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Characterizations of inner product spaces involving homogeneity of isosceles orthogonality

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Abstract. In this paper by using the notion of homogeneity property of the isosceles orthogonality, we derive some characterizations of inner product spaces. We also prove that a weakened hypothesis of the homogeneity of the isosceles orthogonality and a weakened reformulation of the Ficken characterization can still characterize inner product spaces. Finally, we present a characterization of inner product spaces related to an angular distance equality.

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1. Introduction. There are a lot of significant natural geometric properties which fail in general normed linear spaces, such as non Euclidean ones. Some of these interesting properties hold just when the space is an inner product one. This is the most important motivation for studying characterizations of inner product spaces. The first norm characterization of inner product spaces was given by Fréchet in 1935. Since then, the problem of finding necessary and sufficient conditions for a normed linear space to be an inner product one has been investigated by many mathematicians who considered some geometric aspects of underlying spaces (see [4,5,16]).

We recall two orthogonality types introduced in normed linear spaces. In 1945 James [12] introduced the so-called isosceles orthogonality as follows:

$$x \perp_I y \text{ if and only if } \|\mathrm{x}+\mathrm{y}\| = \|\mathrm{x}-\mathrm{y}\|.$$

Birkhoff [2] also introduced Birkhoff orthogonality in 1935 as follows:

$$x \perp_B y \text{ if and only if } \|\mathrm{x}\| \leq \|\mathrm{x}+\mathrm{ty}\| \text{ for all } \mathrm{t} \in \mathbb{R}.$$

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Some other known orthogonalities in normed linear spaces can be found in [2,3,7] and references therein. One way to obtain characterizations of inner product spaces is to force these orthogonalities to fulfill some properties of the orthogonality. For instance James [12] proved that X is an inner product space if and only if the isosceles orthogonality is homogeneous, i.e., if and only if for all $\alpha \in \mathbb{R}$, $x \perp_I y$ implies $x \perp_I \alpha y$. Characterizations of inner product spaces, in which some properties of the orthogonality relations have been used, can also be derived from [9, Theorem 2.2 and Corollary 3.2], [15, Theorem 16(a)] and [6, Corollary 1].

In Section 2, we study homogeneity of the isosceles orthogonality in general normed linear spaces. We prove that a weakened hypothesis of the homogeneity of the isosceles orthogonality and a weakened reformulation of the Ficken characterization can still characterize inner product spaces. In Section 3, using the notion of the homogeneity of the isosceles orthogonality in Minkowski planes, we derive some characterizations of Euclidean planes. Finally, in the last section we present a characterization of inner product spaces related to an angular distance equality. In this paper $(X, \|.\|)$ always denotes a real normed linear space and S_X is the corresponding unit sphere.

2. Homogeneity of the isosceles orthogonality in normed linear spaces. Let $(X, \|.\|)$ be a normed linear space. It has been proved by Ficken that the norm comes from an inner product if and only if $\|x + ty\| = \|y + tx\|$ for all $t \in \mathbb{R}$ and $x, y \in X$ with $\|x\| = \|y\|$. This characterization of inner product spaces is well-known as the Ficken characterization. Using the Ficken characterization, James showed that the isosceles orthogonality is homogeneous only in inner product spaces [12]. The hypothesis of homogeneity of the isosceles orthogonality has been weakened in different ways (see [5,10]). Amir in [5] shows that a normed linear space X is an inner product space if and only if there exists $\alpha \in (0, 1)$ such that for all $x, y \in X$, $x \perp_I y \Rightarrow x \perp_I \alpha y$.

In the sequel we improve the above result in such a way that the above α depends on x and y. First we need the following lemma.

Lemma 1 [5]. *Let $(X, \|.\|)$ be a normed linear space. The norm comes from an inner product if and only if for all $x, y \in S_X$,*

$$x \perp_I y \Longrightarrow x \perp_B y.$$

Theorem 2.1. *Let $(X, \|.\|)$ be a normed linear space and $h \in (0, 1)$. Then the norm comes from an inner product if and only if*

$$\forall x, y \in X, \quad \exists\, \alpha \in (0, h], \quad x \perp_I y \Rightarrow x \perp_I \alpha y.$$

Proof. If X is an inner product space, then the isosceles orthogonality is homogeneous and obviously the result holds.

