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On blow-ups and resolutions of Hermitian-symplectic and strongly Gauduchon metrics

Song Yang

Abstract. In this note, we study the blow-ups of Hermitian-symplectic manifolds and strongly Gauduchon manifolds along a point or compact complex submanifold. We show that any Hermitian-symplectic (resp. strongly Gauduchon) orbifold has a Hermitian-symplectic (resp. strongly Gauduchon) resolution.

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1. Introduction. Let (M^{2n}, J, g) be a complex Hermitian manifold of complex dimension n, and let $\omega(-,-) := g(J-,-)$ be the associated Kähler form. A special Hermitian metric is one whose Kähler form ω satisfies a certain special condition. For example, we say that

- if ω is closed, ω is a *Kähler metric*;
- if ω^{n-1} is closed, ω is a *balanced metric*;
- if $\partial\omega$ is $\bar{\partial}$-closed, ω is a *strong Kähler with torsion* (for short *strong KT*) or *pluriclosed* metric;
- if $\partial\omega^{n-1}$ is $\bar{\partial}$-closed, ω is a *Gauduchon metric*;
- if $\partial\omega$ is $\bar{\partial}$-exact with an additional condition, ω is a *Hermitian-symplectic metric.*
- if $\partial\omega^{n-1}$ is $\bar{\partial}$-exact, ω is a *strongly Gauduchon metric.*

In some sense, these Hermitian metrics are dual to each other: Kähler metrics and balanced metrics; strong KT metrics and Gauduchon metrics; Hermitian-symplectic metrics and strongly Gauduchon metrics. That's why we are interested in these.

It is well known that Hermitian metrics exist on any complex manifold, and Gauduchon [7] showed that there exists a Gauduchon metric (unique up to normalisation) in the conformal class of any Hermitian metric on any compact complex manifold. However, the Kähler, balanced, strong KT, Hermtian-symplectic, and strongly Gauduchon metrics need not exist on an arbitrary complex manifold. A natural issue is to construct complex manifolds with these special Hermitian metrics.

In this note, we are mainly interested in the Hermitian-symplectic metrics. Streets and Tian [16] constructed a parabolic flow for Hermitian-symplectic metrics, analogous to the Kähler-Ricci flow. In loc. cit., Streets and Tian showed that a complex surface is Hermitian-symplectic if and only if it is Kähler (see [16, Proposition 1.6]), and they proposed the following question (see [16, Question 1.7]) which is still an open problem.

Question 1.1. *Does there exist a complex manifold, of dimension* ≥ 3*, which admits a Hermitian-symplectic metric but no Kähler metric?*

The present note is enlightened by the deep and beautiful work of Fino and Tomassini [6]. As far as we know, there exists no general technique for constructing Hermitian-symplectic manifolds. However, we will show that the blow-up of a Hermitian-symplectic manifold at a point or more points also admits a Hermitian-symplectic metric. In this way we can get new Hermitian-symplectic manifolds from a given one. More generally, we have

Proposition 1.2 (Proposition 3.2). *Let* M *be a complex manifold, and let* $\widehat{M}$ *be the blow-up of* M *along a compact complex submanifold* N*. If* M *admits a Hermitian-symplectic* (*resp. strongly Gauduchon*) *metric, then* $\widehat{M}$ *also admits a Hermitian-symplectic* (*resp. strongly Gauduchon*) *metric.*

Notice that Joyce constructed simply-connected compact manifolds with exceptional holonomy G_2 and Spin(7) by resolving orbifolds of quotients of tori in [11]. Recently, Fernández and Muñoz [5] obtained the first example of simply-connected compact non-formal symplectic manifold of dimension 8 by resolving symplectically the singularities; in their joint paper [3] with Cavalcanti, they introduced a method to resolve a symplectic orbifold (see [3, Theorme 3.3]) and obtained the first example of a simply connected compact not-formal symplectic manifold of dimension 8 which satisfies the Lefschetz property. Moreover, Fino and Tomassini [6] proved that any strong KT orbifold has a strong KT resolution (see [6, Theorem 5.4]) by using the *Hironaka resolution of singularities theorem* (see [9]), and they constructed some simply-connected, non-Kähler, compact strong KT manifolds. In the same spirit, it is natural, reasonable, and necessary to investigate the resolution of Hermitian-symplectic orbifolds, and we have

