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On the polynomial Hardy–Littlewood inequality

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Abstract. We investigate the behavior of the constants of the polynomial Hardy–Littlewood inequality.

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1. Introduction. Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ and given $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, define $|\alpha| := \alpha_1 + \cdots + \alpha_n$. Also, $\mathbf{x}^\alpha$ stands for the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{K}^n$. The polynomial Bohnenblust–Hille inequality asserts that, given $m, n \geq 1$, if P is a homogeneous polynomial of degree m on ℓ_∞^n given by

$$P(x_1, \ldots, x_n) = \sum_{|\alpha|=m} a_\alpha \mathbf{x}^\alpha,$$

then

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq B_{\mathbb{K},m}^{\mathrm{pol}} \|P\|$$

for some positive constant $B_{\mathbb{K},m}^{\mathrm{pol}}$ which does not depend on n (the exponent $\frac{2m}{m+1}$ is optimal), where $\|P\| := \sup_{z \in B_{\ell_\infty^n}} |P(z)|$. Precise estimates of the growth of the constants $B_{\mathbb{K},m}^{\mathrm{pol}}$ are crucial for different applications. The following diagram shows the evolution of the estimates of $B_{\mathbb{K},m}^{\mathrm{pol}}$ for complex scalars.

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Authors	Year	Estimate
Bohnenblust and Hille	1931, [6] (*Ann. Math.*)	$B_{\mathbb{C},m}^{\text{pol}} \le m^{\frac{m+1}{2m}} \left(\sqrt{2}\right)^{m-1}$
Defant, Frerick, Ortega-Cerdá, Ounaïes, and Seip	2011, [9] (*Ann. Math.*)	$B_{\mathbb{C},m}^{\text{pol}} \le \left(1+\frac{1}{m-1}\right)^{m-1} \sqrt{m}\left(\sqrt{2}\right)^{m-1}$
Bayart, Pellegrino, and Seoane-Sepúlveda	2014, [5] (*Adv. Math.*)	$B_{\mathbb{C},m}^{\text{pol}} \le C(\varepsilon)\,(1+\varepsilon)^m$

In the table above, $C(\varepsilon)\,(1+\varepsilon)^m$ means that given $\varepsilon > 0$, there is a constant $C(\varepsilon) > 0$ such that $B_{\mathbb{C},m}^{\text{pol}} \le C(\varepsilon)\,(1+\varepsilon)^m$ for all m.

For real scalars it is shown in [7, Theorem 2.2] that

$$(1.1)^m \le B_{\mathbb{R},m}^{\text{pol}} \le C(\varepsilon)\,(2+\varepsilon)^m,$$

and this means that for real scalars the hypercontractivity of $B_{\mathbb{R},m}^{\text{pol}}$ is optimal.

From now on, for any map $f:\mathbb{R}\to\mathbb{R}$ we define

$$f(\infty) := \lim_{p\to\infty} f(p).$$

When replacing ℓ_∞^n by ℓ_p^n, the extension of the polynomial Bohnenblust–Hille inequality is called polynomial Hardy–Littlewood inequality and the optimal exponents are $\frac{2mp}{mp+p-2m}$ for $2m \le p \le \infty$. More precisely, given $m, n \ge 1$, if P is a homogeneous polynomial of degree m on ℓ_p^n with $2m \le p \le \infty$ given by $P(x_1,\ldots,x_n) = \sum_{|\alpha|=m} a_\alpha \mathbf{x}^\alpha$, then there is a constant $C_{\mathbb{K},m,p}^{\text{pol}} \ge 1$ such that

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}} \le C_{\mathbb{K},m,p}^{\text{pol}}\,\|P\|,$$

and $C_{\mathbb{K},m,p}^{\text{pol}}$ does not depend on n, where $\|P\| := \sup_{z\in B_{\ell_p^n}} |P(z)|$.

