform quasi-coercivity) there exist $\alpha_1 > 0$, $\delta \in \mathbb{R}$ such that for all $t \in [0,\tau]$ and all $u, v \in V$ we have $\alpha_1\|u\|_V^2 \le \Re\mathfrak{a}(t;u,u) + \delta\|u\|_H^2$.

For each t, we can associate with the form $\mathfrak{a}(t;\cdot,\cdot)$ an operator $A(t)$ defined as follows

$$\mathscr{D}(A(t)) = \{u \in V, \exists v \in H : \mathfrak{a}(t,u,\varphi) = [v \mid \varphi]_H\ \forall \varphi \in V\}$$
$$A(t)u := v.$$

On the other hand, there exists a linear operator $\mathcal{A}(t) : V \to V'$ such that $\mathfrak{a}(t;u,v) = \langle \mathcal{A}(t)u, v\rangle$ for all $u, v \in V$. The operator $\mathcal{A}(t)$ can be seen as an unbounded operator on V' with domain V and $A(t)$ is the part of $\mathcal{A}(t)$ on H, that is,

$$\mathscr{D}(A(t)) = \{u \in V,\ \mathcal{A}(t)u \in H\}, \quad A(t)u = \mathcal{A}(t)u.$$

It is a known fact that $-A(t)$ and $-\mathcal{A}(t)$ both generate holomorphic semigroups $(e^{-s\,A(t)})_{s\ge 0}$ and $(e^{-s\,\mathcal{A}(t)})_{s\ge 0}$ on H and V', respectively. For each $s \ge 0$, $e^{-s\,A(t)}$ is the restriction of $e^{-s\,\mathcal{A}(t)}$ to H. For all this, we refer to Ouhabaz [10, Chapter 1].

The notion of maximal L_p-regularity for the above Cauchy problem is defined as follows.

Definition 1.1. Fix $u_0 \in H$. We say that (P) has maximal L_p-regularity (in H) if for each $f \in L_p(0,\tau;H)$ there exists a unique $u \in W_p^1(0,\tau;H)$ such that $u(t) \in \mathscr{D}(A(t))$ for almost all t, which satisfies (P) in the L_p-sense.

Recall that under the assumptions [H1]–[H3], J.L. Lions proved maximal L_2-regularity in V' for all initial data $u_0 \in H$, see e.g. [8], [12, page 112]. This means that for every $u_0 \in H$ and $f \in L_2(0,\tau;V')$, the equation

$$\begin{cases} u'(t) + \mathcal{A}(t)\,u(t) = f(t) \\ u(0) = u_0 \end{cases} \tag{P$'$}$$

has a unique solution $u \in W_2^1(0,\tau;V') \cap L_2(0,\tau;V)$. It is a remarkable fact that Lions's theorem does not require any regularity assumption (with respect to t) on the sesquilinear forms apart from measurability. Note however that maximal regularity in H differs considerably from maximal regularity in V'. The fact that the forms have the same domain means that the operators $\mathcal{A}(t)$ have constant domains in V', and this fact plays an important role in proving maximal regularity. The operators $A(t)$ may have different domains as operators on H. The problem of maximal regularity in H for (P) was stated explicitly by Lions, and it is still open in general. Some progress has been made in recent years.

First, recall that Bardos [4] proved maximal L_2-regularity in H with initial data $u_0 \in V$ provided $\mathscr{D}(A(t)^{1/2}) = V$ as space and topologically and assuming that $t \mapsto \mathfrak{a}(t; u, v)$ is C^1 on $[0, \tau]$. His result was extended in Arendt et al. [2] for Lipschitz forms (with respect to t) and allowing a multiplicative perturbation by bounded operators $B(t)$ which are measurable in t. The maximal L_2-regularity is then proved for the evolution problem associated with $B(t)A(t)$. Ouhabaz and Spina [11] proved maximal L_p-regularity on H for all $p \in (1, \infty)$ under the assumption that $t \mapsto \mathfrak{a}(t; u, v)$ is α-Hölder continuous for some $\alpha > 1/2$. The result in [11] concerns the problem (P) with initial data $u(0) = 0$. A simple example was given recently by Dier [5] which shows that in general the answer to Lions' problem is negative. The following positive result was proved by Haak and Ouhabaz [7].

Theorem 1.2. *Suppose that the forms* $(\mathfrak{a}(t))_{0\le t\le\tau}$ *satisfy the hypotheses* [H1]–[H3] *and the regularity condition*

$$|\mathfrak{a}(t; u, v) - \mathfrak{a}(s; u, v)| \le \omega(|t-s|)\, \|u\|_V \|v\|_V, \tag{1.1}$$

where $\omega : [0, \tau] \to [0, \infty)$ *is a non-decreasing function such that*

$$\int_0^\tau \frac{\omega(t)}{t^{3/2}}\, \mathrm{d}t < \infty. \tag{1.2}$$

Then the Cauchy problem (P) *with* $u_0 = 0$ *has maximal* L_p*-regularity in* H *for all* $p \in (1, \infty)$*. If in addition* ω *satisfies the p-Dini condition*