Theorem 1.3 (Theorem 4.3). *Let* (M, J) *be a complex orbifold of complex dimension* n *endowed with a* J*-Hermitian-symplectic (resp. strongly Gauduchon) metric. Then there exists a Hermitian-symplectic (resp. strongly Gauduchon) resolution.*

More recently, Verbitsky [17] attempted to answer the Question 1.1 of Street and Tian [16] for twistor spaces. While he showed that the twistor spaces are not only never Hermitian-symplectic, they never admit a strong KT metric unless they are Kähler (see [17, Proposition 3.3 and Corollary 3.4]). This note is also growing as an attempt to approach Question 1.1 of Street and Tian [16], and we hope that Theorem 4.3 will be useful to construct non-Kähler Hermitian-symplectic manifolds.

2. Preliminaries. In this section, we recall the definitions of Hermitian-symplectic structures and strongly Gauduchon structures on complex manifolds.

Definition 2.1. Let (M^{2n}, J) a be complex manifold.

1. A *Hermitian-symplectic structure* on M is a real 2-form Ω such that $d\Omega = 0$ and the $(1,1)$-component $\Omega^{1,1}$ is strictly positive definite.
2. A *strongly Gauduchon structure* on M is a real $(2n-2)$-form Ω such that $d\Omega = 0$ and the $(n-1, n-1)$-component $\Omega^{n-1,n-1}$ is strictly positive definite.

Equivalently, one has the following

Lemma 2.2. *Let* (M, J) *be a complex manifold. Then*

1. *To give a Hermitian-symplectic structure on* (M, J) *is equivalent to give a Kähler form* ω *satisfying* $\partial\omega = \bar{\partial}\alpha$ *for some* ∂*-closed* $(2, 0)$*-form* α (*see* [4, Proposition 2.1]).
2. *To give a strongly Gauduchon structure on* (M, J) *is equivalent to give a Kähler form* ω *which satisfies* $\partial\omega^{n-1} = \bar{\partial}\beta$ *for some* $(n, n-2)$*-form* β (*see* [15, Proposition 4.2]).

Proof. 1. If a real 2-form $\Omega := \Omega^{2,0} + \Omega^{1,1} + \Omega^{0,2}$ is a Hermitian-symplectic structure, then $d\Omega = 0$ implies that $\partial\Omega^{1,1} = \bar{\partial}(-\Omega^{2,0})$ and $\partial(-\Omega^{2,0}) = 0$. If a Kähler form ω satisfies $\partial\omega = \bar{\partial}\alpha$, where α is a ∂-closed $(2, 0)$-form, we define $\Omega := \omega - \alpha - \bar{\alpha}$, then $d\Omega = 0$ and $\Omega^{1,1} := \omega$ is strictly positive definite.

2. If a real $(2n-2)$-form $\Omega := \Omega^{n,n-2} + \Omega^{n-1,n-1} + \Omega^{n-2,n}$ is a strongly Gauduchon structure, then $d\Omega = 0$ and this implies that $\partial\Omega^{n-1,n-1} = \bar{\partial}(-\Omega^{n,n-2})$. Since $\Omega^{n-1,n-1}$ is a strictly positive definite $(n-1, n-1)$-form, then there exists a Kähler form ω such that $\omega^{n-1} = \Omega^{n-1,n-1}$ (see Michelsohn [13, p. 279–280]). If a Kähler form ω satisfying $\partial\omega^{n-1} = \bar{\partial}\beta$ and β is a $(n, n-2)$-form, we define $\Omega := \omega^{n-1} - \beta - \bar{\beta}$, then $d\Omega = 0$ and $\Omega^{n-1,n-1} := \omega^{n-1}$.

$\square$

Lemma 2.2(1) implies that the Hermitian-symplectic structure is a special strong KT metric. Recently, Enrietti, Fino, and Vezzoni [4] showed that the strong KT metric exists on many complex nilmanifolds, and the Hermitian-symplectic metric never exists on any complex nilmanifold. It turns out that the existence of Hermitian-symplectic metrics on complex manifolds is more restrictive.

