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Nonlocal parabolic equation with conserved spatial integral

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Abstract. In this short note, we investigate the behavior of the solution for a scalar nonlocal semi-linear parabolic equation, in which the nonlocal term acts to conserve the spatial integral of the solution as time evolves. For the solution blowing up in finite time, the blow-up rate is estimated. For the global solution, the global convergence is studied.

Mathematics Subject Classification. Primary 35B35, 35B40, 35K57; Secondary 35K60, 35K65.

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1. Introduction. In this note, we consider the following nonlocal semilinear parabolic equation:

$$\begin{aligned} &u_t = u_{xx} + u^2 - \fint_0^{2\pi} u^2\,dx, (x,t) \in (0,2\pi)\times(0,T),\\ &u_x(0,t) = u_x(2\pi,t) = 0, t \in (0,T),\\ &u(x,0) = u_0(x), x \in [0,2\pi], \end{aligned} \tag{1}$$

where $\fint_0^{2\pi} u^2\,dx = \frac{1}{2\pi}\int_0^{2\pi} u^2\,dx$, T denotes the maximal existence time of the solution, and $u_0(x) \in C^1([0,\pi])$ with $u_{0x}(0) = u_{0x}(2\pi) = 0$. If we denote

$$\bar{u} = \fint_0^{2\pi} u\,dx,$$

it is immediately seen that $\bar{u}$ is conserved as time evolves. In the sequel we always assume that

$$\bar{u} = C_0.$$

Problem (1) arises in nuclear science, where the growth of temperature is known to be very fast, like u^2, but some absorption catalytic material is put into the system in a way such that the total mass is conserved. It can also

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be used to model other phenomena in population dynamics and biological sciences, where the total mass is often conserved or known, but the growth of a certain cell is known to be of some form. Budd et al. [1] have considered (1) on the interval $[0,1]$ with the condition of $\fint_0^1 u\,dx = 0$. They obtained the global existence result for small initial data as well as the nonglobal existence result for the initial data whose Fourier coefficients satisfy an infinite number of conditions. For some reasons explained in [1], Problem (1) is also related to Navier–Stokes equations on an infinite slab.

Problem (1) has been studied by many other authors [2–7,10,14] in a more general form:

$$\begin{aligned}
&u_t = \Delta u + f(u) - \fint_\Omega f(u)\,dx, (x,t) \in \Omega \times (0,T),\\
&\frac{\partial u}{\partial \nu} = 0, (x,t) \in \partial\Omega \times (0,T),\\
&u(x,0) = u_0(x), \quad x \in \Omega; \quad \bar{u} = C_0,
\end{aligned} \tag{2}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N (N \geq 1)$, ν is the outer normal vector of $\partial\Omega$, and the function $f(u)$ is usually taken to be a power of u. In contrast with usual nonlocal problems such as discussed in [13], the comparison principle is not always valid for problem (2) (see Section 4 in [3]). So it is often necessary to introduce some new techniques.

The case when $f(u) = u|u|^{p-1}$ and $C_0 > 1$ is studied by Hu and Yin [7], and the nonglobal existence result is established under an energy condition:

$$E(u_0) = \int\limits_\Omega \left[\frac{1}{2}|\nabla u_0|^2 - \frac{1}{p+1}|u_0|^{p+1}\right] dx \leq -C$$

by using a convexity argument, where $C > 0$ is a constant depending on the measure of Ω. When $C_0 = 0$, their result is refined by Gao and Han [6] where the energy $E(u_0)$ is not required to be negative.

In 2007, Jazar et al. [2,10] considered (2) where $f(u) = |u|^p$ and $C_0 = 0$. They show for any $p > 1$ that the solution must blow up if the initial data satify

$$E(u_0) = \int\limits_\Omega \left[\frac{1}{2}|\nabla u_0|^2 - \frac{1}{p+1} u_0|u_0|^p\right] dx < 0.$$

A global existence result under an explicit smallness condition on the initial data is also established in [2].

In the work mentioned above, the emphasis is on sufficient conditions which guarantee the existence or non-existence of global solutions. To the best of our knowledge, there are very few results on the blow-up rate for nonglobal solutions. In this work, we consider the simple case (1) and employ the Gagliardo–Nirenberg interpolation inequality to establish an estimate for the blow-up rate of nonglobal solutions. Also, the convergence result is given for a global solution via the Lyapunov functional method. The main theorem is:

Theorem 1. 1. *If the solution of* (1) *only exists for a finite time* $T > 0$*, there is a constant* $C > 0$*, only depending on initial data, such that*

$$\int_0^{2\pi} (u - \bar{u})^2 \, dx \geq C(T - t)^{-1}.$$

2. *If the solution of* (1) *exists for all time, then for any time sequence* $\{t_j\}_{j=1}^{\infty} \to \infty$*, there is a subsequence* $\{t_{j_k}\}_{k=1}^{\infty}$ *such that* $u(x, t_{j_k})$ *converges uniformly to a function* $w(x)$ *satisfying*

$$w_{xx} + w^2 - \fint_0^{2\pi} w^2 dx = 0.$$

We will prove the main theorem in the following sections. Before the end, we also would like to note that for more general $f(u)$, Problem (2) is studied in [3,5] from the view point of the stability of stationary solutions. Moreover, the existence theory is established in [5]. The negative solutions to (2) with $f(u) = |u|^p$ are discussed in [14]. For more about related problems, one may refer to [12] and the references therein.