Now let $x_0, y_0 \in X$ and $x_0 \perp_I y_0$. Define $A := \{t \in \mathbb{R}, \quad x_0 \perp_I ty_0\}$. The set A is nonempty, symmetric, and closed. We will show that $\inf\{t > 0,\ t \in A\} = 0$. By assumption, we know that there exists $\alpha_1 \in (0, h]$ such that $x_0 \perp_I \alpha_1 y_0$. Applying the assumption again for x_0 and $\alpha_1 y_0$, there exists $\alpha_2 \in (0, h]$ such that $x_0 \perp_I \alpha_2(\alpha_1 y_0)$. We can proceed to obtain a sequence $\{\alpha_n\}$ such that

$x_0 \perp_I \alpha_n\alpha_{n-1}\ldots\alpha_1 y$. Now define the sequence $t_n := \alpha_1\alpha_2\ldots\alpha_n$, $(n \geq 1)$. Clearly for all $n \in \mathbb{N}$, $t_n \in A$, $t_n \leq h^n$ and hence $t_n \to 0$. So $\inf\{t > 0, \quad t \in A\} = 0$. The function $g(t) := \|x_0 + ty_0\|$ is convex. Since $t_n \in A$, we obtain that $g(t_n) = g(-t_n)$. Therefore the line $y = \|x_0\|$ supports $g(t)$ at the point $(0, \|x_0\|)$ and $\|x_0\| \leq \|x_0 + ty_0\|$, for all $t \in \mathbb{R}$, i.e. $x_0 \perp_B y_0$ and the result holds by Lemma 1. □

Remark 1. In the proof of [11, Lemma 5], maybe there exists a strictly decreasing sequence $\{\gamma_n\}$ of positive numbers which does not converge to 0. Let $\epsilon \in (0,1)$, we define the following set

$$H'_X(\epsilon) := \{x \in S_X, \quad \forall y \in X,\ x \perp_I y\ \exists\ \alpha \in (0,\epsilon],\ x \perp_I \alpha y\}.$$

By choosing $H'_X(\epsilon)$ instead of H'_X in [11], the sequence $\{\gamma_n\}$ in [11, Lemma 5] should be converged to 0 and by the same method as in [11, Theorem 6], we obtain that $H'_X(\epsilon) \subseteq H_X$. Therefore, Theorem 2.1 is a special case of the revised version of [11, Corollary 7] which is stated as follows:
Let X be a Banach space whose dimension is at least two. If the relative interior of $H'_X(\epsilon)$ in S_X is not empty, then X is a Hilbert space.

Now we state a formulation of the Ficken characterization which was proved by Lorch [5] as follows: A normed linear space X is an inner product space if and only if there exists $c \geq 1$, such that for all $u, v \in S_X$, $\|u + cv\| = \|v + cu\|$. In the following theorem, we show that the above c does not need to be a fixed number.

Theorem 2.2. *Let X be a normed linear space and $M > 1$ be a given number. Then the norm comes from an inner product if and only if*

$$\forall u, v \in S_X,\ \exists\ c \in (1, M], \quad \|u + cv\| = \|v + cu\|.$$

Proof. If X is an inner product space, then clearly for all $u, v \in S_X$ and $t \in \mathbb{R}$, we have $\|u + tv\| = \|v + tu\|$.
For the coverse, define the real function $f(t) := \frac{1+t}{1-t}$. Since the function $f(t)$ is one to one, by taking $h := f^{-1}(M)$, we have $h \in (0,1)$. Let $x, y \in X$ and $x \perp_I y$. Without loss of generality, we assume that $\|x \pm y\| = 1$. Let $u := x - y$, $v := x + y$, and $\alpha := \frac{c-1}{c+1}$. Then $\alpha \leq h$ and

$$\begin{aligned}\|x + \alpha y\| &= \|x + \frac{c-1}{c+1}y\| = \frac{1}{c+1}\|(c+1)x + (c-1)y\| = \frac{1}{c+1}\|cv + u\| \\ &= \frac{1}{c+1}\|u + cv\| = \frac{1}{c+1}\|(c+1)x + (1-c)y\| = \|x - \alpha y\|.\end{aligned}$$

So by Theorem 2.1, the result holds. □

Using the above Theorem, we have the following corollary.