Next, we discuss the stability of Hermitian-symplectic metrics under the deformation of the complex structures.

Definition 2.3. A *complex analytic family* of compact complex manifolds is a proper holomorphic submersion $\pi\colon \mathfrak{X} \to \Delta(\varepsilon)$ from an arbitrary complex analytic manifold $\mathfrak{X}$ to $\Delta(\varepsilon) := \{t \in \mathbb{C} | |t| < \varepsilon\}$.

In this note, we only consider the small deformation, that means we may as well assume ε is small enough. It is well-known that the Kähler metric is stable under the small deformation of the complex structures (see [12, Theorem 15] or cf. [18, Theorem 9.23]). In loc. cit., Fino and Tomassini proved that the strong KT is not stable under small deformations of the complex structures (see [6, Theorem 2.2]). There is a simple observation that the Hermitian-symplectic metric is stable under small deformations of the complex structures, analogous to the strongly Gauduchon metric (see [15]).

Proposition 2.4. *Let $\pi\colon \mathfrak{X} \to \Delta(\varepsilon)$ be a small deformation of a compact Hermitian-symplectic manifold M, and put $M_t := \pi^{-1}(t)$ and $M = M_0$. Then for small $t \in \Delta - \{0\}$, M_t also has a Hermitian-symplectic metric.*

Proof. Let $\varphi_t\colon M_t \to M$ be a diffeomorphism which varies smoothly with $t \in \Delta(\varepsilon)$ and $\varphi_0 = id$. Suppose Ω is a Hermitian-symplectic structure on M_0, i.e. $d\Omega = 0$ and its $(1,1)$-component $\Omega^{1,1}$ is strictly positive definite. Let $\Omega_t = \varphi_t^*\Omega$. Then Ω_t is a real 2-form and $d\Omega_t = 0$. We decompose it as $\Omega_t := \Omega_t^{2,0} + \Omega_t^{1,1} + \Omega_t^{0,2}$. Since Ω_t is real, thus $\Omega_t^{1,1}$ is real and approaches $\Omega^{1,1}$ as $t \to 0$. In conclusion, $d\Omega_t = 0$ and $\Omega_t^{1,1}$ is strictly positive definite for sufficiently small t, hence Ω_t is Hermitian-symplectic. $\square$

3. Blow-ups. In this section, we start by showing that the blow-up of a Hermitian-symplectic (resp. strongly Gauduchon) manifold at a point is also Hermitian-symplectic (resp. strongly Gauduchon), as in the Kähler case (see [2] or [18, Proposition 3.24]) and the strong KT case (see [6, Proposition 3.1]).

Proposition 3.1. *Let M be a complex manifold, and let $\widehat{M}_p$ be the blow-up of M at a point $p \in M$. If M admits a Hermitian-symplectic (resp. strongly Gauduchon) metric, then $\widehat{M}_p$ also admits a Hermitian-symplectic (resp. strongly Gauduchon) metric.*

Proof. Let $\pi : \widehat{M} \to M$ be a blow-up of M at a point $p \in M$. We denote by $E := \pi^{-1}(p)$ the exceptional divisor of the blowing up. Then, there exists a Hermitian metric in the holomorphic line bundle $\mathcal{O}_{\widehat{M}}(-E)$ on $\hat{M}$ associated to the exceptional divisor E, such that the Chern form $\hat{c}$ satisfies (i) $\hat{c}$ is strictly positive definite along E; (ii) $\hat{c}$ is positive semi-definite at points of E; (iii) there exists a relatively compact neighborhood W of E such that $\hat{c}$ is zero outside W (we refer the readers to Griffiths–Harris [8, p. 185–187] for more details).

Then there exists a big enough $\kappa \gg 0$, such that the real $(1,1)$-form

$$\hat{\omega} := \kappa\pi^*\omega + \hat{c} \tag{3.1}$$

is a strictly positive definite $(1,1)$-form, and

$$\hat{\Omega} := \kappa\pi^*\omega^{n-1} + \hat{c}^{n-1} \tag{3.2}$$

is a strictly positive definite $(n-1,n-1)$-form.