This is a consequence of the multilinear Hardy–Littlewood inequality (see [2,10]). More precisely, given an integer $m \ge 1$, the multilinear Hardy–Littlewood inequality (see [1,12,14]) asserts that for $2m \le p \le \infty$ there exists a constant $C_{\mathbb{K},m,p}^{\text{mult}} \ge 1$ such that, for all continuous m–linear forms $T:\ell_p^n\times\cdots\times\ell_p^n\to\mathbb{K}$ and all positive integers n,

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} |T(e_{j_1},\ldots,e_{j_m})|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}} \le C_{\mathbb{K},m,p}^{\text{mult}}\,\|T\|$$

and the exponents $\frac{2mp}{mp+p-2m}$ are optimal, where $\|T\| := \sup_{z^{(1)},\ldots,z^{(m)}\in B_{\ell_p^n}} |T(z^{(1)},\ldots,z^{(m)})|$. When $p=\infty$ we recover the classical multilinear Bohnenblust–Hille inequality (see [6]). More precisely, it asserts that there exists a constant $B_{\mathbb{K},m}^{\text{mult}}$ such that for all continuous m-linear forms $T:\ell_\infty^n\times\cdots\times\ell_\infty^n\to\mathbb{K}$ and all positive integers n,

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} \left| T(e_{j_1},\ldots,e_{j_m}) \right|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq B_{\mathbb{K},m}^{\mathrm{mult}} \|T\| .$$

In this paper we look for upper and lower estimates for $C_{\mathbb{K},m,p}^{\mathrm{pol}}$. The notation of the constants $C_{\mathbb{K},m,p}^{\mathrm{mult}}$ and $B_{\mathbb{K},m}^{\mathrm{mult}}$ above will be used in all this paper.

2. First (and probably bad) upper estimates for $C_{\mathbb{K},m,p}^{\mathrm{pol}}$. Given $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, define $\binom{m}{\alpha} := \frac{m!}{\alpha_1! \cdots \alpha_n!}$ for $|\alpha| = m \in \mathbb{N}^*$. A straightforward consequence of the multinomial formula yields the following relationship between the coefficients of a homogeneous polynomial and the polar of the polynomial (this lemma appears in [8] and is essentially *folklore*).

Lemma 2.1. *If P is a homogeneous polynomial of degree m on $\mathbb{K}^n$ given by*

$$P(x_1,\ldots,x_n) = \sum_{|\alpha|=m} a_\alpha \mathbf{x}^\alpha$$

and L is the polar of P (i.e., the unique symmetric m-linear form associated to P), then

$$L(e_1^{\alpha_1},\ldots,e_n^{\alpha_n}) = \frac{a_\alpha}{\binom{m}{\alpha}},$$

where $\{e_1,\ldots,e_n\}$ is the canonical basis of $\mathbb{K}^n$ and $e_k^{\alpha_k}$ stands for e_k repeated α_k times.

The following result is also essentially known. We present here the details of its proof for the sake of completeness of the paper.

Proposition 2.2. *If P is a homogeneous polynomial of degree m on ℓ_p^n with $p \geq 2m$ given by $P(x_1,\ldots,x_n) = \sum_{|\alpha|=m} a_\alpha \mathbf{x}^\alpha$, then*

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq C_{\mathbb{K},m,p}^{\mathrm{pol}} \|P\|$$

with

$$C_{\mathbb{K},m,p}^{\mathrm{pol}} \leq C_{\mathbb{K},m,p}^{\mathrm{mult}} \frac{m^m}{(m!)^{\frac{mp+p-2m}{2mp}}},$$

where $C_{\mathbb{K},m,p}^{\mathrm{mult}}$ are the constants of the multilinear Hardy–Littlewood inequality.

Proof. From Lemma 2.1 we have

$$\begin{aligned} \sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}} &= \sum_{|\alpha|=m} \left(\binom{m}{\alpha} \left| L\left(e_1^{\alpha_1},\ldots,e_n^{\alpha_n}\right) \right| \right)^{\frac{2mp}{mp+p-2m}} \\ &= \sum_{|\alpha|=m} \binom{m}{\alpha}^{\frac{2mp}{mp+p-2m}} \left| L\left(e_1^{\alpha_1},\ldots,e_n^{\alpha_n}\right) \right|^{\frac{2mp}{mp+p-2m}} . \end{aligned}$$