2. The proof of Theorem 1(1). Consider the following functional:

$$\Psi(u) = \int_0^{2\pi} (u - \bar{u})^2 \, dx.$$

From the equation in (1), we compute to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\Psi(u) &= \frac{1}{2}\frac{d}{dt}\int_0^{2\pi} u^2 \, dx \\ &= \int_0^{2\pi} u u_t \, dx \\ &= \int_0^{2\pi} u(u_{xx} + u^2 - \fint_0^{2\pi} u^2 \, dx) \, dx \\ &= -\int_0^{2\pi} u_x^2 \, dx + \int_0^{2\pi} u^2 (u - \bar{u}) \, dx, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \int_0^{2\pi} u^2(u - \bar{u}) \, dx &= \int_0^{2\pi} [(u - \bar{u})^2 + 2u\bar{u} - \bar{u}^2](u - \bar{u}) \, dx \\ &= \int_0^{2\pi} (u - \bar{u})^3 \, dx + 2C_0 \int_0^{2\pi} (u - \bar{u})^2. \end{aligned} \tag{2.2}$$

We shall use the Gagliardo–Nirenberg inequalities [11]: For a periodic function w with zero mean,

$$||w^{(j)}||_{L^r} \leq C||w||_{L^p}^{1-\theta}||w^{(k)}||_{L^q}^{\theta}, \quad \theta \in (0,1),$$

where r, p, q, j, k and θ satisfy $p, q, r > 1, j \geq 0$,

$$\frac{1}{r} = j + \theta\left(\frac{1}{q} - k\right) + (1-\theta)\frac{1}{p},$$

and

$$\frac{j}{k} \leq \theta \leq 1.$$

Here the constant C depends on r, p, q, j, and k only. Using the above interpolation inequality, we have (by taking $j = 0, r = 3, \theta = \frac{1}{3}, p = \frac{4}{3}, k = 1, q = 2$)

$$\left(\int_0^{2\pi} |u-\bar{u}|^3\,dx\right)^{1/3} \leq C\left(\int_0^{2\pi} |u-\bar{u}|^{4/3}\,dx\right)^{1/2}\left(\int_0^{2\pi} u_x^2\,dx\right)^{1/6},$$

that is,

$$\int_0^{2\pi} |u-\bar{u}|^3\,dx \leq C^3\left(\int_0^{2\pi} |u-\bar{u}|^{4/3}\,dx\right)^{3/2}\left(\int_0^{2\pi} u_x^2\,dx\right)^{1/2}.$$

By the Cauchy inequality and the Hölder inequality, we have

$$\begin{aligned}\int_0^{2\pi} |u-\bar{u}|^3\,dx &\leq \int_0^{2\pi} u_x^2\,dx + \frac{C^6}{4}\left(\int_0^{2\pi} |u-\bar{u}|^{4/3}\,dx\right)^3 \\ &\leq \int_0^{2\pi} u_x^2\,dx + \frac{C^6(2\pi)^{1/3}}{4}\left(\int_0^{2\pi} |u-\bar{u}|^2\,dx\right)^2.\end{aligned}$$

Combining the above inequality with (2.1) and (2.2), we obtain the following lemma:

Lemma 2.1. *For the functional Ψ, we have*

$$\frac{d}{dt}\Psi(u) \leq 2C_1\Psi^2 + 4C_0\Psi,$$

where $C_1 = \frac{C^6(2\pi)^{1/3}}{4}$.

Furthermore, we can estimate u_x by virtue of Ψ.

Lemma 2.2. *For the solution $u(x,t)$ to (1), there holds*

$$\int_0^{2\pi} u_x^2\,dx \leq 2C_1\Psi^2 + 6C_0\Psi + C_2,$$

where C_2 only depends on initial data.

Proof. Recall that

$$E(u(\cdot,t)) = \frac{1}{2}\int_0^{2\pi} u_x^2\,dx - \frac{1}{3}\int_0^{2\pi} u^3\,dx.$$

A direct computation yields

$$\frac{d}{dt}E(u(\cdot,t)) = -\int_0^{2\pi} u_t^2\,dx \le 0,$$

which implies

$$\int_0^{2\pi} u_x^2\,dx \le \frac{2}{3}\int_0^{2\pi} u^3 + 2E(u_0). \tag{2.3}$$

Combining (2.1), (2.3) with Lemma 2.1, we obtain the desired result by taking $C_2 := 4\pi C_0^3 + 2E(u_0)$. □