Hermitian-symplectic case. If ω is a Hermitian-symplectic metric, i.e., there exits a ∂-closed $(2,0)$-form α such that

$$\partial\omega = \bar{\partial}\alpha.$$

We put $\rho := \pi^*\alpha$, then ρ is a ∂-closed $(2,0)$-form since π is a holomorphic map and $\partial\alpha = 0$. By Eq. (3.1), we have

$$\begin{aligned}\partial\hat{\omega} &= \kappa\pi^*\partial\omega = \kappa\pi^*\bar{\partial}\alpha = \kappa\bar{\partial}(\pi^*\alpha)\\ &= \kappa\bar{\partial}\rho.\end{aligned}$$

In summary, we have a Kähler form $\hat{\omega}$ satisfying $\partial\hat{\omega} = \bar{\partial}\hat{\alpha}$ and $\hat{\alpha}$ is a ∂-closed $(2,0)$-form, where $\hat{\alpha} := \kappa\rho$.

Strongly Gauduchon case. If ω is a strongly Gauduchon metric, i.e., there exits a $(n, n-2)$-form β satisfying

$$\partial\omega^{n-1} = \bar{\partial}\beta.$$

Denote $\varrho := \pi^*\beta$, then ϱ is a $(n, n-2)$-form since π is a holomorphic map. By Eq. (3.2), we get that

$$\begin{aligned}\partial\hat{\Omega} &= \kappa\pi^*\partial\omega^{n-1} = \kappa\pi^*\bar{\partial}\beta = \kappa\bar{\partial}(\pi^*\beta)\\ &= \kappa\bar{\partial}\varrho.\end{aligned}$$

Since $\hat{\Omega}$ is strictly positive definite $(n-1, n-1)$-form, then there exists a Kähler form $\hat{\omega}$ such that $\hat{\omega}^{n-1} = \hat{\Omega}$ (see Michelsohn [13, p. 279–280]). In summary, we have a Kähler form $\hat{\omega}$ satisfying $\partial\hat{\omega}^{n-1} = \bar{\partial}\hat{\beta}$ and $\hat{\beta} := \kappa\varrho$.

This completes the proof of Proposition 3.1. □

An immediate consequence of Proposition 3.1 is that it is possible to get new Hermitian-symplectic (or strongly Gauduchon) manifolds by blowing-up points of a given Hermitian-symplectic (resp. strongly Gauduchon) manifold. Indeed, Proposition 3.1 can be generalized to the blow-up of a Hermitian-symplectic (resp. strongly Gauduchon) manifold along a compact complex submanifold. More precisely, we obtain.

Proposition 3.2. *Let M be a complex manifold, and let $\widehat{M}$ be the blow-up of M along a compact complex submanifold N. If M has a Hermitian-symplectic (resp. strongly Gauduchon) metric, then $\widehat{M}$ also admits a Hermitian-symplectic (resp. strongly Gauduchon) metric.*

Proof. Let $\pi : \widehat{M} \to M$ be the blow-up of M along N. We let $E := \pi^{-1}(N)$ denote the exceptional divisor which is isomorphic to $\mathbb{P}(\mathcal{N}_{N/M})$ the projective bundle of the normal bundle of N in M. Then there exists a holomorphic line bundle $\mathcal{L}$ on $\widehat{M}$ such that $\mathcal{L}$ is trivial out of E and $\mathcal{L}|_E \cong \mathcal{O}_{\mathbb{P}(\mathcal{N}_{N/M})}(1)$ (cf. [18, Lemma 3.25]).