However, for every choice of α, the term $|L(e_1^{\alpha_1},\ldots,e_n^{\alpha_n})|^{\frac{2mp}{mp+p-2m}}$ is repeated $\binom{m}{\alpha}$ times in the sum $\sum_{i_1,\ldots,i_m=1}^{n}|L(e_{i_1},\ldots,e_{i_m})|^{\frac{2mp}{mp+p-2m}}$. Thus

$$\begin{aligned}&\sum_{|\alpha|=m}\binom{m}{\alpha}^{\frac{2mp}{mp+p-2m}}\left|L\left(e_1^{\alpha_1},\ldots,e_n^{\alpha_n}\right)\right|^{\frac{2mp}{mp+p-2m}}\\&\quad=\sum_{i_1,\ldots,i_m=1}^{n}\binom{m}{\alpha}^{\frac{2mp}{mp+p-2m}}\frac{1}{\binom{m}{\alpha}}\left|L\left(e_{i_1},\ldots,e_{i_m}\right)\right|^{\frac{2mp}{mp+p-2m}}\end{aligned}$$

and, since $\binom{m}{\alpha}\le m!$, we have

$$\begin{aligned}&\sum_{|\alpha|=m}\binom{m}{\alpha}^{\frac{2mp}{mp+p-2m}}\left|L\left(e_1^{\alpha_1},\ldots,e_n^{\alpha_n}\right)\right|^{\frac{2mp}{mp+p-2m}}\\&\quad\le(m!)^{\frac{mp-p+2m}{mp+p-2m}}\sum_{i_1,\ldots,i_m=1}^{n}|L(e_{i_1},\ldots,e_{i_m})|^{\frac{2mp}{mp+p-2m}}.\end{aligned}$$

We finally obtain

$$\begin{aligned}&\left(\sum_{|\alpha|=m}|a_\alpha|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}}\\&\le\left((m!)^{\frac{mp-p+2m}{mp+p-2m}}\sum_{i_1,\ldots,i_m=1}^{n}|L(e_{i_1},\ldots,e_{i_m})|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}}\\&=(m!)^{\frac{mp-p+2m}{2mp}}\left(\sum_{i_1,\ldots,i_m=1}^{n}|L(e_{i_1},\ldots,e_{i_m})|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}}\\&\le(m!)^{\frac{mp-p+2m}{2mp}}C_{\mathbb{K},m,p}^{\text{mult}}\|L\|.\end{aligned}$$

On the other hand, it is well-known that

$$\|L\|\le\frac{m^m}{m!}\|P\|$$

and hence

$$\begin{aligned}\left(\sum_{|\alpha|=m}|a_\alpha|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}}&\le C_{\mathbb{K},m,p}^{\text{mult}}(m!)^{\frac{mp-p+2m}{2mp}}\frac{m^m}{m!}\|P\|\\&=C_{\mathbb{K},m,p}^{\text{mult}}\frac{m^m}{(m!)^{\frac{mp+p-2m}{2mp}}}\|P\|.\end{aligned}$$

□

Remark 2.3. Let us define the polarization constants for polynomials on ℓ_p spaces as

$$\mathbb{K}(m,p):=\inf\{M>0\,:\,\|L\|\le M\|P\|\},$$

where the infimum is taken over all $P \in \mathcal{P}(^m\ell_p^n)$ and L is the polar of P. Notice that using $\mathbb{K}(m,p)$ instead of $\frac{m^m}{m!}$ we may improve the inequality

$$\|L\| \leq \frac{m^m}{m!}\|P\|$$

used in the proof of Proposition 2.2. For details on polarization constants we refer to the excellent book of Dineen [11, Section 1.3].

3. The real polynomial Hardy–Littlewood inequality: lower bounds for the constants. As mentioned in the Introduction, in [7, Theorem 2.2] it is proved that $C_{\mathbb{R},m,\infty}^{\mathrm{pol}} \geq (1.1)^m$ for all $m \geq 2$. In this section we show that a similar result holds for the constants $C_{\mathbb{R},m,p}^{\mathrm{pol}}$ of the polynomial Hardy–Littlewood inequality.

Theorem 3.1. *For all positive integers $m \geq 2$ and $2m \leq p < \infty$, we have*

$$\left(\sqrt[16]{2}\right)^m \leq 2^{\frac{mp+p-6m+4}{4p}\cdot\frac{m-1}{m}} \leq C_{\mathbb{R},m,p}^{\mathrm{pol}}.$$

Proof. Let m be an even integer. Consider the m-homogeneous polynomial $P_m : \ell_p^m \to \mathbb{R}$ given by

$$P_m(x_1,\ldots,x_m) = \left(x_1^2 - x_2^2\right)\left(x_3^2 - x_4^2\right)\cdots\left(x_{m-1}^2 - x_m^2\right).$$

Notice that

$$\|P_m\| = P_m\left(\frac{1}{\sqrt[p]{m/2}}, 0, \frac{1}{\sqrt[p]{m/2}}, \ldots, \frac{1}{\sqrt[p]{m/2}}, 0\right) = \left(\frac{1}{\sqrt[p]{m/2}}\right)^m.$$