$$\int_0^\tau \left(\frac{\omega(t)}{t}\right)^p \mathrm{d}t < \infty, \tag{1.3}$$

then (P) *has maximal* L_p*-regularity for all* $u_0 \in (H, \mathscr{D}(A(0)))_{1-1/p,p}$*. Moreover, there exists a positive constant* C *such that*

$$\begin{aligned}&\|u\|_{L_p(0,\tau;H)} + \|u'\|_{L_p(0,\tau;H)} + \|A(\cdot)u(\cdot)\|_{L_p(0,\tau;H)}\\ &\quad\le C\left[\|f\|_{L_p(0,\tau;H)} + \|u_0\|_{(H,\mathscr{D}(A(0)))_{1-1/p,p}}\right].\end{aligned}$$

In this theorem, $(H, \mathscr{D}(A(0)))_{1-1/p,p}$ denotes the classical real-interpolation space, see [13, Chapter 1.13] or [9, Proposition 6.2].

In the case where $p = 2$, we obtain maximal L_2-regularity for $u(0) \in \mathscr{D}((\delta + A(0))^{1/2})$. The theorem can be used in the case where $t \mapsto \mathfrak{a}(t; u, v)$ is α-Hölder continuous for some $\alpha > \frac{1}{2}$. The case of piecewise α-Hölder continuous is also covered. See [7] for the details.

The aim of the present paper is to weaken the regularity assumption measured by (1.2) and (1.3) in some situations. More precisely, we assume in addition to [H1]–[H3] that there exist $\beta, \gamma \in [0, 1]$ such that for all $u, v \in V$

$$|\mathfrak{a}(t;u,v) - \mathfrak{a}(s;u,v)| \le \omega(|t-s|)\,\|u\|_{V_\beta}\|v\|_{V_\gamma}, \tag{1.4}$$

where $V_\beta := [H,V]_\beta$ is the classical complex interpolation space for $\beta \in [0,1]$ with $V_0 = H$ and $V_1 = V$. If $\beta, \gamma \in (0,1)$, the assumption (1.4) means that the difference of the forms is defined on a larger space than the common form domain V.

Our main result is the following.

Theorem 1.3. *Suppose that the forms $(\mathfrak{a}(t))_{0\le t\le\tau}$ satisfy the hypotheses* [H1]–[H3] *and* (1.4)*, where $\omega : [0,\tau] \to [0,\infty)$ is a non-decreasing function such that*

$$\int_0^\tau \frac{\omega(t)}{t^{1+\frac{\gamma}{2}}}\,\mathrm{d}t < \infty. \tag{1.5}$$

Then the Cauchy problem (P) *with $u_0 = 0$ has maximal L_p-regularity in H for all $p \in (1,\infty)$. If in addition ω satisfies the p-Dini condition*

$$\int_0^\tau \left(\frac{\omega(t)}{t^{\frac{\beta+\gamma}{2}}}\right)^p \mathrm{d}t < \infty, \tag{1.6}$$

then (P) *has maximal L_p-regularity for all $u_0 \in (H, \mathscr{D}(A(0)))_{1-1/p,p}$. Moreover, there exists a positive constant C such that*

$$\|u\|_{L_p(0,\tau;H)} + \|u'\|_{L_p(0,\tau;H)} + \|A(\cdot)u(\cdot)\|_{L_p(0,\tau;H)} \le C\left[\|f\|_{L_p(0,\tau;H)} + \|u_0\|_{(H,\mathscr{D}(A(0)))_{1-1/p,p}}\right].$$

A related result was proved recently by Arendt and Monniaux [3] who prove maximal L_2-regularity under the additional condition that $\beta = \gamma$ in (1.4). We observe that in our result β does not come into play if $u_0 = 0$. We expect the theorem to be true with $\min(\beta,\gamma)$ in place of γ in (1.5).

The following two corollaries follow immediately from the theorem.

Corollary 1.4. *Suppose that the forms $(\mathfrak{a}(t))_{0\le t\le\tau}$ satisfy the hypotheses* [H1]–[H3] *and α-Hölder continuous in the sense that*

$$|\mathfrak{a}(t,u,v) - \mathfrak{a}(s,u,v)| \le C|t-s|^\alpha \|u\|_{V_\beta}\|v\|_{V_\gamma}. \tag{1.7}$$

Then the Cauchy problem (P) *with $u_0 = 0$ has maximal L_p-regularity in H for all $p \in (1,\infty)$ provided $\alpha > \frac{\gamma}{2}$. If in addition $\alpha > \frac{\beta+\gamma}{2} - \frac{1}{p}$, then* (P) *has maximal L_p-regularity for all $u_0 \in (H, \mathscr{D}(A(0)))_{1-1/p,p}$. Moreover, there exists a positive constant C such that*

$$\|u\|_{L_p(0,\tau;H)} + \|u'\|_{L_p(0,\tau;H)} + \|A(\cdot)u(\cdot)\|_{L_p(0,\tau;H)} \le C\left[\|f\|_{L_p(0,\tau;H)} + \|u_0\|_{(H,\mathscr{D}(A(0)))_{1-1/p,p}}\right].$$