Suppose h is a Hermitian metric on $\mathcal{O}_{\mathbb{P}(\mathcal{N}_{N/M})}(1)$, and let c_h be the associated Chern form. By using the partition of unity, we extend h to be a Hermitian metric $h_{\mathcal{L}}$ on $\mathcal{L}$ which, outside of a compact neighborhood of N, is the flat Hermitian metric for the given trivialization of $\mathcal{L}$ over $\widehat{M}\backslash E$. Hence, the Chern form $\hat{c}_{\mathcal{L}}$ of $\mathcal{L}$ is zero outside of a compact neighborhood of N, and $\hat{c}_{\mathcal{L}}|_{\mathbb{P}(\mathcal{N}_{N/M})} = c_h$ (we refer the reader to [18, Section 3.3.3] for more details).

Let ω be the Kähler form of a Hermitian metric on M. The compactness of the complex submanifold N implies that there exists a big enough $\kappa \gg 0$, such that the real $(1,1)$-form

$$\hat{\omega} := \kappa\pi^*\omega + \hat{c}_{\mathcal{L}}$$

is strictly positive definite, and

$$\hat{\Omega} := \kappa\pi^*\omega^{n-1} + \hat{c}_{\mathcal{L}}^{n-1}$$

is a strictly positive definite $(n-1, n-1)$-form.

By the same strategy as for Proposition 3.1, it follows that Proposition 3.2 holds. $\square$

We know that the blow-up of a compact complex manifold along a compact complex submanifold is a special case of modifications (i.e., proper, holomorphic, bimeromorphic maps). Alessandrini and Bassanelli [1] proved that the existence of a balanced metric on compact complex manifolds is stable under modifications. Parallelling, Popovici [14] showed that the existence of strongly Gauduchon metrics on compact complex manifolds is stable under modifications (see [14, Theorem 1.3]). Similarly, for a Hermitian-symplectic metric, we propose the following question.

Question 3.3. *Is the existence of Hermitian-symplectic metrics on compact complex manifolds stable under modifications?*

A well-known example of Hironaka [10] showed that the Kähler metric is not stable under modifications. Theoretically, if Question 3.3 is true, then one can give a positive answer to the Question 1.1 of Streets and Tian [16].

4. Resolution Theorem. In this section, we consider the Hermitian-symplectic orbifolds and strongly Gauduchon orbifolds and study their resolutions of singularities.

First, let us recall the definition of complex orbifolds and the notion of Hermitian metrics on complex orbifolds (see e.g. [11, Section 6.5] or [6, Section 5]).

A *complex orbifold* of complex dimension n is a singular complex manifold (M, J) of complex dimension n whose singularities are locally isomorphic to quotient singularities $\mathbb{C}^n/G$, $G \subset \mathrm{Gl}(n, \mathbb{C})$ being a finite subgroup. Furthermore, the set S of singular points of the complex orbifold (M, J) has real codimension at least 2.

We say that g is a *Hermitian metric on the complex orbifold* (M, J) if it is a J-Hermitian metric in the usual sense on the nonsingular part of M, and G-invariant in any orbifold chart U/G. A *Hermitian complex orbifold* (M, J, g) of complex dimension n is a complex orbifold (M, J) of complex dimension n endowed with a J-Hermitian metric g.

Although the complex orbifolds have some singularities, many good properties for manifolds, e.g., the notions of differential k-forms and (p, q)-forms, still hold on the complex orbifolds. By definition, the differential k-forms and

(p, q)-forms on a complex orbifold (M, J) are defined locally at a singularity $p \in M$ as $G(p)$-invariant forms on $\mathbb{C}^n$, where $G(p) \subset \mathrm{Gl}(n, \mathbb{C})$ is that M is locally isomorphic to $\mathbb{C}^n/G(p)$. Then we have the differential d on orbifold differential forms, and the differential d splits as $d = \partial + \bar{\partial}$ as usual.

Definition 4.1. Let (M, J, g) be a Hermitian complex n-orbifold with Kähler form ω. We say that ω is a *Hermitian-symplectic* if there exists a ∂-closed $(2, 0)$-form α such that $\partial\omega = \bar{\partial}\alpha$. We say that ω is a *strongly Gauduchon*, if there exists a $(n, n-2)$-form β such that $\partial\omega^{n-1} = \bar{\partial}\beta$.

Next, we review the resolution of Hermitian metric singularities.