From the Hardy–Littlewood inequality for P_m, we have

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}} \leq C_{\mathbb{R},m,p}^{\mathrm{pol}}\|P_m\|,$$

i.e.,

$$C_{\mathbb{R},m,p}^{\mathrm{pol}} \geq \frac{\left(2^{\frac{m}{2}}\right)^{\frac{mp+p-2m}{2mp}}}{\left(\frac{1}{\sqrt[p]{m/2}}\right)^m} = 2^{\frac{mp+p-2m}{4p}}\left(\frac{m}{2}\right)^{\frac{m}{p}} = 2^{\frac{mp+p-6m}{4p}} m^{\frac{m}{p}} \geq 2^{\frac{mp+p-6m}{4p}}.$$

If m is odd, define

$$Q_m(x_1,\ldots,x_m) = \left(x_1^2 - x_2^2\right)\left(x_3^2 - x_4^2\right)\cdots\left(x_{m-2}^2 - x_{m-1}^2\right)x_m.$$

Then

$$\|Q_m\| \leq \|P_{m-1}\| = \left(\frac{1}{\sqrt[p]{(m-1)/2}}\right)^{m-1}.$$

From the Hardy–Littlewood inequality for Q_m, we have

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}} \leq C^{\mathrm{pol}}_{\mathbb{R},m,p} \|Q_m\|,$$

i.e.,

$$C^{\mathrm{pol}}_{\mathbb{R},m,p} \geq \frac{\left(2^{\frac{m-1}{2}}\right)^{\frac{mp+p-2m}{2mp}}}{\left(\frac{1}{\sqrt[p]{(m-1)/2}}\right)^{m-1}} = 2^{\frac{(mp+p-2m)(m-1)}{4mp}} \left(\frac{m-1}{2}\right)^{\frac{m-1}{p}}$$
$$= 2^{\frac{mp+p-6m+4}{4p}\cdot\frac{m-1}{m}} (m-1)^{\frac{m-1}{p}} \geq 2^{\frac{mp+p-6m+4}{4p}\cdot\frac{m-1}{m}}.$$

□

Remark 3.2. From the estimates of the last proof, note that if m is even and $m \geq 4$, then

$$C^{\mathrm{pol}}_{\mathbb{R},m,p} \geq 2^{\frac{mp+p-6m}{4p}} m^{\frac{m}{p}} \geq 2^{\frac{mp+p-6m}{4p}} \left(2^{\frac{3}{2}}\right)^{\frac{m}{p}} = \left(\sqrt[4]{2}\right)^{m+1}. \tag{3.1}$$

If m is odd, and $m \geq 5$, then

$$C^{\mathrm{pol}}_{\mathbb{R},m,p} \geq 2^{\frac{mp+p-6m+4}{4p}\cdot\frac{m-1}{m}} (m-1)^{\frac{m-1}{p}} \geq 2^{\frac{mp+p-6m+4}{4p}\cdot\frac{m-1}{m}} \left(2^{\frac{3}{2}}\right)^{\frac{m-1}{p}}$$
$$= 2^{\frac{pm^2+4m-p-4}{4mp}} \geq \left(\sqrt[4]{2}\right)^{m-\frac{1}{m}}. \tag{3.2}$$

Thus, by (3.1) and (3.2), if $m \geq 4$,

$$C^{\mathrm{pol}}_{\mathbb{R},m,p} \geq \left(\sqrt[4]{2}\right)^{m-\frac{1}{m}}.$$

4. The complex polynomial Hardy–Littlewood inequality: upper estimates. The following multi-index notation will come in handy for us: for positive integers m, n, we set

$$\mathcal{M}(m,n) := \{\mathbf{i} = (i_1, \ldots, i_m);\, i_1, \ldots, i_m \in \{1, \ldots, n\}\},$$
$$\mathcal{J}(m,n) := \{\mathbf{i} \in \mathcal{M}(m,n);\, i_1 \leq i_2 \leq \cdots \leq i_m\},$$