Corollary 1.5. *Suppose that the forms $(\mathfrak{a}(t))_{0\le t\le\tau}$ satisfy the hypotheses* [H1]–[H3] *and are α-Hölder continuous in the sense that*

$$|\mathfrak{a}(t,u,v) - \mathfrak{a}(s,u,v)| \le C|t-s|^\alpha \|u\|_{V_\beta}\|v\|_{V_\gamma}, \tag{1.8}$$

for some $\alpha > \frac{\gamma}{2}$. *Then the Cauchy problem* (P) *has maximal* L_2*-regularity in* H *for all* $u_0 \in \mathscr{D}((\delta + A(0))^{1/2})$. *Moreover, there exists a positive constant* C *such that*

$$\begin{aligned}&\|u\|_{L_2(0,\tau;H)} + \|u'\|_{L_2(0,\tau;H)} + \|A(\cdot)u(\cdot)\|_{L_2(0,\tau;H)}\\ &\quad\leq C\left[\|f\|_{L_2(0,\tau;H)} + \|(\delta + A(0))^{1/2}u_0\|_H\right].\end{aligned}$$

Notation We shall often use C or C' to denote all inessential constants. We use $W_p^1(0,\tau;E)$ as well as $H^s(\Omega) := W_2^s(\Omega)$ for the classical Sobolev spaces. The first one is the Sobolev space of order one of L_p-functions on $(0,\tau)$ with values in a Banach space E, and the second one is the Sobolev space of order s of L_2 scalar-valued functions acting on a domain Ω.

2. Proof of the main result. Throughout this section we adopt the notation of the introduction. We shall use the strategy and ideas of Proof of Theorem 1.2 in [7] with some modifications in order to incorporate the additional assumption (1.4).

Recall that the solution u to (P) exists in V' by Lions' theorem mentioned in the introduction. The aim is to prove that $u(t) \in \mathscr{D}(A(t))$ for almost all $t \in [0,\tau]$ and $A(.)u(.) \in L_p(0,\tau;H)$. From this and the Cauchy problem (P), it follows that $u \in W_p^1(0,\tau;H)$.

From now on we assume without loss of generality that the forms are coercive, that is [H3] holds with $\delta = 0$. The reason is that by replacing $A(t)$ by $A(t) + \delta$, the solution v of (P) is $v(t) = u(t)e^{-\delta t}$ and it is clear that $u \in W_p^1(0,\tau;H)$ *if and only if* $v \in W_p^1(0,\tau;H)$.

First we have the representation formula (see [7] for all what follows)

$$\begin{aligned}u(t) = &\int_0^t e^{-(t-s)\mathcal{A}(t)}(\mathcal{A}(t)-\mathcal{A}(s))u(s)\,\mathrm{d}s\\ &+\int_0^t e^{-(t-s)\mathcal{A}(t)}f(s)\,\mathrm{d}s + e^{-t\,\mathcal{A}(t)}u_0.\end{aligned} \tag{2.1}$$

In addition,

$$\mathcal{A}(t)u(t) = (Q\mathcal{A}(\cdot)u(\cdot))(t) + (Lf)(t) + (Ru_0)(t), \tag{2.2}$$

where

$$(Qg)(t) := \int_0^t \mathcal{A}(t)e^{-(t-s)\mathcal{A}(t)}(\mathcal{A}(t) - \mathcal{A}(s))\,\mathcal{A}(s)^{-1}g(s)\,\mathrm{d}s$$

$$(Lg)(t) := \mathcal{A}(t)\int_0^t e^{-(t-s)\mathcal{A}(t)}g(s)\,\mathrm{d}s \quad \text{and} \quad (Ru_0)(t) := \mathcal{A}(t)e^{-t\,\mathcal{A}(t)}u_0.$$

The aim is to prove boundedness on $L_p(0,\tau;H)$ of the operators L, R, and Q and then, by a simple scaling argument, the norm of Q is less than 1.

This allows us to invert $(I - Q)$ on $L_p(0, \tau; H)$ and conclude from (2.2) that $A(.)u(.) \in L_p(0, \tau; H)$.

We start with the operator L. The following result is Lemma 11 in [7].

Lemma 2.1. *Suppose that in addition to the assumptions* [H1]–[H3] *that* (1.4) *holds for some* $\beta, \gamma \in [0, 1]$ *and* $\omega : [0, \tau] \to [0, \infty)$ *a non-decreasing function such that*

$$\int_0^\tau \frac{\omega(t)^2}{t}\,\mathrm{d}t < \infty. \tag{2.3}$$

Then L *is a bounded operator on* $L_p(0, \tau; H)$ *for all* $p \in (1, \infty)$.

Now we deal with the operator R.