Corollary 1. *Let X be a normed linear space and $0 < m < 1$ be a given number. Then the norm comes from an inner product if and only if*

$$\forall u, v \in S_X,\ \exists\ c \in [m, 1), \quad \|u + cv\| = \|v + cu\|.$$

3. Homogeneity of the isosceles orthogonality in Minkowski planes. In order to characterize inner product spaces, sometimes it is more convenient to consider two-dimensional normed linear spaces, since there is a theorem which implies that a normed linear space X is an inner product space if and only if each two-dimensional subspace of X is an inner product space [5]. A two-dimensional real normed linear space is called a Minkowski plane and a two-dimensional real inner product space is called a Euclidean plane.

The concept of angular distance between nonzero elements x and y in a normed linear space X was defined as $\alpha[x,y] := \|\frac{x}{\|x\|} - \frac{y}{\|y\|}\|$. There are interesting characterizations of inner product spaces connected with the concept of angular distance; see [5,8,14] and references therein. In 2009, Wu proved a characterization of Minkowski planes which deals with the concept of angular distance [17, Theorem 5.3.1]. In the following, we state some results which improve the results due to Wu.

The following lemma deals with the uniqueness property of the isosceles orthogonality in Minkowski planes. For more information about the uniqueness property of the isosceles orthogonality, see [1,13].

Lemma 2 [1, Corollary 4]. *Let $(X, \|.\|)$ be a Minkowski plane. For any $x \in S_X$ and $0 \le \alpha \le 1$, there exists a point $y \in \alpha S_X$ which is unique up to the sign and satisfies $x \perp_I y$.*

Let $(X, \|.\|)$ be a Minkowski plane with a fixed orientation ω and $x \in S_X$. Let us H_x^+ and H_x^- be the two open half-planes, bounded by the line passing through x and $-x$, such that the orientation from $(-x)$ to z and z to x are given by ω for any point $z \in H_x^+$, and the orientations from x to z and z to $(-x)$ are also given by ω for any point $z \in H_x^-$. By the uniqueness property of the isosceles orthogonality (Lemma 2), for any $t \in [0,1]$ there exists a unique point $F_x(t)$ such that $x \perp_I F_x(t)$ and $F_x(t) \in tS_X \bigcap \overline{H_x^+}$. Also we denote the unit vector $\frac{F_x(t)}{\|F_x(t)\|}$ by $T_x(t)$, for all $t \in (0,1]$.

Lemma 3 [17, Lemma 3.2.6]. *Let $(X, \|.\|)$ be a Minkowski plane, $\{t_n\} \subseteq (0,1]$ be a sequence such that $\lim_{n\to\infty} t_n = 0$, and that $\{T_x(t_n)\}$ is a Cauchy sequence. Then $x \perp_B \lim_{n\to\infty} T_x(t_n)$.*

Using the notion of homogeneity property of the isosceles orthogonality, we state the following theorem which provides some characterizations of inner product spaces.

Theorem 3.1. *A Minkowski plane X is Euclidean if for any $x \in S_X$ and any $n \in \mathbb{N}$, we have $x \perp_I \beta_x(n)T_x(1/n)$ for some sequence $\{\beta_x(n)\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \beta_x(n) = 1$.*

Proof. Let $x \in S_X$, and let $y \in S_X$ be a vector such that $x \perp_I y$. Since $\{T_x(1/n)\}_{n=1}^{\infty}$ is a bounded sequence in S_X, there exists a convergent subsequence $\{T_x(1/n_k)\}_{k=1}^{\infty}$ in S_X and by Theorem 3, $x \perp_B \lim_{k\to\infty} T_x(1/n_k)$. Now let $a_k = \beta_x(n_k)T_x(1/n_k)$, we have

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \beta_x(n_k)T_x(1/n_k) = \lim_{k\to\infty} T_x(1/n_k).$$

Using the hypothesis, we have $x \perp_I \beta_x(n_k)T_x(1/n_k)$ for any $k \in \mathbb{N}$. So for any $k \in \mathbb{N}$, $x \perp_I a_k$ and therefore $x \perp_I \lim_{k\to\infty} a_k = \lim_{k\to\infty} T_x(1/n_k)$. But since $x \perp_I y$, the uniqueness property of the isosceles orthogonality implies that either $\lim_{k\to\infty} T_x(1/n_k) = y$ or $\lim_{k\to\infty} T_x(1/n_k) = -y$. So $x \perp_B y$ and by Lemma 1, X is Euclidean. □

The results in the following corollaries are derived from the above theorem as its special cases.