and for $k = 1, \ldots, m$, $\mathcal{P}_k(m)$ denotes the set of the subsets of $\{1, \ldots, m\}$ with cardinality k. For $S = \{s_1, \ldots, s_k\} \in \mathcal{P}_k(m)$, its complement will be $\widehat{S} := \{1, \ldots, m\} \setminus S$, and $\mathbf{i}_S$ shall mean $(i_{s_1}, \ldots, i_{s_k}) \in \mathcal{M}(k,n)$. For a multi-index $\mathbf{i} \in \mathcal{M}(m,n)$, we denote by $|\mathbf{i}|$ the cardinality of the set of multi-indexes $\mathbf{j} \in \mathcal{M}(m,n)$ such that there is a permutation σ of $\{1, \ldots, m\}$ with $i_{\sigma(k)} = j_k$, for every $k = 1, \ldots, m$. The equivalence class of $\mathbf{i}$ is denoted by $[\mathbf{i}]$. When we write $c_{[\mathbf{i}]}$ for $\mathbf{i} \in \mathcal{M}(m,n)$, we mean $c_{\mathbf{j}}$ for $\mathbf{j} \in \mathcal{J}(m,n)$ and $\mathbf{j}$ equivalent to $\mathbf{i}$.

The following very recent generalization of the famous Blei inequality will be crucial for our estimates (see [5, Remark 2.2]).

Lemma 4.1 (Bayart, Pellegrino, Seoane [5]). *Let m, n be positive integers, $1 \leq k \leq m$ and $1 \leq s \leq q$, satisfying $\frac{m}{\rho} = \frac{k}{s} + \frac{m-k}{q}$. Then for all scalar matrices $(a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,n)}$,*

$$\left(\sum_{\mathbf{i}\in\mathcal{M}(m,n)} |a_{\mathbf{i}}|^{\rho}\right)^{\frac{1}{\rho}} \leq \prod_{S\in\mathcal{P}_k(m)} \left(\sum_{\mathbf{i}_S}\left(\sum_{\mathbf{i}_{\hat{S}}} |a_{\mathbf{i}}|^{q}\right)^{\frac{s}{q}}\right)^{\frac{1}{s}\cdot\frac{1}{\binom{m}{k}}}.$$

Let us use the following notation: $S_{\ell_p^n}$ denotes the unit sphere on ℓ_p^n if $p < \infty$, and $S_{\ell_\infty^n}$ denotes the n-dimensional torus. More precisely: for $p \in (0, \infty)$

$$S_{\ell_p^n} := \left\{\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n : \|\mathbf{z}\|_{\ell_p^n} = 1\right\},$$

and

$$S_{\ell_\infty^n} := \mathbb{T}^n = \{\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_i| = 1\}.$$

Let μ^n be the normalized Lebesgue measure on the respective set. The following lemma is a particular instance ($1 \leq p = s \leq 2$ and $q = 2$) of the Khinchin–Steinhaus polynomial inequalities (for polynomials homogeneous or not) and $p \leq q$.

Lemma 4.2. *Let $1 \leq s \leq 2$. For every m-homogeneous polynomial $P(\mathbf{z}) = \sum_{|\alpha|=m} a_\alpha \mathbf{z}^\alpha$ on $\mathbb{C}^n$ with values in $\mathbb{C}$, we have*

$$\left(\sum_{|\alpha|=m} |a_\alpha|^2\right)^{\frac{1}{2}} \leq \left(\frac{2}{s}\right)^{\frac{m}{2}} \left(\int_{\mathbb{T}^n} |P(\mathbf{z})|^s \, d\mu^n(\mathbf{z})\right)^{\frac{1}{s}}.$$

When $n = 1$ a result due to Weissler (see [15]) asserts that the optimal constant for the general case is $\sqrt{2/s}$. In the n-dimensional case, the best constant for m-homogeneous polynomials is $(\sqrt{2/s})^m$ (see also [4]).

For $m \in [2, \infty]$ let us define $p_0(m)$ as the infimum of the values of $p \in [2m, \infty]$ such that for all $1 \leq s \leq \frac{2p}{p-2}$ there is a $K_{s,p} > 0$ such that

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2p}{p-2}}\right)^{\frac{p-2}{2p}} \leq K_{s,p}^m \left(\int_{S_{\ell_p^n}} |P(\mathbf{z})|^s \, d\mu^n(\mathbf{z})\right)^{\frac{1}{s}} \tag{4.1}$$

for all positive integers n and all m-homogeneous polynomials $P : \mathbb{C}^n \to \mathbb{C}$. For the sake of simplicity, $p_0(m)$ will be simply denoted by p_0. From Lemma 4.2 we know that this definition makes sense, since from this lemma we know that (4.1) is valid for $p = \infty$. We conjecture that $p_0 \leq m^2$.