Definition 4.2. Let (M, J, g) be a Hermitian-symplectic (resp. strongly Gauduchon) orbifold with Hermitian-symplectic (resp. strongly Gauduchon) metric g. A *Hermitian-symplectic* (resp. *strongly Gauduchon*) resolution of (M, J, g) consists of a smooth complex manifold $(\widehat{M}, \widehat{J})$ equipped with a $\widehat{J}$-Hermitian-symplectic (resp. strongly Gauduchon) metric $\hat{g}$ and of a map $\pi\colon \widehat{M} \to M$ which satisfies

(i) the map $\pi\colon \widehat{M}\setminus E \to M\setminus S$, where S is the set of singular points of M and $E := \pi^{-1}(S)$ is the *exceptional set*;

(ii) the metric $\hat{g} = \pi^* g$ on the complement of a neighborhood of E.

In his fundamental paper [9], Hironaka proved that the singularities of any complex algebraic variety can be resolved by a finite sequence of blow-ups, which is now called *Hironaka resolution of singularities theorem*.

We are now ready to prove the resolution theorem.

Theorem 4.3. *Let (M, J) be a complex orbifold of complex dimension n which admits a J-Hermitian metric g.*

(a) *If g is a J-Hermitian-symplectic metric, then there exists a Hermitian-symplectic resolution.*

(b) *If g is a J-strongly Gauduchon metric, then there exists a strongly Gauduchon resolution.*

Proof. We follow the same strategy as in [3,6]. For a singular point $p \in S$ of M, taking an orbifold chart $U(p) := B^n/G(p)$, where $B^n = \{z \in \mathbb{C}^n \mid |z| < 1\}$ is the standard ball of radius 1 of $\mathbb{C}^n$ and $G(p) \subset GL(n, \mathbb{C})$ is a finite subgroup. Then $X := \mathbb{C}^n/G(p)$ is an affine complex algebraic variety with the only singular point 0. According to the *Hironaka resolution of singularities theorem*, there is a resolution $\pi_X : \widehat{X} \to X$ which is a smooth complex variety obtained by a finite sequence of blow-ups. We denote $E := \pi_X^{-1}(0)$ the exceptional set and $\widehat{U(p)} := \pi_X^{-1}(U(p))$, and then E is a complex submanifold of $\widehat{X}$. By identifying $\widehat{U(p)}\setminus E$ with $U(p)\setminus\{p\}$, we define a smooth complex manifold

$$\widehat{M} := (M - \{p\}) \bigcup \widehat{U(p)}.$$

Then there exists a natural projection $\pi\colon \widehat{M} \to M$.

Next, we present two strictly positive definite forms on $\widehat{M}$, by the classical way, one is a $(1, 1)$-form; the others one is a $(n-1, n-1)$-form.

We denote by $\omega_0 := \sqrt{-1}\partial\bar\partial(\sum_{i=1}^n |z_i|)$ the standard Kähler form on $\mathbb{C}^n$, and by $j\colon B^n \hookrightarrow \mathbb{C}^n$ the natural inclusion. Let b be a bump function, i.e., a smooth non-negative real-valued function which equals 0 in $(B^n - B^n(\frac{3}{4})/G(p)$, and which equals 1 in $B^n(\frac{1}{4})/G(p)$.

Suppose ω is the Kähler form of g, then there exists a real big enough $\lambda \gg 0$ such that

$$\tilde\omega := \lambda\pi^*\omega + \sqrt{-1}\partial\bar\partial\left(bj^*\sum_{i=1}^n |z_i|\right) \tag{4.1}$$

is a strictly positive definite $(1,1)$-form, and

$$\tilde\Omega := \lambda\pi^*\omega^{n-1} + \left(\sqrt{-1}\partial\bar\partial\left(bj^*\sum_{i=1}^n |z_i|\right)\right)^{n-1} \tag{4.2}$$

is a strictly positive definite $(n-1,n-1)$-form.