Corollary 2. *A Minkowski plane X is Euclidean if and only if for all $u, v \in S_X$ with $u \neq v$, $\alpha[u - v, v] = \alpha[v - u, u]$.*

Proof. If X is Euclidean plane, then clearly $\alpha[u - v, v] = \alpha[v - u, u]$ for all $u, v \in S_X$ with $u \neq v$. Conversely, let $x \in S_X$. By Lemma 2 for any $n \in \mathbb{N}$, $x \perp_I F_x(1/n)$. Let $u = \frac{x+F_x(1/n)}{\|x+F_x(1/n)\|}$ and $v = \frac{x-F_x(1/n)}{\|x-F_x(1/n)\|}$, we have

$$\begin{aligned}
&\big\|\big(\|x + F_x(1/n)\| + \|F_x(1/n)\|\big)T_x(1/n) + x\big\| \\
&\quad = \|x + F_x(1/n)\|\left\|\left(\frac{1}{\|F_x(1/n)\|} + \frac{1}{\|x + F_x(1/n)\|}\right)F_x(1/n) + \frac{1}{\|x + F_x(1/n)\|}x\right\| \\
&\quad = \|x + F_x(1/n)\|\alpha[v - u, u] = \|x - F_x(1/n)\|\alpha[u - v, v] \\
&\quad = \|x - F_x(1/n)\|\left\|\left(\frac{1}{\|F_x(1/n)\|} + \frac{1}{\|x - F_x(1/n)\|}\right)F_x(1/n) - \frac{1}{\|x - F_x(1/n)\|}x\right\| \\
&\quad = \big\|\big(\|x - F_x(1/n)\| + \|F_x(1/n)\|\big)T_x(1/n) - x\big\|.
\end{aligned}$$

Taking $\beta_x(n) = \|x + F_x(1/n)\| + \|F_x(1/n)\|$, we have $x \perp_I \beta_x(n)T_x(1/n)$ and $\lim_{n\to\infty} \beta_x(n) = 1$. So by Theorem 3.1, X is Euclidean. □

By taking $u = \frac{x+F_x(1/n)}{\|x+F_x(1/n)\|}$, $v = \frac{F_x(1/n)-x}{\|x-F_x(1/n)\|}$, $\beta_x(n) = \|x - F_x(1/n)\| + \|F_x(1/n)\|$ and using Theorem 3.1, we have the following corollary which was proved by Wu.

Corollary 3 [17, Theorem 5.3.1]. *A Minkowski plane X is Euclidean if and only if for all $u, v \in S_X$ with $u \neq v$, $\alpha[u + v, v] = \alpha[v + u, u]$.*

In a normed linear space X, a vector x is said to be orthogonal to y in the sense of Singer [2] ($x \perp_S y$) if either $\|x\|\|y\| = 0$ or $\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\| = \left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|$. We know that for all $n \in \mathbb{N}$, $x \perp_I F_x(1/n)$. So by taking $\beta_x(n) = 1$ and using Theorem 3.1, we have the following corollary, which was proved by Alonso.

Corollary 4 [2]. *A Minkowski plane X is Euclidean if and only if for all nonzero $x, y \in X$, $x \perp_I y$ implies $x \perp_S y$.*

4. An angular distance equality. The results in Corollary 2 and 3 in the previous section have been based on satisfying the following angular distance equality

$$\alpha[u + tv, v] = \alpha[v + tu, u] \tag{4.1}$$

for $t = -1$ and $t = 1$ respectively. In this section we present a new characterization of inner product spaces involving the equality (4.1).