Recall first that $-A(t)$ is the generator of a bounded holomorphic semigoup of angle $\frac{\pi}{2} - \arctan(\frac{M}{\alpha_0})$ where α_0 and M are as in the assumptions [H2] and [H3]. See [10, Chapter 1] or [7]. In addition we have

Lemma 2.2. *Let* $\omega : \mathbb{R} \to \mathbb{R}_+$ *be some function, and assume that*

$$|\mathfrak{a}(t; u, v) - \mathfrak{a}(s; u, v)| \le \omega(|t-s|)\|u\|_{V_\beta}\|v\|_{V_\gamma}$$

for all $u, v \in V$*. Then*

$$\|R(z, A(t)) - R(z, A(s))\|_{\mathcal{B}(H)} \le \frac{c_\theta}{|z|^{1-\frac{\beta+\gamma}{2}}}\omega(|t-s|)$$

for all $z \notin S_\theta$ *with any fixed* $\theta > \arctan(M/\alpha)$*. The constant* c_θ *is independent of* $z, t,$ *and* s.

Proof. Fix $\theta > \arctan(M/\alpha)$. Note that (see [7], Proposition 6d)

$$\|(z - A(t))^{-1}x\|_V \le \frac{C_\theta}{\sqrt{|z|}}\|x\|_H \text{ for all } \ z \notin S_\theta. \tag{2.4}$$

Observe that for $u, v \in V$,

$$\begin{aligned}
&|[R(z, A(t))u - R(z, A(s))u \mid v]_H| \\
&\quad = |[R(z, A(t))(A(s) - A(t))R(z, A(s))u \mid v]_H| \\
&\quad = |[A(s)R(z, A(s))u \mid R(z, A(t))^* v]_H - [A(t)R(z, A(s))u \mid R(z, A(t))^* v]_H| \\
&\quad = |\mathfrak{a}(s; R(z, A(s))u, R(z, A(t))^* v) - \mathfrak{a}(t; R(z, A(s))u, R(z, A(t))^* v)| \\
&\quad \le \omega(|t - s|)\|R(z, A(s))u\|_{V_\beta}\|R(z, A(t))^* v\|_{V_\gamma} \\
&\quad \le \frac{c_\theta}{|z|^{2-\frac{\beta+\gamma}{2}}}\omega(|t-s|)\,\|u\|_H\,\|v\|_H.
\end{aligned}$$

Here we used the estimate $\|R(z, A(s))u\|_{V_\beta} \le \frac{c_\theta}{|z|^{1-\frac{\beta}{2}}}\|u\|_H$ which follows from (2.4) and $\|R(z, A(s))u\|_H \le \frac{c_\theta}{|z|}\|u\|_H$ by complex interpolation since $V_\beta := [H, V]_\beta$. A similar estimate holds for $\|R(z, A(t))^* v\|_{V_\gamma}$. □

Lemma 2.3. *Assume* (1.6)*. Then there exists* $C > 0$ *such that*

$$\|Ru_0\|_{L_p(0,\tau;H)} \le C\|u_0\|_{(H,\mathscr{D}(A(0)))_{1-1/p,p}}$$

for all $u_0 \in (H, \mathscr{D}(A(0)))_{1-1/p,p}$.

Proof. Recall that the operator R is given by $(Rg)(t) = A(t)e^{-t\,A(t)}g$ for $g \in H$. Let

$$(R_0 g)(t) := A(0)e^{-t\,A(0)}g.$$

We estimate the difference $(R - R_0)g$. Let $v \in H$ and $\Gamma = \partial S_\theta$ with $\theta < {}^{\pi}\!/_2$ as in (2.4). Then the functional calculus for the sectorial operators $A(t)$ and $A(0)$ gives

$$\begin{aligned}
&\Big[A(t)e^{-t\,A(t)}g - A(0)e^{-t\,A(0)}g \,|\, v\Big]_H \\
&\quad = \tfrac{1}{2\pi i}\int_\Gamma \big[ze^{-tz}\big[R(z,A(t)) - R(z,A(0))\big]g \,|\, v\big]_H \,\mathrm{d}z \\
&\quad = \tfrac{1}{2\pi i}\int_\Gamma \big[ze^{-tz}R(z,\mathcal{A}(t))\big[\mathcal{A}(0) - \mathcal{A}(t)\big]R(z,A(0))g \,|\, v\big]_H \,\mathrm{d}z \\
&\quad = \tfrac{1}{2\pi i}\int_\Gamma \big[ze^{-tz}\big[\mathcal{A}(0) - \mathcal{A}(t)\big]R(z,A(0))g \,|\, R(z,A(t))^* v\big]_H \,\mathrm{d}z \\
&\quad = \tfrac{1}{2\pi i}\int_\Gamma ze^{-tz}\big[\mathfrak{a}(0; R(z,A(0))g, R(z,A(t))^* v) \\
&\qquad - \mathfrak{a}(t; R(z,A(0))g, R(z,A(t))^* v)\big]\,\mathrm{d}z.
\end{aligned}$$