Claim 4.4 (Hermitian-symplectic case). *If g is a J-Hermitian-symplectic metric, then there exists a Hermitian-symplectic metric $\hat g$ on $\widehat M$ and a neighborhood W of E, such that*

$$\hat g|_{\widehat M - W} = g.$$

In fact, if ω is a Hermitian-symplectic metric, i.e., there exits a ∂-closed $(2,0)$-form α such that

$$\partial\omega = \bar\partial\alpha.$$

We denote $\sigma := \pi^*\alpha$ and decompose σ as $\sigma := \sigma^{2,0} + \sigma^{1,1} + \sigma^{0,2}$. By Eq. (4.1), we have

$$\partial\tilde\omega = \lambda\pi^*\partial\omega = \lambda\pi^*\bar\partial\beta = \lambda(\bar\partial\sigma^{2,0} + \bar\partial\sigma^{1,1} + \bar\partial\sigma^{0,2}).$$

Since $\partial\tilde\omega$ is a pure $(2,1)$-form, we get that $\partial\tilde\omega = \lambda\bar\partial\sigma^{2,0}$. As σ is ∂-closed, $\partial\sigma^{2,0} = 0$. In summary, we have a Kähler form $\tilde\omega$ satisfying $\partial\tilde\omega = \bar\partial\tilde\beta$ and $\tilde\beta$ is a ∂-closed $(2,0)$-form, where $\tilde\beta := \lambda\sigma^{2,0}$. It follows that $\hat\omega = \frac{1}{\lambda}\tilde\omega$ is a Hermitian-symplectic metric on $\widehat M$ and $\hat\omega = \omega$ on $\widehat M - W$, where $W := \pi^{-1}((B^n - B^n(\frac{3}{4}))/G(p))$ is a neighborhood of E.

Claim 4.5 (Strongly Gauduchon case). *If g is a J-strongly Gauduchon metric, then there exists a strongly Gauduchon metric $\hat g$ on $\widehat M$ and a neighborhood W of E such that*

$$\hat g|_{\widehat M - W} = g.$$

If ω is a strongly Gauduchon metric, that is there exits a $(n, n-2)$-form β such that

$$\partial\omega^{n-1} = \bar\partial\beta,$$

we decompose $\varsigma := \pi^*\beta$ as $\varsigma := \varsigma^{n,n-2} + \varsigma^{n-1,n-1} + \varsigma^{n-2,n}$. By Eq. (4.2), we obtain

$$\partial\tilde\Omega = \lambda\pi^*\partial\omega^{n-1} = \lambda\pi^*\bar\partial\beta = \lambda(\bar\partial\varsigma^{n,n-2} + \bar\partial\varsigma^{n-1,n-1} + \bar\partial\varsigma^{n-2,n}).$$

Since $\partial\tilde\Omega$ is a pure $(n, n-1)$-form, we get that $\partial\tilde\omega = \lambda\bar\partial\varsigma^{n,n-2}$. As $\tilde\Omega$ is a strictly positive definite $(n-1, n-1)$-form, thus there is a Kähler form $\tilde\omega$ such that $\tilde{\omega}^{n-1} = \tilde{\Omega}$ (see Michelsohn [13, p. 279–280]), we have a Kähler form $\tilde{\omega}$ such that $\tilde{\omega}^{n-1} = \tilde{\Omega}$ and which satisfies $\partial\tilde{\omega}^{n-1} = \bar{\partial}\tilde{\beta}$ and $\tilde{\beta} := \lambda\varsigma^{n,n-2}$ is a $(n, n-2)$-form. It follows that $\hat{\omega} = \frac{1}{\sqrt[n-1]{\lambda}}\tilde{\omega}$ is a strongly Gauduchon metic on $\widehat{M}$ and $\hat{\omega} = \omega$ on $\widehat{M} - W$, where $W := \pi^{-1}((B^n - B^n(\frac{3}{4}))/G(p))$ is a neighborhood of E. This completes the proof of Theorem 4.3. $\square$

Remark 4.6. The idea of the proof is essentially the same that of [6, Theorem 5.4] and [3, Theorem 3.3]. This resolution theorem may allow us to construct new examples of Hermitian-symplectic manifolds, and we wish to do this in the near future.

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Song Yang
Department of Mathematics,
Sichuan University,
Chengdu, 610064 Sichuan,
People's Republic of China
e-mail: syang.math@gmail.com

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