Proof of Theorem 1(1) If the time span T of the solution to (1) is finite, from the local existence result in [2] we know it must hold that

$$\lim_{t\to T} ||u(\cdot,t)||_\infty = \infty.$$

In fact, we can also deduce that $\lim_{t\to T}\sup\Psi(u(\cdot,t)) = \infty$. Otherwise, Ψ is bounded, say

$$\Psi(u(\cdot,t)) \le C_3$$

for some constant C_3. Then from Lemma 2.2 we have

$$\int_0^{2\pi} u_x^2\,dx \le C_4$$

for some constant C_4. Let $|u|_{\max}(t) = \max_{x\in[0,2\pi]}|u(x,t)|$. Assume that $|u|_{\max}(t) = |u(x_t,t)|$ for some $x_t \in [0,2\pi]$. Notice that

$$\begin{aligned}\Big|u(x_t,t) - u(x,t)\Big| &= \left|\int_x^{x_t} u_x(x,t)\,dx\right| \\ &\le |x_t - x|^{\frac{1}{2}}\left(\int_x^{x_t} u_x^2\,dx\right)^{\frac{1}{2}} \\ &\le |x_t - x|^{\frac{1}{2}} C_4^{1/2}.\end{aligned}$$

Therefore, for any $\varepsilon > 0$, there exists a number $\delta = C_4^{-1/2}\varepsilon$ such that

$$|u(x_t,t) - u(x,t)| \le \varepsilon$$

for all $x \in (x_t - \delta^2, x_t + \delta^2)$ and all $t \in (0,T)$. Since $\lim_{t\to T}|u|_{\max}(t) = \infty$, we have $\lim_{t\to T}\int_0^{2\pi} u^2\,dx = \infty$. Furthermore, we have $\lim_{t\to T}\Psi = \infty$. A contradiction! Thus we have shown that $\lim_{t\to T}\sup\Psi(u(\cdot,t)) = \infty$.

From Lemma 2.1,

$$\frac{d}{dt}\Psi(u) \leq 2C_1\Psi^2 + 4C_0\Psi \leq \begin{cases} 2C_1\Psi^2, & \text{if } C_0 \leq 0; \\ (2C_1 + 4C_0)(\Psi^2 + \Psi), & \text{if } C_0 > 0. \end{cases}$$

Then integration yields the desired result

$$\Psi(u(\cdot, t)) \geq C_5(T - t)^{-1},$$

for some constant $C_5 > 0$ only depending on initial data. □

3. The proof of Theorem 1(2). If the solution exists globally, that is, $T = \infty$, then we have the following estimate, which is from [4].

Lemma 3.1. *For a global solution $u(x, t)$ of (1), there is a constant C, independent of time, such that*

$$||u(\cdot, t)||_{C^1} < C, \quad \forall\, t \geq 0.$$

Proof of Theorem 1(2) For any time sequence $\{t_j\}_{j=1}^{\infty} \to \infty$, we can use Lemma 3.1 and the Arzela–Ascoli theorem to conclude that there is a subsequence $\{t_{j_k}\}_{k=1}^{\infty}$ such that

$$||u(x, t_{j_k}) - w(x)||_\infty \to 0, \quad \text{as} \quad k \to \infty,$$

for some Lipschitz continuous function $w(x)$. Now we claim that $w(x)$ is in fact a stationary solution of (1). Recall that

$$E(u(\cdot, t)) = \frac{1}{2}\int_0^{2\pi} u_x^2\, dx - \frac{1}{3}\int_0^{2\pi} u^3\, dx$$

and

$$\int_0^{2\pi} u_t^2\, dx = -\frac{d}{dt}E(u(\cdot, t)).$$

Thus we have

$$\int_0^t \int_0^{2\pi} u_t^2\, dxdt = E(u(\cdot, t)) - E(u(\cdot, 0)).$$

By Lemma 3.1, we know that there is a time-independent constant C_6 such that

$$E(u(\cdot, t)) \leq C_6, \quad \forall\ t \geq 0.$$

Hence the integral $\int_0^\infty \int_0^{2\pi} u_t^2\, dxdt$ exists, or equivalently,

$$\int_0^\infty \int_0^{2\pi} u_t^2\, dxdt < +\infty. \tag{3.1}$$

By a routine method of regularity estimate [8,9], Lemma 3.1 guarantees that all the orders of the derivatives of $u(x,t)$ in x and t are uniformly bounded. So there is a time-independent constant C_7 such that

$$\Big|\frac{d}{dt}\int_0^{2\pi} u_t^2\,dx\Big| \le C_7.$$

Thus (3.1) implies that $\lim_{t\to\infty}\int_0^{2\pi} u_t^2\,dx = 0$. Hence $\lim_{t\to\infty} u_t = 0$ holds uniformly on $[0,2\pi]$. Then taking the limit in both sides of the equation of (1) implies that $w(x)$ satisfies $w_{xx} + w^2 - \fint_0^{2\pi} w^2 dx = 0$. $\square$

We remark that there are infinitely many stationary solutions to (1) (see [1]). Whether the global solution of (1) converges to a unique stationary solution or not is an open problem.

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