It follows from (1.4) and Lemma 2.2 that

$$\begin{aligned}
&|[(Rg - R_0 g)(t) \,|\, v]_H| \\
&\quad \le \tfrac{1}{2\pi}\int_\Gamma \omega(t)|z|e^{-t\,\Re(z)}\|R(z,A(0))g\|_{V_\beta}\|R(z,A(t))^* v\|_{V_\gamma}\,|\mathrm{d}z| \\
&\quad \le C\omega(t)\|g\|_H\|v\|_H \int_\Gamma |z|^{\frac{\beta+\gamma}{2}-1}e^{-t\,\Re z}\,|\mathrm{d}z| \\
&\quad \le C' \frac{\omega(t)}{t^{\frac{\beta+\gamma}{2}}}\|g\|_H\|v\|_H.
\end{aligned}$$

Since this is true for all $v \in H$, we conclude that

$$\|(Ru_0)(t) - (R_0 u_0)(t)\|_H \le C' \frac{\omega(t)}{t^{\frac{\beta+\gamma}{2}}}\|u_0\|_H. \tag{2.5}$$

From the hypothesis (1.6), it follows that $Ru_0 - R_0 u_0 \in L_p(0,\tau;H)$. On the other hand, since $A(0)$ is invertible, it is well-known that $A(0)e^{-t\,A(0)}u_0 \in L_p(0,\tau;H)$ if and only if $u_0 \in (H, \mathscr{D}(A(0)))_{1-{}^1\!/_p,p}$ (see Triebel [13, Section 1.14.5]). Therefore, $Ru_0 \in L_p(0,\tau;H)$ and the lemma is proved. □

Proof of Theorem 1.3 As we already mentioned before, the arguments are essentially the same as in [7] in which we use the additional assumption (1.4) to weaken the required regularity on the forms.

We start with the case $u_0 = 0$ and let $f \in C_c^\infty(0,\tau;H)$. From (2.2) we have

$$(I - Q)A(\cdot)u(\cdot) = Lf(\cdot). \tag{2.6}$$

Recall that L is bounded on $L_p(0,\tau;H)$ by Lemma 2.1. We shall now prove that Q is bounded on $L_p(0,\tau;H)$. Let $g \in L_2(0,\tau;H)$ and $v \in H$. We have

$$
\begin{aligned}
&|\,[Qg(t)\,|\,v]_H\,| \\
&\quad= \int_0^t \big[\mathfrak{a}(t;\mathcal{A}(s)^{-1}g(s),\mathcal{A}(t)^* e^{-(t-s)\mathcal{A}(t)^*)}v) \qquad (2.7)\\
&\qquad -\mathfrak{a}(s;\mathcal{A}(s)^{-1}g(s),\mathcal{A}(t)^* e^{-(t-s)\mathcal{A}(t)^*)}v)\big]\,\mathrm{d}s \\
&\quad\le \int_0^t \omega(|t-s|)\|\mathcal{A}(s)^{-1}g(s)\|_{V_\beta}\|\mathcal{A}(t)^* e^{-(t-s)\mathcal{A}(t)^*)}v\|_{V_\gamma}\,\mathrm{d}s. \qquad (2.8)
\end{aligned}
$$

By the coercivity assumption, one has for all $s > 0$

$$
\begin{aligned}
\alpha_1\|\mathcal{A}(t)e^{-s\mathcal{A}(t)}v\|_V^2 &\le \Re\mathfrak{a}(t, A(t)e^{-sA(t)}v, A(t)e^{-sA(t)}v)\\
&\le \|A(t)^2e^{-sA(t)}v\|_H\|A(t)e^{-sA(t)}v\|_H.
\end{aligned}
$$

On the other hand, $\|A(t)e^{-sA(t)}v\|_H \le \frac{C}{s}\|v\|_H$ (see Proposition 6b) in [7]) and $\|A(t)^2e^{-sA(t)}v\|_H = \|A(t)e^{-\frac{s}{2}A(t)}A(t)e^{-\frac{s}{2}A(t)}v\|_H \le \frac{C'}{s^2}\|v\|_H$. Hence

$$\|\mathcal{A}(t)e^{-s\mathcal{A}(t)}v\|_V \le \frac{C}{s^{\frac{3}{2}}}\|v\|_H.$$

Using this and again $\|A(t)e^{-sA(t)}v\|_H \le \frac{C}{s}\|v\|_H$, it follows by complex interpolation that

$$\|\mathcal{A}(t)e^{-s\mathcal{A}(t)}v\|_{V_\gamma} \le \frac{C}{s^{1+\frac{\gamma}{2}}}\|v\|_H. \qquad (2.9)$$

The constant C is independent of $t, s,$ and v. The adjoint operators $\mathcal{A}(t)^*$ satisfy the same estimates.