Our basic tools in this section are norm derivatives. In a normed linear space $(X, \|.\|)$, the norm derivatives are given for fixed $x, y \in X$ by the following expression

$$\rho'_{\pm}(x,y) := \lim_{\lambda \to \pm 0} \frac{\|x+\lambda y\|^2 - \|x\|^2}{2\lambda}. \tag{4.2}$$

Some characterizations of inner product spaces given in terms of norm derivatives were reported in [4,5]. The next two theorems describe several properties of ρ'_+ and ρ'_-.

Theorem 4.1 [4]. *Let $(X, \|.\|)$ be a normed linear space and ρ'_+ and ρ'_- be given by (4.2). Then*

(i) $\rho'_{\pm}(0,y) = \rho'_{\pm}(x,0) = 0$ *for all* $x, y \in X$,
(ii) $\rho'_{\pm}(x,x) = \|x\|^2$ *for all* $x \in X$,
(iii) $\rho'_{\pm}(\alpha x,y) = \rho'_{\pm}(x,\alpha y) = \alpha\rho'_{\pm}(x,y)$ *for all* $x, y \in X$ *and* $\alpha \geq 0$,
(iv) $\rho'_{\pm}(\alpha x,y) = \rho'_{\pm}(x,\alpha y) = \alpha\rho'_{\mp}(x,y)$ *for all* $x, y \in X$ *and* $\alpha \leq 0$.

Theorem 4.2 [4,5]. *Let $(X, \|.\|)$ be a normed linear space. Then the following statements are equivalent:*

(i) $\rho'_+(x,y) = \rho'_+(y,x)$ *for all* $x, y \in X$,
(ii) $\rho'_-(x,y) = \rho'_-(y,x)$ *for all* $x, y \in X$,
(iii) $(X, \|.\|)$ *is an inner product space.*

In the following proposition, we state a necessary condition for our characterization.

Proposition 1. *Let $(X, \|.\|)$ be an inner product space and x, y be two linearly independent vectors such that $\|x\| = \|y\|$. Then $\alpha[x+ty, y] = \alpha[y+tx, x]$ for all $t \in \mathbb{R}$.*

Proof. Let $\langle \cdot, \cdot \rangle$ be the inner product on X, $x, y \in X$ be linearly independent vectors with the same norm, and $t \in \mathbb{R}$. Then

$$\begin{aligned}\alpha^2[x+ty,y] &= \left\| \frac{x+ty}{\|x+ty\|} - \frac{y}{\|y\|} \right\|^2 = \left\langle \frac{x+ty}{\|x+ty\|} - \frac{y}{\|y\|}, \frac{x+ty}{\|x+ty\|} - \frac{y}{\|y\|} \right\rangle \\ &= 2 - \frac{2}{\|x+ty\|\|y\|}\left(\langle x,y\rangle + t\|y\|^2\right). \end{aligned} \tag{4.3}$$

Similarly we get

$$\alpha^2[y+tx,x] = 2 - \frac{2}{\|y+tx\|\|x\|}\left(\langle x,y\rangle + t\|x\|^2\right). \tag{4.4}$$

From (4.3), (4.4) and the Ficken characterization, we deduce that $\alpha[x+ty, y] = \alpha[y+tx, x]$. □

By Corollary 2 or 3, the reverse of Proposition 1 holds. The next result also provides a reverse of Proposition 1. The proof has been based on the concept of norm derivatives.

Theorem 4.3. *Let $(X, \|.\|)$ be a normed linear space. If for any linearly independent vectors x and y with the same norm, there exists a real number $h \in \mathbb{R}$ such that*

$$\|x + hy\| = \|y + hx\| \text{ and } \alpha[x + ty, y] = \alpha[y + tx, x] \quad (t \leq h), \tag{4.5}$$

then $(X, \|.\|)$ is an inner product space.