Now, let us state and prove the main result of this section. The argument of the proof follows the lines of that in [5,9]. We will use the following result due L. Harris (see [11, Exercise 1.68]):

Lemma 4.3 (Harris). *Let X be a complex normed linear space. If P is a homogeneous polynomial of degree m on X and L is the polar of P, then, for* *any nonnegative integers* $m_1,\ldots,m_k$ *with* $m_1+\cdots+m_k=m$ *and for any* $x^{(1)},\ldots,x^{(k)}$ *unit vectors in* X,

$$\left|L(\underbrace{x^{(1)},\ldots,x^{(1)}}_{m_1\ times},\ldots,\underbrace{x^{(k)},\ldots,x^{(k)}}_{m_k\ times})\right|\leq\frac{m_1!\cdots m_k!\cdot m^m}{m_1^{m_1}\cdots m_k^{m_k}\cdot m!}\|P\|.$$

Theorem 4.4. *Let* $m\in[2,\infty]$ *and* $1\leq k\leq m-1$. *If* $p_0(m-k)<p\leq\infty$ *(and* $p=\infty$ *if* $p_0(m-k)=\infty$*) then, for every* m*-homogeneous polynomial* $P:\ell_p^n\to\mathbb{C}$ *defined by* $P(\mathbf{z})=\sum_{|\alpha|=m}a_\alpha\mathbf{z}^\alpha$, *we have*

$$\begin{aligned}\left(\sum_{|\alpha|=m}|a_\alpha|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}}&\leq K^{m-k}_{\frac{2kp}{kp+p-2k},p}\cdot\frac{m^m}{(m-k)^{m-k}}\\&\cdot\left(\frac{(m-k)!}{m!}\right)^{\frac{p-2}{2p}}\left(\frac{2}{\sqrt{\pi}}\right)^{\frac{2k(k-1)}{p}}\\&\cdot\left(B^{\mathrm{mult}}_{\mathbb{C},k}\right)^{\frac{p-2k}{p}}\|P\|,\end{aligned}$$

where $B^{\mathrm{mult}}_{\mathbb{C},k}$ *is the optimal constant of the multilinear Bohnenblust–Hille inequality associated with* k*-linear forms.*

Proof. We can also write

$$P(\mathbf{z})=\sum_{\mathbf{i}\in\mathcal{J}(m,n)}c_{\mathbf{i}}z_{i_1}\ldots z_{i_m}.$$

Consider

$$\rho=\frac{2mp}{mp+p-2m},\ s_k=\frac{2kp}{kp+p-2k},\quad\text{and}\quad q=\frac{2p}{p-2}.$$

Note that

$$s_k\leq 2<q\quad\text{and}\quad\frac{m}{\rho}=\frac{mp+p-2m}{2p}$$

and

$$\begin{aligned}\frac{k}{s_k}+\frac{m-k}{q}&=\frac{kp+p-2k}{2p}+\frac{(m-k)(p-2)}{2p}\\&=\frac{kp+p-2k}{2p}+\frac{mp-kp-2m+2k}{2p}\\&=\frac{mp+p-2m}{2p}.\end{aligned}$$

Thus

$$\frac{m}{\rho}=\frac{k}{s_k}+\frac{m-k}{q}$$

and we can use Lemma 4.1.

Let $L : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{C}$ be the unique symmetric m-linear map associated to P. Note that

$$L\left(\mathbf{z}^{(1)}, \ldots, \mathbf{z}^{(m)}\right) = \sum_{\mathbf{i} \in \mathcal{M}(m,n)} \frac{c_{[\mathbf{i}]}}{|\mathbf{i}|} z_{i_1}^{(1)} \ldots z_{i_m}^{(m)}.$$

Thus

$$\begin{aligned}
\sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}} &= \sum_{\mathbf{i} \in \mathcal{J}(m,n)} |c_{\mathbf{i}}|^{\frac{2mp}{mp+p-2m}} \\
&= \sum_{\mathbf{i} \in \mathcal{M}(m,n)} |\mathbf{i}|^{\frac{-p}{mp+p-2m}} \left(\frac{\left|c_{[\mathbf{i}]}\right|}{|\mathbf{i}|^{\frac{1}{q}}}\right)^{\frac{2mp}{mp+p-2m}} \\
&\leq \sum_{\mathbf{i} \in \mathcal{M}(m,n)} \left(\frac{\left|c_{[\mathbf{i}]}\right|}{|\mathbf{i}|^{\frac{1}{q}}}\right)^{\frac{2mp}{mp+p-2m}}.
\end{aligned}$$