Now we estimate $\|\mathcal{A}(s)^{-1}g(s)\|_{V_\beta}$. By coercivity

$$
\begin{aligned}
\alpha_1\|\mathcal{A}(s)^{-1}g(s)\|_V^2 &\le \Re\mathfrak{a}(s;\mathcal{A}(s)^{-1}g(s),\mathcal{A}(s)^{-1}g(s))\\
&= \Re\langle\mathcal{A}(s)\mathcal{A}(s)^{-1}g(s),\mathcal{A}(s)^{-1}g(s)\rangle\\
&= \Re\left[g(s)\,|\,\mathcal{A}(s)^{-1}g(s)\right]_H\\
&\le \|g(s)\|_H^2\|\mathcal{A}(s)^{-1}\|_{\mathcal{B}(H)}.
\end{aligned}
$$

Hence

$$\|\mathcal{A}(s)^{-1}g(s)\|_{V_\beta}^2 \le C\|\mathcal{A}(s)^{-1}g(s)\|_V^2 \le C\|g(s)\|_H^2\|\mathcal{A}(s)^{-1}\|_{\mathcal{B}(H)}.$$

Inserting this and (2.9) (for the adjoint operators) in (2.8), we obtain

$$\|(Qg)(t)\|_H \le \int_0^t \frac{C'}{(t-s)^{1+\gamma/2}}\,\omega(t-s)\,\|\mathcal{A}(s)^{-1}\|_{\mathcal{B}(H)}^{1/2}\|g(s)\|_H\,\mathrm{d}s. \qquad (2.10)$$

Now, once we replace $A(s)$ by $A(s)+\mu$, (2.9) is valid with a constant independent of $\mu \ge 0$ and using the estimate

$$\|(\mathcal{A}(s)+\mu)^{-1}\|_{\mathcal{B}(H)} \le \frac{1}{\mu},$$

in (2.10) for $A(s)+\mu$ we see that

$$\|(Qg)(t)\|_H \leq \frac{C'}{\sqrt{\mu}} \int_0^t \frac{\omega(t-s)}{(t-s)^{1+\gamma/2}} \|g(s)\|_H \,\mathrm{d}s.$$

The operator S defined by

$$Sh(t) := \int_0^t \frac{\omega(t-s)}{(t-s)^{1+\gamma/2}} h(s)\,\mathrm{d}s$$

is bounded on $L_p(0,\tau;\mathbb{R})$ as a convolution by an L_1-kernel (here we use (1.5)). It follows that Q is bounded on $L_p(0,\tau;H)$ with norm of at most $\frac{C''}{\sqrt{\mu}}$ for some constant C''. Taking then μ large enough makes Q strictly contractive such that $(I-Q)^{-1}$ is bounded on $L_p(0,\tau;H)$. Then, for $f \in C_c^\infty(0,\tau;H)$, (2.6) can be rewritten as

$$A(\cdot)u(\cdot) = (I-Q)^{-1}Lf(\cdot).$$

This shows that $u(t) \in \mathscr{D}(A(t))$ for almost all t and $A(\cdot)u(\cdot) \in L_p(0,\tau;H)$.

For general $u_0 \in (H, \mathscr{D}(A(0)))_{1-1/p,p}$ we suppose in addition to (1.5) that (1.6) holds. Lemma 2.3 shows that $Ru_0 \in L_p(0,\tau;H)$. As previously we conclude that

$$A(\cdot)u(\cdot) = (I-Q)^{-1}(Lf + Ru_0)$$

whenever $f \in C_c^\infty(0,\tau;H)$. Thus taking the L_p norm yields

$$\|A(\cdot)u(\cdot)\|_{L_p(0,\tau;H)} \leq C\|(Lf + Ru_0)\|_{L_p(0,\tau;H)}.$$

We use again the previous estimates on L and R to obtain

$$\|A(\cdot)u(\cdot)\|_{L_p(0,\tau;H)} \leq C' \left[\|f\|_{L_p(0,\tau;H)} + \|u_0\|_{(H,\mathscr{D}(A(0)))_{1-1/p,p}}\right].$$

Using the Eq. (P), we obtain a similar estimate for u' and so

$$\begin{aligned}&\|u'(\cdot)\|_{L_p(0,\tau;H)} + \|A(\cdot)u(\cdot)\|_{L_p(0,\tau;H)}\\ &\quad\leq C'' \left[\|f\|_{L_p(0,\tau;H)} + \|u_0\|_{(H,\mathscr{D}(A(0)))_{1-1/p,p}}\right].\end{aligned}$$

We write $u(t) = A(t)^{-1}A(t)u(t)$ and use once again the fact that the norms of $A(t)^{-1}$ on H are uniformly bounded, we obtain

$$\begin{aligned}&\|u(t)\|_{L_p(0,\tau;H)} \leq C_1\|A(\cdot)u(\cdot)\|_{L_p(0,\tau;H)}\\ &\quad\leq C_2 \left[\|f\|_{L_p(0,\tau;H)} + \|u_0\|_{(H,\mathscr{D}(A(0)))_{1-1/p,p}}\right].\end{aligned}$$

We conclude therefore that the following a priori estimate holds

$$\begin{aligned}&\|u\|_{L_p(0,\tau;H)} + \|u'\|_{L_p(0,\tau;H)} + \|A(\cdot)u(\cdot)\|_{L_p(0,\tau;H)}\\ &\quad\leq C \left[\|f\|_{L_p(0,\tau;H)} + \|u_0\|_{(H,\mathscr{D}(A(0)))_{1-1/p,p}}\right],\end{aligned} \tag{2.11}$$

where the constant C does not depend on $f \in C_c^\infty(0,\tau;H)$.