Proof. Assume that a and b are two arbitrary linearly independent vectors in X. We want to show that $\rho'_{-}(a, b) = \rho'_{-}(b, a)$. Let $x := \|b\|a$ and $y := \|a\|b$. Since $\|x\| = \|y\|$, by assumption there exists $h \in \mathbb{R}$ such that (4.5) holds. Define $H := \{t \in \mathbb{R} : \|x + ty\| = \|y + tx\|\}$. If $t \in H \cap (-\infty, h]$, then

$$\begin{aligned}\alpha[x + ty, y] = \alpha[y + tx, x] &= \left\| \frac{x + ty}{\|x + ty\|} - \frac{y}{\|y\|} \right\| = \left\| \frac{y + tx}{\|y + tx\|} - \frac{x}{\|x\|} \right\| \\ &= \left\| x + \left(t - \frac{\|x + ty\|}{\|x\|}\right) y \right\| = \left\| y + \left(t - \frac{\|x + ty\|}{\|x\|}\right) x \right\|.\end{aligned}$$

In fact, we showed that

$$\text{If } \ t \in H \cap (-\infty, h], \quad \text{then } \ t - \frac{\|x + ty\|}{\|x\|} \in H \cap (-\infty, h]. \tag{4.6}$$

We define the real sequence $\{c_n\}$ as follows:

$$c_1 := h, \quad c_n := -\frac{\|x + (c_1 + \cdots + c_{n-1})y\|}{\|x\|} \quad (n \geq 2).$$

We will show that $\lim_{n\to\infty} c_n \neq 0$, and hence $\sum_{n=1}^{\infty} c_n$ is a divergent series. First we define a real valued function f as $f(t) := \|x + ty\|$. It is clear that $\lim_{t\to\pm\infty} f(t) = \infty$, so there exists $M > 0$ such that $f(t) > 1$ for all t satisfying $|t| > M$. Putting $I = [-M, M]$ one can observe that the continuous function f takes a minimum at some point $t_0 \in I$. Therefore $f(t) \geq \min\{1, f(t_0)\}$ for all $t \in \mathbb{R}$. But due to the linearly independence of x and y, $\min\{1, f(t_0)\} > 0$. So for all $n \geq 2$, $|c_n| \geq \frac{\min\{1, f(t_0)\}}{\|x\|}$, this shows that $\lim_{n\to\infty} c_n \neq 0$.

Let $\{s_n\}$ be the sequence of partial sums of the series $\sum_{n=1}^{\infty} c_n$. Clearly $\{s_n\}$ is decreasing. We may assume that $\{s_n\}$ is a sequence of negative real numbers and so $s_n \longrightarrow -\infty$ as $n \longrightarrow \infty$. Applying (4.6) to

$$s_1 = h, \ s_2 = h - \frac{\|x + hy\|}{\|x\|}, \ s_3 = h - \frac{\|x + hy\|}{\|x\|} - \frac{\|x + (h - \frac{\|x+hy\|}{\|x\|})y\|}{\|x\|}, \ldots$$

frequently, we deduce that $s_n \in H$ $(n = 1, 2, \ldots)$. Putting $t_n = s_n^{-1}$ $(n = 1, 2, \ldots)$ and the simple fact that H is closed under inversion we have $\{t_n\}$ is a sequence of negative real numbers in H such that $\lim_{n\to\infty} t_n = 0$. Since $t_n \in H$ $(n = 1, 2, \ldots)$ and $\|x\| = \|y\|$, we have

$$
\begin{aligned}
\rho'_-(a,b) &= \frac{1}{\|a\|\|b\|}\rho'_-(x,y) && \text{(by Theorem 4.1)}\\
&= \frac{1}{\|a\|\|b\|}\lim_{t\to 0^-}\frac{\|x+ty\|^2-\|x\|^2}{2t}\\
&= \frac{1}{\|a\|\|b\|}\lim_{n\to\infty}\frac{\|x+t_ny\|^2-\|x\|^2}{2t_n}\\
&= \frac{1}{\|a\|\|b\|}\lim_{n\to\infty}\frac{\|y+t_nx\|^2-\|y\|^2}{2t_n} && (t_n\in H \text{ and } \|\mathrm{x}\|=\|\mathrm{y}\|)\\
&= \frac{1}{\|a\|\|b\|}\lim_{t\to 0^-}\frac{\|y+tx\|^2-\|y\|^2}{2t}\\
&= \frac{1}{\|a\|\|b\|}\rho'_-(y,x)=\rho'_-(b,a). && \text{(by Theorem 4.1)}
\end{aligned}
$$

If a and b are linearly dependent, then clearly $\rho'_-(a,b)=\rho'_-(b,a)$ and by Theorem 4.2, X is an inner product space. □

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