Using Lemma 4.1 with $s_k = \frac{2kp}{kp+p-2k}$ and $q = \frac{2p}{p-2}$, we get

$$\begin{aligned}
&\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}} \\
&\leq \left[\prod_{S \in \mathcal{P}_k} \left(\sum_{\mathbf{i}_S \in \mathcal{M}(k,n)} \left(\sum_{\mathbf{i}_{\hat{S}} \in \mathcal{M}(m-k,n)} \left(\frac{\left|c_{[\mathbf{i}]}\right|}{|\mathbf{i}|^{\frac{1}{q}}}\right)^q\right)^{\frac{s_k}{q}}\right)^{\frac{1}{s_k}}\right]^{\frac{1}{\binom{m}{k}}}.
\end{aligned}$$

Note that $|\mathbf{i}| \leq |\mathbf{i}_{\hat{S}}| \frac{m!}{(m-k)!}$, and thus

$$\begin{aligned}
\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}} &\leq \left(\frac{m!}{(m-k)!}\right)^{\frac{q-1}{q}} \\
&\times \left[\prod_{S \in \mathcal{P}_k} \left(\sum_{\mathbf{i}_S \in \mathcal{M}(k,n)} \left(\sum_{\mathbf{i}_{\hat{S}} \in \mathcal{M}(m-k,n)} \frac{\left|c_{[\mathbf{i}]}\right|^q}{|\mathbf{i}|} \left(\frac{\left|\mathbf{i}_{\hat{S}}\right|}{|\mathbf{i}|}\right)^{q-1}\right)^{\frac{s_k}{q}}\right)^{\frac{1}{s_k}}\right]^{\frac{1}{\binom{m}{k}}} \\
&= \left(\frac{m!}{(m-k)!}\right)^{\frac{q-1}{q}} \\
&\times \left[\prod_{S \in \mathcal{P}_k} \left(\sum_{\mathbf{i}_S \in \mathcal{M}(k,n)} \left(\sum_{\mathbf{i}_{\hat{S}} \in \mathcal{M}(m-k,n)} \frac{\left|c_{[\mathbf{i}]}\right|^q}{|\mathbf{i}|^q} \left|\mathbf{i}_{\hat{S}}\right|^{q-1}\right)^{\frac{s_k}{q}}\right)^{\frac{1}{s_k}}\right]^{\frac{1}{\binom{m}{k}}}.
\end{aligned}$$

Let us fix $S \in \mathcal{P}_k(m)$. There is no loss of generality in supposing $S = \{1, \ldots, k\}$. We then fix some $\mathbf{i}_S \in \mathcal{M}(k,n)$ and we introduce the following $(m-k)$-homogeneous polynomial on ℓ_p^n:

$$P_{\mathbf{i}_S}(\mathbf{z}) = L\left(e_{i_1},\ldots,e_{i_k},\mathbf{z},\ldots,\mathbf{z}\right).$$

Observe that

$$P_{\mathbf{i}_S}(\mathbf{z}) = \sum_{\mathbf{i}_{\hat{S}}\in\mathcal{M}(m-k,n)} \frac{c_{[\mathbf{i}]}}{|\mathbf{i}|} z_{\mathbf{i}_{\hat{S}}} = \sum_{\mathbf{i}_{\hat{S}}\in\mathcal{J}(m-k,n)} \frac{c_{[\mathbf{i}]}}{|\mathbf{i}|}\left|\mathbf{i}_{\hat{S}}\right| z_{\mathbf{i}_{\hat{S}}}$$

and so

$$\|P_{\mathbf{i}_S}(\mathbf{z})\|_q = \left(\sum_{\mathbf{i}_{\hat{S}}\in\mathcal{J}(m-k,n)} \frac{\left|c_{[\mathbf{i}]}\right|^q}{|\mathbf{i}|^q}\left|\mathbf{i}_{\hat{S}}\right|^q\right)^{\frac{1}{q}} = \left(\sum_{\mathbf{i}_{\hat{S}}\in\mathcal{M}(m-k,n)} \frac{\left|c_{[\mathbf{i}]}\right|^q}{|\mathbf{i}|^q}\left|\mathbf{i}_{\hat{S}}\right|^{q-1}\right)^{\frac{1}{q}}.$$