The latter estimate extends by density to all $f \in L_p(0,\tau;H)$ (see [7]). This proves the desired maximal L_p-regularity property. □

3. Examples.

3.1. Schrödinger operators with time dependent potentials. We consider on $H = L^2(\mathbb{R}^d)$ Schrödinger operators $A(t) = -\Delta + m(t, .)$ with time dependent potentials $m(t, x)$. We make the following assumptions:

- There exists a non-negative function $m_0 \in L_{1,loc}$ and two positive constants c_1, c_2 such that
$$c_1 m_0(x) \le m(t, x) \le c_2 m_0(x), \quad x \in \mathbb{R}^d, \quad t \in [0, \tau]. \tag{3.1}$$
- There exists a function $p_0 \in L_{1,loc}$ such that
$$|m(t, x) - m(s, x)| \le |t - s|^\alpha p_0(x), \quad x \in \mathbb{R}^d, \quad t, s \in [0, \tau]. \tag{3.2}$$
- There exists $C > 0$ and $s \in [0, 1]$ such that
$$\int_{\mathbb{R}^d} p_0(x)|u(x)|^2 \,\mathrm{d}x \le C\|u\|_{H^s(\mathbb{R}^d)}, \quad u \in C_c^\infty. \tag{3.3}$$

Note that assumption (3.3) is satisfied for several weights p_0. For example, this is the case for $p_0 = \frac{1}{|x|^2}$ and $s = 1$ by Hardy's inequality. On the other hand, by Hölder's inequality and classical Sobolev embeddings for H^s, one finds r_s such that (3.3) holds for $p_0 \in L_{r_s}$. Obviously, (3.3) holds with $s = 0$ if $p_0 \in L_\infty$.

The operator $A(t) = -\Delta + m(t, x)$ is defined as the operator associated with the form
$$\mathfrak{a}(t; u, v) = \int_{\mathbb{R}^d} \nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\mathbb{R}^d} m(t, .) uv \,\mathrm{d}x$$
defined on
$$V = \left\{ u \in H^1(\mathbb{R}^d), \int_{\mathbb{R}^d} m_0|u|^2 \,\mathrm{d}x < \infty \right\}.$$
The forms $\mathfrak{a}(t; \cdot, \cdot)$ satisfy the standard assumptions [H1]–[H3]. Using the additional assumption (3.3), we can estimate the difference $\mathfrak{a}(t; u, v) - \mathfrak{a}(s; u, v)$ as follows
$$\begin{aligned}
|\mathfrak{a}(t; u, v) - \mathfrak{a}(s; u, v)| &= |\int_{\mathbb{R}^d} [m(t, .) - m(s, .)] uv \,\mathrm{d}x| \\
&\le |t - s|^\alpha \int_{\mathbb{R}^d} p_0(x)|uv| \,\mathrm{d}x \\
&\le |t - s|^\alpha (\int_{\mathbb{R}^d} p_0(x)|u|^2 \,\mathrm{d}x)^{1/2} \left(\int_{\mathbb{R}^d} p_0(x)|v|^2 \,\mathrm{d}x \right)^{1/2} \\
&\le C|t - s|^\alpha \|u\|_{H^s(\mathbb{R}^d)} \|v\|_{H^s(\mathbb{R}^d)}.
\end{aligned}$$
Therefore we can apply Theorem 1.3 to obtain maximal L_p-regularity for the evolution equation associated with $A(t) = -\Delta + m(t, .)$ under the condition

$\alpha > s/2$, where α and s are as in (3.2) and (3.3). For $p = 2$, the initial data u_0 can be taken in $V = \mathscr{D}(A(0)^{1/2})$. For $p \neq 2$ we assume $u_0 \in (H, \mathscr{D}(A(0)))_{1-1/p,p}$ and $\alpha > \max(s/2, s - 1/p)$ by condition (1.6).

3.2. Elliptic operators with Robin boundary conditions. Let Ω be a bounded domain of $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$. We denote by Tr the classical trace operator. Let $\beta : [0,\tau] \times \partial\Omega \to [0,\infty)$ and $a_k : [0,\tau] \times \Omega \to \mathbb{R}$ be bounded measurable functions for $k = 1, \cdots, d$ such that

$$|\beta(t,x) - \beta(s,x)| \leq C|t-s|^\alpha, \quad t,s \in [0,\tau], \quad x \in \partial\Omega$$

and

$$|a_k(t,x) - a_k(s,x)| \leq C|t-s|^\alpha, \quad t,s \in [0,\tau], \quad x \in \Omega.$$

We define the form

$$\mathfrak{a}(t;u,v) := \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \sum_{k=1}^d \int_\Omega a_k(t,x)\partial_k u \cdot v \,\mathrm{d}x + \int_{\partial\Omega} \beta(t,\cdot)\mathrm{Tr}(u)\mathrm{Tr}(v)\,\mathrm{d}\sigma,$$

for all $u, v \in H^1(\Omega)$. The associated operator $A(t)$ is formally given by

$$A(t) = -\Delta + \sum_{k=1}^d a_k(t,x)\partial_k u$$

and subject to the time dependent Robin boundary condition:

$$\tfrac{\partial u}{\partial n} + \beta(t,\cdot)u = 0 \quad \text{on} \quad \partial\Omega.$$

Here $\frac{\partial u}{\partial n}$ denotes the normal derivative.