By the definition of p_0, we have

$$\|P_{\mathbf{i}_S}(\mathbf{z})\|_q^{s_k} \le K_{s_k,p}^{(m-k)s_k}\int\limits_{S_{\ell_p^n}} \left|L\left(e_{i_1},\ldots,e_{i_k},\mathbf{z},\ldots,\mathbf{z}\right)\right|^{s_k} d\mu^n(\mathbf{z}).$$

Thus,

$$\sum_{\mathbf{i}_S}\left(\sum_{\mathbf{i}_{\hat{S}}} \frac{\left|c_{[\mathbf{i}]}\right|^q}{|\mathbf{i}|^q}\left|\mathbf{i}_{\hat{S}}\right|^{q-1}\right)^{\frac{1}{q}\times s_k} \le K_{s_k,p}^{(m-k)s_k}\int\limits_{S_{\ell_p^n}}\sum_{\mathbf{i}_S} \left|L\left(e_{i_1},\ldots,e_{i_k},\mathbf{z},\ldots,\mathbf{z}\right)\right|^{s_k} d\mu^n(\mathbf{z}).$$

Now fixing $\mathbf{z}\in S_{\ell_p^n}$ we apply the multilinear Hardy–Littlewood inequality to the $k-$linear form $\left(\mathbf{z}^{(1)},\ldots,\mathbf{z}^{(k)}\right)\mapsto L\left(\mathbf{z}^{(1)},\ldots,\mathbf{z}^{(k)},\mathbf{z},\ldots,\mathbf{z}\right)$ and we obtain, from [3, Theorem 1.1] and Lemma 4.3,

$$\begin{aligned}
&\sum_{\mathbf{i}_S}\left|L\left(e_{i_1},\ldots,e_{i_k},\mathbf{z},\ldots,\mathbf{z}\right)\right|^{s_k}\\
&\le\left(\left(\frac{2}{\sqrt{\pi}}\right)^{\frac{2k(k-1)}{p}}\cdot\left(B_{\mathbb{C},k}^{\text{mult}}\right)^{\frac{p-2k}{p}}\cdot\sup_{\mathbf{z}^{(1)},\ldots,\mathbf{z}^{(k)}\in S_{\ell_p^n}}\left|L\left(\mathbf{z}^{(1)},\ldots,\mathbf{z}^{(k)},\mathbf{z},\ldots,\mathbf{z}\right)\right|\right)^{s_k}\\
&\le\left(\left(\frac{2}{\sqrt{\pi}}\right)^{\frac{2k(k-1)}{p}}\cdot\left(B_{\mathbb{C},k}^{\text{mult}}\right)^{\frac{p-2k}{p}}\cdot\frac{(m-k)!\cdot m^m}{(m-k)^{m-k}\cdot m!}\|P\|\right)^{s_k}.
\end{aligned}$$

Thus

$$\begin{aligned}
\left(\sum_{|\alpha|=m}|a_\alpha|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}} &\le \left(\frac{m!}{(m-k)!}\right)^{\frac{q-1}{q}}\\
&\quad\cdot K_{s_k,p}^{m-k}\cdot\frac{(m-k)!\cdot m^m}{(m-k)^{m-k}\cdot m!}\cdot\left(\frac{2}{\sqrt{\pi}}\right)^{\frac{2k(k-1)}{p}}\\
&\quad\cdot\left(B_{\mathbb{C},k}^{\text{mult}}\right)^{\frac{p-2k}{p}}\|P\|.
\end{aligned}$$

□

5. Real versus complex estimates. As it happens with the constants of the Bohnenblust–Hille inequality, we observe that

$$C_{\mathbb{R},m,p}^{\mathrm{pol}} \leq 2^{m-1} C_{\mathbb{C},m,p}^{\mathrm{pol}}. \tag{5.1}$$

In fact, from [13] we know that if $P : \ell_p \to \mathbb{R}$ is an m-homogeneous polynomial and $P_{\mathbb{C}} : \ell_p \to \mathbb{C}$ is the same polynomial, then

$$\|P_{\mathbb{C}}\| \leq 2^{m-1} \|P\|.$$

We thus obtain (5.1). So if one succeeds in proving that $C_{\mathbb{R},m,p}^{\mathrm{pol}} \leq C^m$ (for all $p \geq 2m$) for a certain $C \geq 1$, as it happens with the constants of the Bohnenblust–Hille inequality, then we immediately conclude that a similar result holds for real scalars (with the constant multiplied by two).

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