Now we check (1.4). We have for $u, v \in H^1(\Omega)$,

$$\begin{aligned}
&|\mathfrak{a}(t;u,v) - \mathfrak{a}(s;u,v)| \\
&\quad = \left| \sum_{k=1}^d \int_\Omega [a_k(t,\cdot) - a_k(s,\cdot)]\partial_k u \cdot v \,\mathrm{d}x + \int_{\partial\Omega} [\beta(t,\cdot) - \beta(s,\cdot)]\mathrm{Tr}(u)\mathrm{Tr}(v)\,\mathrm{d}\sigma \right| \\
&\quad \leq C|t-s|^\alpha \left(\|u\|_{H^1(\Omega)} + \|u\|_{H^{1/2+\varepsilon}(\Omega)} \|v\|_{H^{1/2+\varepsilon}(\Omega)} \right),
\end{aligned}$$

where we used the fat that the trace operator is bounded from $H^{1/2+\varepsilon}(\Omega)$ into $L_2(\partial\Omega)$ for $\varepsilon > 0$. Hence

$$|\mathfrak{a}(t;u,v) - \mathfrak{a}(s;u,v)| \leq C|t-s|^\alpha \|u\|_{H^1(\Omega)} \|v\|_{H^{1/2+\varepsilon}(\Omega)}.$$

We apply Theorem 1.3 or the subsequent corollaries to obtain maximal L_2-regularity for the corresponding evolution equation under the condition $\alpha > 1/4$ for initial data $u(0) \in H^1(\Omega) = \mathscr{D}(A(0)^{1/2})$. We also have maximal L_p-regularity for $1 < p < \infty$ if $\alpha > \max(\frac{1}{4}, \frac{3}{4} - \frac{1}{p})$ and $u(0) \in (H, \mathscr{D}(A(0)))_{1-1/p,p}$. In the case $p = 2$ and $a_k = 0$, this result was proved in [3].

3.3. Elliptic operators with Wentzell boundary conditions. We wish to consider the heat equation with time dependent Wentzell boundary conditions:

$$\beta(t,\cdot)u + \frac{\partial u}{\partial n} + \Delta u = 0 \quad \text{on} \quad \partial\Omega. \tag{3.4}$$

As in the previous example, we assume that Ω is a bounded Lipschitz domain and $\beta : [0,\tau] \times \partial\Omega \to [0,\infty)$ is a bounded measurable function such that

$$|\beta(t,x) - \beta(s,x)| \le C|t-s|^{\alpha}, \quad t,s \in [0,\tau], \quad x \in \partial\Omega.$$

In order to consider the Laplacian with Wentzell boundary conditions, it is convenient to work on $H := L_2(\Omega) \oplus L_2(\partial\Omega)$ (see [1] or [6]). Set

$$V = \{(u, \mathrm{Tr}(u)), u \in H^1(\Omega)\}.$$

The Hilbert space V is endowed with the norm

$$\|(u, \mathrm{Tr}(u))\|_V := \left(\|u\|^2_{H^1(\Omega)} + \|\mathrm{Tr}(u)\|^2_{L_2(\partial\Omega)}\right)^{1/2}.$$

We define the forms

$$\mathfrak{a}(t; (u, \mathrm{Tr}(u)), (v, \mathrm{Tr}(v)) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x + \int_{\partial\Omega} \beta(t,\cdot)\mathrm{Tr}(u)\mathrm{Tr}(v)\, \mathrm{d}\sigma,$$

for $u, v \in H^1(\Omega)$. The forms $\mathfrak{a}(t)$ are well defined on V and satisfy the assumptions [H1]–[H3]. In addition,

$$\begin{aligned}
&|\mathfrak{a}(t; (u, \mathrm{Tr}(u)), (v, \mathrm{Tr}(v)) - \mathfrak{a}(s; (u, \mathrm{Tr}(u)), (v, \mathrm{Tr}(v))| \\
&\le \int_{\partial\Omega} |\beta(t,\cdot) - \beta(s,\cdot)||\mathrm{Tr}(u)\mathrm{Tr}(v)|\, \mathrm{d}\sigma \\
&\le C|t-s|^{\alpha} \|\mathrm{Tr}(u)\|_{L_2(\partial\Omega)} \|\mathrm{Tr}(v)\|_{L_2(\partial\Omega)} \\
&\le C|t-s|^{\alpha} \|(u, \mathrm{Tr}(u))\|_H \|(v, \mathrm{Tr}(v))\|_H.
\end{aligned}$$

We apply again Theorem 1.3 and obtain maximal L_p-regularity on $L_2(\Omega) \oplus L_2(\partial\Omega)$ for all $p \in (1,\infty)$ and $u(0) \in H^1(\Omega)$ under the sole condition that $\alpha > 0$.

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